THE UNIVERSITY OF CHICAGO

TOROIDAL COMPACTIFICATIONS OF INTEGRAL MODELS OF SHIMURA VARIETIES OF HODGE TYPE

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
KEERTHI SHYAM MADAPUSI SAMPATH

CHICAGO, ILLINOIS
JUNE 2011

To Amma, Appa and Emily;
my three pillars.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

The work for this thesis was begun (and might even be ended) under the guidance of my advisor, Mark Kisin. Without his support, optimism, patience and insight, it would never seen the light of day; and, even after it did, it would have been in a state much shoddier than its present (tolerable) condition, were it not for his very careful reading. For all this, I am sincerely grateful.

I owe much to Kazuya Kato, for his generosity with his time, and his ideas, many of which form the backbone of this work. I have also benefited greatly from mathematical conversations and correspondence with: Brian Conrad, Robert Kottwitz, Madhav Nori, Benoit Stroh, Kai-Wen Lan, Sug Woo Shin, Tasho Kaletha, Adrian Iovita, Avner Ash, and many others. I am thankful to all of them for their kindness and their insight, as well as to the Department of Mathematics at the University of Chicago, for all the opportunities and resources it has provided me over these long years...

...years that would have been intolerable without the emotional and social bulwark provided by friends and family. I am in great debt:

To my friends and colleagues, Khalid, Avan, Alan, Timur, Tasho: I have learned something valuable from each of them.

To Mark and Leslie, for all their support, and for having so kindly and readily accepted me into their family.

To Emily, my constant companion, for all her love and caring, and for kicking some sense and discipline into me. This thesis is dedicated to her...

...and to my parents, to whom I owe my being, for their countless kindnesses and sacrifices, and their infinite capacity for love.

INTRODUCTION

The Grothendieck-Lefschetz trace formula tells us that, to understand the Galois module structure on the l-adic cohomology of a variety defined over a number field, it is important to have a good grasp of its reductions at the finite places of that field. Moreover, in the situation where the variety is not proper, we would like to have good compactifications of these reductions that can facilitate the computation of the cohomology. A class of varieties that plays a very important role in the realization of the Langlands program, and whose cohomology we would like to understand well, are Shimura varieties. In [Kis10], Kisin constructed good integral models for a large class of Shimura varieties; these can then be used to study their reductions. The goal of this thesis is to construct good compactifications for these integral models.

Integral models of Shimura varieties

A Shimura variety $\mathrm{Sh}_K(G, X)$ (see [Del79]) arises from three pieces of data:

- A reductive group G over $\mathbb{Q}$.
- A $G(\mathbb{R})$-conjugacy class X of homomorphisms

$$\mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$$

  satisfying certain properties.
- A compact open sub-group $K \subset G(\mathbb{A}_f)$ of the finite adélic points of G.

It is a quasi-projective variety defined over a number field $E(G, X) \subset \mathbb{C}$ associated with the pair (G, X), and its $\mathbb{C}$-points are given by

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/K.$$

The existence of canonical integral models[1] for Shimura varieties at places where G is unramified was conjectured by Langlands in [Lan76]. For certain Shimura varieties, those of PEL type, which parametrize abelian varieties equipped with **P**olarization, **E**ndomorphisms and **L**evel structure, it is possible to construct integral models using the moduli of abelian schemes; see, for instance, [Kot84].

Since, for arbitrary Shimura data, we no longer have moduli interpretations of the associated Shimura varieties, there is no global moduli-theoretic method that will work in

1. where 'canonical' was later made precise by Milne. See [Moo98]: the original version had to be corrected slightly.

general. Even when the Shimura variety parametrizes polarized abelian varieties with certain additional Hodge cycles, since the Hodge conjectures are still beyond current expertise, this moduli interpretation cannot be extended in any natural way to finite characteristic. Nonetheless, consider the following conditions:

- $v|p$ is a place of $E = E(G, X)$, with $p > 2$;
- The p-primary component $K_p \subset G(\mathbb{Q}_p)$ is of the form $\mathscr{G}(\mathbb{Z}_p)$, for a reductive model $\mathscr{G}$ over $\mathbb{Z}_p$ for G. Such a maximal compact is called **hyperspecial** and exists whenever G is quasi-split at p and split over an unramified extension $\mathbb{Z}_p$.
- The prime-to-p level K^p is small enough;
- (G, X) is of *Hodge* type: that is, we have an embedding $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$ into the symplectic Shimura datum.

Under these hypotheses, Kisin, employing an idea of Faltings, has constructed a canonical model for $\mathrm{Sh}_K(G, X)$ over $\mathscr{O}_{E,v}$ in [Kis10].[2] [3] Let us denote this model by $\mathscr{S}_K(G, X)_v$.

Integral compactifications

The work in this article arose from thinking about the following natural question:

- Suppose $\mathrm{Sh}_K(G, X)$ is projective (this happens precisely when $G/Z(G)$ is anisotropic over $\mathbb{Q}$, and is equivalent to saying that there are no non-trivial unipotent elements in $G(\mathbb{Q})$; see [BB66]); then is the canonical model $\mathscr{S}_K(G, X)_v$ proper over $\mathscr{O}_{E,v}$?

We show that the answer is in the affirmative.

More generally, we have the following result:

Theorem 1. *The integral canonical model $\mathscr{S} = \mathscr{S}_K(G, X)_v$ admits a good toroidal compactification $\overline{\mathscr{S}} = \overline{\mathscr{S}}_K(G, X)_v$ over $\mathscr{O}_{E,v}$. In particular, étale locally around any point, the embedding $\mathscr{S} \subset \overline{\mathscr{S}}$ is isomorphic to a torus embedding $T \subset \overline{T}$ (see* [AMRT10]*), and the boundary $\overline{\mathscr{S}} \setminus \mathscr{S}$ is an effective Cartier divisor over $\mathscr{O}_{E,v}$. Moreover, the boundary admits a stratification parametrized by a conical complex that can be described explicitly in terms of the Shimura datum (G, X, K).*

---

2. A proof along these lines of the existence of these models has also been claimed by Vasiu in [Vas99].

3. In fact, Kisin has extended this construction to Shimura varieties of the more general *abelian* type as well. We will consider the problem of their compactification in a future paper. He can also work with $p = 2$ under some further restrictions, but we will not have anything to say about the even prime in this article.

In particular, the boundary divisor is flat over $\mathscr{O}_{E,v}$, and is therefore empty if and only if it is already empty over E; that is, $\mathscr{S}_K(G,X)_v$ is proper if and only if $\mathrm{Sh}_K(G,X)$ is.

Remark 1. The compactification we construct is log smooth, but not necessarily smooth, and the boundary divisor is not necessarily one with normal crossings. This is an artifact of our method of construction. But the singularities of the compactification can be systematically resolved by the general theory of torus embeddings from [KKMSD73]. Once this is done, we will have an affirmative answer to Conjecture 2.18 in [Mil92] on the existence of smooth toroidal compactifications of $\mathscr{S}$.

Such good compactifications were originally constructed in complete generality over $\mathbb{C}$ by Mumford and his collaborators in [AMRT10]. Later on, Faltings-Chai constructed integral compactifications for the Siegel modular variety in [FC90], and their methods were amplified and extended to the case of Shimura varieties of PEL type by Kai-Wen Lan in [Lan08]. We note that our construction is the first (that we are aware of) that works for spin groups associated with odd-dimensional quadratic forms (type B_n, $n > 2$). Indeed, by the Kottwitz classification in [Kot92], such groups can never appear in Shimura data of PEL type.

Morita's conjecture

Theorem 1 has the following pleasant consequence:

Theorem 2. *Suppose A is an abelian variety defined over a number field F, and suppose its Mumford-Tate group G is anisotropic modulo its center. Then, for every $p > 2$ such that G has a reductive model over $\mathbb{Z}_p$ and for every finite place $v|p$ of F, A has potentially good reduction over F_v.*

The hypothesis on the Mumford-Tate group ensures that A does not 'degenerate in characteristic 0'. The theorem says that this is enough to keep it from degenerating in finite characteristic as well. This result proves a good part of Yasuo Morita's conjecture (see [Mor75]), in whose statement there is no restriction on p. Other such partial results can be found in [Pau04],[Vas08] and [Lan10b]. The first two papers, as part of their hypotheses, impose certain local conditions on G, but prove the full conjecture of Morita, without restriction on p, for G that satisfy these constraints. In the last cited paper, Lan also proves the full conjecture as long as A appears in the family of abelian varieties over a compact Shimura variety of PEL type. This is a consequence of a more general group-theoretic bound on the toric rank of semi-stable abelian varieties appearing at the boundary of Shimura varieties of PEL type.

On the other hand, in the statement of Theorem 2 above, there is no restriction whatsoever on the Mumford-Tate group G; but we have nothing to say about the reduction of A at primes where G is ramified.

Method of construction

Let us now give a short description of the method used in the proof of Theorem 1. It is more or less a direct generalization of that used in [Kis10] to construct the canonical model $\mathscr{S}$. Since (G, X) is of Hodge type, we can embed the Shimura variety $\mathrm{Sh}_K(G, X)$ inside $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$, for some compact open $K' \subset \mathrm{GSp}(\mathbb{A}_f)$ containing K. The latter Shimura variety has a natural integral model $\mathscr{S}'$ over $\mathscr{O}_{E_v}$, representing a certain moduli problem for principally polarized abelian varieties equipped with level structures. By the theory of [FC90] and [Lan08], $\mathscr{S}'$ admits a smooth toroidal compactification $\overline{\mathscr{S}'}$ with boundary divisor $D' = \overline{\mathscr{S}'} \setminus \mathscr{S}'$. Let $\overline{\mathscr{S}}$ be the normalization of the Zariski closure of $\mathrm{Sh}_K(G, X)$ in $\overline{\mathscr{S}'}$: this is our candidate for a good compactification of $\mathscr{S}$.

Take a closed point $s_0 \in \overline{\mathscr{S}} \setminus \mathscr{S}$ at the boundary with residue field k, and consider the completions $\hat{\mathscr{O}}_{\overline{\mathscr{S}}, s_0}$ and $\hat{\mathscr{O}}' = \hat{\mathscr{O}}_{\overline{\mathscr{S}}', s_0}$ of $\overline{\mathscr{S}}$ and $\overline{\mathscr{S}}'$ at s_0. Choose some lift $\tilde{s}_0 \in \overline{\mathscr{S}}(\mathscr{O}_K)$, for some finite extension $K/\mathbb{Q}_p$ with residue field k, and let $\hat{\mathscr{O}}$ be the co-ordinate ring of the normalization of the irreducible component of $\mathrm{Spf}\, \hat{\mathscr{O}}_{\overline{\mathscr{S}}, s_0}$ containing $\tilde{s}_0$. We show that $\mathrm{Spf}\, \hat{\mathscr{O}}$ intersects the boundary divisor transversally and that it is log smooth with respect to the log structure induced from the boundary divisor.

To do this, just as in the construction in [Kis10], we build an explicit model $R \to R_G$ for the map $\hat{\mathscr{O}}' \to \hat{\mathscr{O}}$ that, by its very definition, has the properties that we need. In loc. cit., this is accomplished, using an idea of Faltings, through the deformation theory of p-divisible groups and its relation via Dieudonné theory to certain linear algebraic objects. For us, it would have been natural to push the analogous story through for their degenerate cousins, the log p-divisible groups, defined and studied by Kato (cf. [Kata],[Katb]). Unfortunately, this theory and its Dieudonné theoretic counterpart are yet to be fully published (though, see [BCC04]). For our purposes, however, we are able to get by with the use of log 1-motifs (cf. [KKN08b]), which are essentially a generalization to the logarithmic situation of the data used to construct degenerating abelian varieties in [FC90]. We show that the local models at the boundary of the toroidal compactifications of Faltings-Chai are essentially deformation rings (in an appropriate sense) for log 1-motifs. We can associate logarithmic F-crystals with log 1-motifs and study their deformations using log crystalline theory. In this way, we obtain an explicit description of the Faltings-Chai local models.

The first step towards our construction is an extension of the 'Key Lemma' of Kisin [Kis10, 1.3.4], which allows us to perform integral transfer of absolute Hodge cycles from étale cohomology to log crystalline cohomology. More precisely, let A be the semi-stable abelian variety over K attached to the lift $\tilde{s}_0$, and set $\Lambda = H^1(A_{\overline{K}}, \mathbb{Z}_p)$: since A is semi-stable, Λ is equipped with a natural weight-monodromy filtration $W_\bullet\Lambda$. The Hodge tensors over $\mathrm{Sh}_K(G, X)$ give rise to Galois-invariant tensors $\{s_\alpha\} \subset \Lambda^{\otimes}$ (see (2.1.2.2) for an explanation of this notation) defining a reductive sub-group $G_{\mathbb{Z}_p} \subset \mathrm{GL}(\Lambda)$ (the p-adic realization of G). Let k be the residue field of K, let $W = W(k)$ be the ring of Witt vectors over k, and let $K_0 = W\left[\frac{1}{p}\right]$. By Fontaine's theory, we have a canonical p-adic comparison isomorphism

$$\Lambda \otimes_{\mathbb{Z}_p} B_{\mathrm{st}} \xrightarrow{\simeq} D \otimes_{K_0} B_{\mathrm{st}},$$

where $D \coloneqq D_{\mathrm{st}}(\Lambda)$ is the weakly admissible filtered (φ, N)-module covariantly associated with Λ. Under the isomorphism, the tensors $\{s_\alpha\}$ go over to tensors $\{s_{\alpha,\mathrm{st}}\} \subset D^{\otimes}$. Our result is then the following:

Proposition 1. *There is a natural W-lattice $M_0 \subset D$ such that $\{s_{\alpha,\mathrm{st}}\} \subset M_0^{\otimes}$. Moreover, let $\bar{k}$ be an algebraic closure of k; then there exists an isomorphism*

$$\Lambda \otimes_{\mathbb{Z}_p} W(\bar{k}) \xrightarrow{\simeq} M_0 \otimes_W W(\bar{k})$$

that takes $s_\alpha \otimes 1$ to $s_{\alpha,\mathrm{st}} \otimes 1$, and preserves the weight filtrations on both sides. In particular, the point-wise stabilizer $G_W \subset \mathrm{GL}(M_0)$ of $\{s_{\alpha,\mathrm{st}}\}$ is a pure inner form of $G_{\mathbb{Z}_p} \otimes W$, and is therefore itself reductive.

In fact, the lattice M_0 (after a Frobenius twist) arises from the logarithmic F-crystal attached to A.

Once we have this in hand, with some more input from p-adic Hodge theory, we can build our model using the 'log crystalline' realizations $s_{\alpha,\mathrm{st}}$ and additional information from the degeneration data associated with A. Essentially, our local model will be the sub-space of $\mathscr{O}'$ where the tensors $\{s_{\alpha,\mathrm{st}}\}$ propagate to parallel, φ-invariant tensors in Fil^0 of the filtered log F-crystal associated with the family of degenerating abelian varieties over $\mathscr{O}'$. This is in perfect analogy with the global situation over $\mathbb{C}$, where variations of log Hodge structures (cf. [KU09, KKN08a]) replace filtered log F-crystals.

For simplicity, let us explain this in the case where A is principally polarized and has multiplicative reduction to a split torus T with character group Y. Let $V = \Lambda \otimes \mathbb{Q}_p$; the weight-monodromy filtration arises from a short exact sequence:

$$0 \to \mathrm{Hom}(Y, \mathbb{Q}_p) \to V \to Y \otimes \mathbb{Q}_p \to 0,$$

attached to an analytic uniformization $T^{\mathrm{an}}/Y \xrightarrow{\simeq} A^{\mathrm{an}}$ of rigid analytic K-varieties. Let $U \subset \mathrm{GL}(V)$ be the unipotent sub-group associated with the weight-monodromy filtration. Then $B(Y)_{\mathbb{Q}}$, the vector space of rational symmetric bilinear forms on Y, embeds naturally inside the Lie algebra $\mathrm{Lie}\, U$. The local model for $\hat{\mathscr{O}}'$ is the completion at a closed point of a normal, affine torus embedding E_σ of a torus E over W; the co-character group $X_*(E)$ is naturally a $\mathbb{Z}$-lattice within $B(Y)_{\mathbb{Q}}$. We would then like our local model R_G for $\hat{\mathscr{O}}$ to be the completion of a torus embedding for a quotient torus E_G of E. The only natural possibility for the co-character group of E_G is $X_*(E_G) = \mathrm{Lie}(G_{\mathbb{Z}_p}) \cap X_*(E) \subset B(Y)_{\mathbb{Q}}$. *A priori*, this could even be empty! But we have the important:

Lemma 1. *$X_*(E_G)$ generates the $\mathbb{Z}_p$-module $\mathrm{Lie}(G_{\mathbb{Z}_p}) \cap \mathrm{Lie}(U)$.*

We note that a closely related statement has been considered by André; cf. [And90, V.1.6].

To explain the subtlety and interest of this lemma, suppose that A is actually defined over a field embedded in $\mathbb{C}$. Then we are relating two different rational structures on Λ: one coming from the complex analytic uniformization of $A_{\mathbb{C}}$ via Artin's comparison

isomorphism, embodied by the Hodge tensors $\{s_\alpha\}$ and the group $G_{\mathbb{Z}_p}$; and another arising from the p-adic analytic uniformization of A over K, embodied in the weight filtration on Λ. These *a priori* have little to do with each other. If all Hodge tensors on A were generated by endomorphisms and polarizations, then everything would follow trivially from functoriality.

However, in the generality we need, we end up having to again appeal to the global theory of toroidal compactifications. We situate A within a family of degenerating abelian varieties appearing at the boundary of a toroidal compactification of $\mathscr{S}'$ (perhaps different from the family over $\hat{\mathscr{O}}'$), and then show that, for all semi-stable abelian varieties A' in this family arising from points of $\mathrm{Sh}_K(G,X)$, the monodromy $N_{A'}$, naturally an element of $B(Y)\otimes\mathbb{Q}$, kills the tensors s_α, and thus gives us an element of $X_*(E_G)\otimes\mathbb{Q}$. Then, a simple lemma (cf. 3.3.2.2) shows that this gives us sufficiently many linearly independent elements of this vector space to fill up $\mathrm{Lie}(G_{\mathbb{Q}_p})\cap\mathrm{Lie}(U)$.

To show the claim about $N_{A'}$, we use results of Coleman-Iovita (cf. [CI99]). They construct an explicit Hyodo-Kato type isomorphism (cf. [HK94])

$$D_{\mathrm{st}}(\Lambda)\otimes_{K_0}K\xrightarrow{\simeq}H^1_{\mathrm{dR}}(A)$$

under which Fontaine's monodromy operator on the left hand side is taken to the map induced by N_A on the right hand side. Moreover, if $\Lambda'=H^1(A'_{\overline{K}},\mathbb{Z}_p)$, then there is a canonical φ-equivariant isomorphism $D_{\mathrm{st}}(\Lambda')\xrightarrow{\simeq}D_{\mathrm{st}}(\Lambda)$. To now show that $N_{A'}$ kills s_α, it is enough to see that, under the isomorphisms:

$$\Lambda'\otimes B_{\mathrm{dR}}\xrightarrow{\simeq}D_{\mathrm{st}}(\Lambda')\otimes B_{\mathrm{dR}}\xrightarrow{\simeq}D_{\mathrm{st}}(\Lambda)\otimes B_{\mathrm{dR}},$$

the étale realizations $\{s_{\alpha,A'}\}$ in Λ' of the Hodge tensors over $\mathrm{Sh}_K(G,X)$ are carried over to $\{s_{\alpha,\mathrm{st}}\}$; for the crystalline realizations $\{s_{\alpha,\mathrm{st},A'}\}$ in $D_{\mathrm{st}}(\Lambda')$ are killed by $N_{A'}$. By the main result of [Bla94], we only have to check that the isomorphism marked ? in the diagram below takes the de Rham realizations $\{s_{\alpha,A,\mathrm{dR}}\}$ to $\{s_{\alpha,A',\mathrm{dR}}\}$.

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(A) & \xrightarrow[\simeq]{?} & H^1_{\mathrm{dR}}(A') \\ \simeq\downarrow & & \downarrow\simeq \\ D_{\mathrm{st}}(\Lambda)\otimes K & \xrightarrow[\simeq]{} & D_{\mathrm{st}}(\Lambda')\otimes K. \end{array}$$

This is accomplished by showing that ? is given by parallel transport along the Gauss-Manin connection on the de Rham cohomology with log poles of the degenerating family of abelian varieties.

When the reduction of A has a non-trivial abelian factor, our local models R and R_G will be completed torus embeddings over certain explicit deformation spaces for p-divisible groups similar to the ones employed in [Kis10]. To finish the proof, we identify $\hat{\mathscr{O}}'$ with the

explicit model R, and show[4] that every point of the analytic space $(\mathrm{Spf}\,\hat{\mathscr{O}})^{\mathrm{an}}$ corresponding to a semi-stable abelian variety factors through $(\mathrm{Spf}\,R_G)^{\mathrm{an}}$. We can now conclude that $\hat{\mathscr{O}}$ and R_G are isomorphic via Zariski-density of such points and dimension counting. The ingredients that go into this last step are very similar and closely related to the ones sketched above

Tour of contents

In Chapter 1, we begin with a straight-forward extension of Serre-Tate theory to the study of deformations of 1-motives. The main result is (1.1.3.2). Then we introduce the notion of a log 1-motif, following Kato, and show how to associate a log Dieudonné crystal to such a gadget. We also prove a version of Grothendieck-Messing theory for log 1-motives. We end with a few more results from Dieudonné theory that we require later. Most, if not all, of this material is well-known to experts, but we have included it for lack of an adequate reference. We would suggest that the reader skip this chapter on a first reading.

In Chapter 2, we study the p-adic Hodge theory of semi-stable abelian varieties over p-adic fields. After some technical Tannakian preliminaries in § 2.1, we prove the version of the Key Lemma stated above in § 2.2. Extending a result of Kisin in the good reduction case, we give in § 2.3 an explicit description of the log F-crystal associated with a semi-stable abelian variety (cf. 2.3.2.2). We end the chapter in § 2.4 by looking at the log F-crystal associated with a family of degenerating abelian varieties. Here we study the relationship between parallel transport between the fibers of the log crystal *á la* Coleman and the Hyodo-Kato type isomorphism constructed by Coleman-Iovita. The main result here is (2.4.1.1).

Chapter 3 is the technical cornerstone of this thesis. It is here that we construct our explicit local model $R \to R_G$ and devise conditions under which it has the right properties. We direct the reader to the introductions to its various sections for a detailed description of its contents.

In Chapter 4, we finally carry out the strategy sketched above for building our toroidal compactifications. Once the definitions are all in place, this amounts to simply checking that the conditions listed in Chapter 3 are valid for the completion at a point on the boundary of the Zariski closure $\overline{\mathscr{S}}$ above. This turns out to be a reasonably pleasant task. We end with a couple of immediate applications of our result, including Theorem 2 above.

Conventions

- p will always denote an odd prime.
- All schemes will be separated.
- All rings will be commutative and unital.

4. There are some slight complications, but this is more or less the idea.

- All duals will be denoted by the super-script $\vee$, including: the R-linear dual of a module over a ring R; the dual of an abelian scheme; the Cartier dual of a p-divisible group.
- For any finite extension of $\mathbb{Q}_p$ denoted by an upper case letter (e.g. K), we will denote its residue field by the corresponding lower case letter (e.g. k), and its maximal absolutely unramified sub-extension with the addition of the sub-script 0 (e.g. K_0).
- All monoids will be commutative and with identity.

CHAPTER 1
LOG 1-MOTIFS AND LOG F-CRYSTALS

1.1 1-motives

1.1.1

Let S_0 be a scheme in which p is nilpotent, and let $S_0 \hookrightarrow S$ be a nilpotent thickening, by which we mean that S_0 is defined by an ideal $\mathscr{I} \subset \mathscr{O}_S$ such that $\mathscr{I}^n = 0$, for some $n \geq 1$. For any fppf sheaf of abelian groups H over S, let $\hat{H}$ and $H_{\mathscr{I}}$ be the sub-functors

$$\hat{H} : T \mapsto H(T^{\mathrm{red}});$$

$$H_{\mathscr{I}} : T \mapsto \ker\left(H(T) \to H(T \times_S S_0)\right),$$

for any fppf S-scheme T. Here T^{red} denotes the reduced scheme underlying T. Also set $H_0 = H \times_S S_0$.

Let $H^1_{\mathscr{I}}(S, H)$ be the group of isomorphism classes of pairs (E, ι), where E is an fppf H-torsor over S and $\iota : H_0 \xrightarrow{\simeq} E \times_S S_0$ is a trivialization. Since H is commutative this is the same as the group of isomorphism classes of fppf H-torsors over S reducing to the trivial H_0-torsor over S_0. Assume that H is p-divisible; then, for any $n \in \mathbb{Z}_{>0}$, we have the Kummer map $\partial_n : H_{\mathscr{I}}(S) \to H^1_{\mathscr{I}}(S, H[p^n])$ arising from the short exact sequence of fppf sheaves

$$0 \to H[p^n] \to H \xrightarrow{p^n} H \to 0.$$

Lemma 1.1.1.1. *Suppose also that $\hat{H}$ is representable by a formal group law over S. Then, for n large enough, ∂_n is injective. If, in addition, H is itself representable by a smooth group scheme over S, then ∂_n is an isomorphism.*

Proof. We first observe that $p^n H_{\mathscr{I}}(S) = 0$, for n large enough; this follows from [Kat81, 1.1.1]. Choose such an n; we can now easily show the injectivity of ∂_n. Indeed, suppose $h \in H_{\mathscr{I}}(S)$ is such that $\partial_n(h) = 0$. Explicitly, this means that we can find $h_n \in H_{\mathscr{I}}(S)$ such that $p^n h_n = h$, which of course implies that h is 0.

Now suppose H is representable by a smooth group scheme over S; then $H^1_{\mathscr{I}}(S, H) = 0$ and surjectivity is immediate, since the cokernel of ∂_n embeds inside this group. To see the asserted vanishing, we remark first that every fppf H-torsor is in fact locally trivial in the étale topology; this follows from [Mil80, III.3.9][1]. Since the map $S_0 \hookrightarrow S$ is purely

1. Milne only proves this when H is quasi-projective over S, but as pointed out in the remark following the proof in *loc. cit.*, this is valid in the generality we have stated. In any case, we will only require its validity when H is quasi-projective.

inseparable (or radiciel), it follows that an H-torsor over S is trivial if and only if its reduction over S_0 is so; cf. [FK88, I.3.13]. □

1.1.2

Definition 1.1.2.1. A **1-motif** over a scheme S is a complex $M = [Y \xrightarrow{u} G]$ of fppf sheaves of abelian groups over S:

- Y is a locally constant sheaf of free abelian groups, sitting in degree -1.
- G is represented by a semi-abelian scheme over S of constant toric rank, and sits in degree 0.

We will always assume that Y is in fact constant and that G is split; that is, it is an extension

$$1 \to T \to G \to A \to 0,$$

of an abelian scheme A by a split torus T.

Definition 1.1.2.2. For any $n \in \mathbb{Z}_{>0}$, the **p^n-torsion** $M[p^n]$ of a 1-motif M is the derived tensor product (of fppf sheaves) $M \otimes^L \mathbb{Z}/p^n\mathbb{Z}[-1]$. It follows from [Ray94] that $M[p^n]$ is concentrated in degree 0 and sits in a short exact sequence:

$$0 \to G[p^n] \to M[p^n] \to Y/p^nY \to 0.$$

The **p-divisible group** $M[p^\infty]$ associated with a 1-motif M is the direct limit $\lim_n M[p^n]$. It is an extension of the form

$$0 \to G[p^\infty] \to M[p^\infty] \to Y \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Remark 1.1.2.3. More explicitly, we have, for each n,

$$M[p^n] = \frac{\{(g, y) \in G \oplus Y : p^n g = u(y)\}}{\{(u(y), p^n y) : y \in Y\}},$$

as an fppf quotient.

Using this description, the proof of the next lemma is straightforward.

Lemma 1.1.2.4. *Let $M = [Y \xrightarrow{u} G]$ and $M' = [Y \xrightarrow{u'} G]$ be two 1-motives over S and let $M'' = [Y \xrightarrow{u+u'} G]$. Then the extension class of $M[p^\infty]$ in $\mathrm{Ext}^1_S\left(Y \otimes \mathbb{Q}_p/\mathbb{Z}_p, G[p^\infty]\right)$ is the sum of the classes associated with $M[p^\infty]$ and $M'[p^\infty]$.*

□

1.1.3

Let $S_0 \hookrightarrow S$ be a nilpotent thickening of schemes defined by a nilpotent ideal $\mathscr{I} \subset \mathscr{O}_S$, and suppose that p is nilpotent in S. Let $M_0 = [Y \xrightarrow{u_0} G_0]$ be a 1-motif over S_0, and let $\mathscr{G}_0 = M_0[p^\infty]$ be its associated p-divisible group over S_0. Let $\mathrm{Def}_{M_0}(S)$ be the category of tuples (M, ι), where M is a 1-motif over S and $\iota : M \times_S S_0 \xrightarrow{\simeq} M_0$ is an isomorphism, and let $\mathrm{Def}_{\mathscr{G}_0}(S)$ be the category of tuples $(\mathscr{G}, \iota)$, where $\mathscr{G}$ is a p-divisible group over S and $\iota : \mathscr{G} \times_S S_0 \xrightarrow{\simeq} \mathscr{G}_0$ is an isomorphism.

Suppose that G_0 is an extension of the form

$$0 \to T_0 \to G_0 \to A_0 \to 0,$$

where T_0 is a split torus with character group X and A_0 is an abelian scheme over S_0. Let $\mathscr{G}_0^{\mathrm{sab}} = G_0[p^\infty]$ and let $\mathscr{G}_0^{\mathrm{ab}} = A_0[p^\infty]$. We can define analogous deformation categories $\mathrm{Def}_{G_0}(S)$, $\mathrm{Def}_{A_0}(S)$, $\mathrm{Def}_{\mathscr{G}_0^{\mathrm{sab}}}(S)$ and $\mathrm{Def}_{\mathscr{G}_0^{\mathrm{ab}}}(S)$; this gives us the following diagram:

$$\begin{array}{ccc}
\mathrm{Def}_{M_0}(S) & \xrightarrow{\mathscr{G} : M \mapsto M[p^\infty]} & \mathrm{Def}_{\mathscr{G}_0}(S) \\
\downarrow & & \downarrow \\
\mathrm{Def}_{G_0}(S) & \xrightarrow{\mathscr{G}^{\mathrm{sab}} : G \mapsto G[p^\infty]} & \mathrm{Def}_{\mathscr{G}_0^{\mathrm{sab}}}(S) \\
\downarrow & & \downarrow \\
\mathrm{Def}_{A_0}(S) & \xrightarrow{\mathscr{G}^{\mathrm{ab}} : A \mapsto A[p^\infty]} & \mathrm{Def}_{\mathscr{G}_0^{\mathrm{ab}}}(S).
\end{array}$$

Some of the vertical arrows in this diagram require a little explanation. For any deformation G of G_0 over S, the embedding $T_0 \to G_0$ lifts uniquely to an embedding $T \to G$, where T is the split torus over S with character group X, and the corresponding quotient A of G will be a deformation of A_0 over S. This follows from [FC90, §II.1], and takes care of the vertical arrows on the left. For the ones on the right, we only have to observe that maps between p-divisible groups whose targets (resp. domains) are étale (resp. multiplicative) lift uniquely over infinitesimal thickenings. So, for any deformation $\mathscr{G}$ of $\mathscr{G}_0$ over S, the map $\mathscr{G}_0 \to Y \otimes \mathbb{Q}_p/\mathbb{Z}_p$ lifts uniquely to a map $\mathscr{G} \to Y \otimes \mathbb{Q}_p/\mathbb{Z}_p$, and its kernel will be a deformation $\mathscr{G}^{\mathrm{sab}}$ of $\mathscr{G}_0^{\mathrm{sab}}$. Similarly, the map $T_0[p^\infty] \to \mathscr{G}_0^{\mathrm{sab}}$ will lift uniquely to a homomorphism $T[p^\infty] \to \mathscr{G}^{\mathrm{sab}}$, whose quotient will be a deformation of $\mathscr{G}_0^{\mathrm{ab}}$. So we see that the vertical arrows on the right make sense as well.

Proposition 1.1.3.1. *The functor $\mathscr{G} : (M, \iota) \mapsto (M[p^\infty], \iota[p^\infty])$ from $\mathrm{Def}_{M_0}(S)$ to $\mathrm{Def}_{\mathscr{G}_0}(S)$ is an equivalence of categories.*

Proof. Let us show first that $\mathscr{G}$ is faithful. Suppose $M = [Y \xrightarrow{u} G]$ and $M' = [Y \xrightarrow{u'} G']$ are two deformations of M_0 over S and suppose that we have a map $f : M \to M'$ reducing to the identity on M_0. It is equivalent to giving a map $f : G \to G'$ reducing to the identity on G_0 and satisfying $fu = u'$. So it is enough to check that the functor $\mathscr{G}^{\mathrm{sab}}$ is faithful. This follows from [Kat81, 1.1.3]. To show that $\mathscr{G}$ is full, we begin with a map $h : M[p^\infty] \to M'[p^\infty]$ lifting the identity on $\mathscr{G}_0$. Any such map has to carry $G[p^\infty]$ into $G'[p^\infty]$ and induce the identity on $Y \otimes \mathbb{Q}_p/\mathbb{Z}_p$. We first claim that $\mathscr{G}^{\mathrm{sab}}$ is full and so there exists a map $f : G \to G'$ inducing $h|_{G[p^\infty]}$; indeed, this follows from the argument in [Kat81, 1.2.1], since G is p-divisible and $\hat{G}'$ is representable by a formal group law over S.

Now, it only remains to check that $fu = u'$. For fixed $y \in Y$, $g = (fu - u')(y)$ is an element of $G'_{\mathscr{I}}(S)$; we want to show that it is 0. For each $n \in \mathbb{Z}_{>0}$, we consider the $G'[p^n]$-torsor

$$E_{g,n} = \{g' \in G : p^n g' = g\}.$$

The reduction of this over S_0 is canonically isomorphic to $G_0[p^n]$ as a $G_0[p^n]$-torsor. The fact that we have the map $h : M[p^\infty] \to M'[p^\infty]$ implies that we have a trivialization $G'[p^n] \xrightarrow{\simeq} E_{g,n}$ reducing to the identity on $G_0[p^n]$. On the other hand, $E_{g,n}$ is simply the torsor corresponding to $\partial_n(g)$ in the notation of (1.1.1.1) above, and, since $\partial_n(g) = 0$, for all n, we conclude from *loc. cit.* that g must be 0.

We move on to showing essential surjectivity. For $\mathscr{G}^{\mathrm{ab}}$, this is a consequence of Serre-Tate theory; cf. [Kat81, 1.2.1]. Fix a deformation A of A_0 over S, and let $c_0^\vee : X \to A_0^\vee$ be the classifying map for G_0. Then the isomorphism classes of lifts G of G_0 over S whose maximal abelian quotient is A correspond to maps $c^\vee : X \to A^\vee$ whose reduction to S_0 is $c_0^\vee$. The collection of such maps is naturally a torsor under $\mathrm{Hom}(X, A^\vee_{\mathscr{I}}(S))$. The isomorphism classes of lifts

$$0 \to T[p^\infty] \to \mathscr{G}^{\mathrm{sab}} \to A[p^\infty] \to 0$$

over S of the extension

$$0 \to T_0[p^\infty] \to G_0[p^\infty] \to A_0[p^\infty] \to 0$$

are naturally a torsor under $\mathrm{Ext}^1_{\mathscr{I}}(A[p^\infty], T[p^\infty])$, the group of extensions that induce trivial extensions of $A_0[p^\infty]$ by $T_0[p^\infty]$. Note that, by definition,

$$\mathrm{Ext}^1_{\mathscr{I}}(A[p^\infty], T[p^\infty]) = \varprojlim_n \mathrm{Ext}^1_{\mathscr{I}}(A[p^n], T[p^n]).$$

By Cartier duality and (1.1.1.1), this extension group is identified with

$$\begin{aligned}\mathrm{Ext}^1_{\mathscr{I}}\left(X \otimes \mathbb{Q}_p/\mathbb{Z}_p, A^\vee[p^\infty]\right) &= \mathrm{Hom}\left(X, \mathrm{Ext}^1_{\mathscr{I}}\left(\mathbb{Q}_p/\mathbb{Z}_p, A^\vee[p^\infty]\right)\right) \\ &= \mathrm{Hom}\left(X, H^1_{\mathscr{I}}(S, A^\vee[p^\infty])\right) \\ &\xrightarrow[(1.1.1.1)]{\simeq} \mathrm{Hom}(X, A^\vee_{\mathscr{I}}(S)).\end{aligned}$$

Now (1.1.2.4) tells us that, on isomorphism classes, $\mathscr{G}^{\mathrm{sab}}$ induces a map of torsors under $\mathrm{Hom}\left(X, A^{\vee}_{\mathscr{J}}(S)\right)$, and so must be a bijection. The argument for essential surjectivity of the full functor $\mathscr{G}$ is similar: For a fixed deformation G of G_0, isomorphism classes of deformation $[Y \xrightarrow{u} G]$ are a torsor under $\mathrm{Hom}(Y, G_{\mathscr{J}}(S))\mathscr{G}$. Similarly, isomorphism classes of deformations of $\mathscr{G}_0$ which are extensions of the form

$$0 \to G[p^{\infty}] \to \mathscr{G} \to Y \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

form a torsor under the group

$$\mathrm{Hom}\left(Y, H^1_{\mathscr{J}}(S, G[p^{\infty}])\right) \xrightarrow[(1.1.1.1)]{\simeq} \mathrm{Hom}(Y, G_{\mathscr{J}}(S)).$$

On these isomorphism classes, again, by (1.1.2.4), $\mathscr{G}$ gives rise to a map of torsors under the group $\mathrm{Hom}\,(Y, G_{\mathscr{J}}(S))$, and is thus a bijection. □

We can extend the above results to the following situation: Suppose we have a ring R and an ideal $I \subset R$ such that R is (I,p)-adically complete. Let $R_0 = R/I$ and set $S = \mathrm{Spec}\, R$ and $S_0 = \mathrm{Spec}\, R_0$. Suppose that we have a 1-motif $M_0 = [Y \xrightarrow{u_0} G_0]$ such that the abelian quotient A_0 of G_0 is equipped with a polarization λ_0^{ab}. Then we have a diagram of functors similar to the one above with the obvious meaning to the categories involved:

$$\begin{array}{ccc}
\mathrm{Def}_{(M_0,\lambda_0^{\mathrm{ab}})}(S) & \xrightarrow{\mathscr{G} : M \mapsto M[p^{\infty}]} & \mathrm{Def}_{(\mathscr{G}_0,\lambda_0^{\mathrm{ab}}[p^{\infty}])}(S) \\
\downarrow & & \downarrow \\
\mathrm{Def}_{(G_0,\lambda_0^{\mathrm{ab}})}(S) & \xrightarrow{\mathscr{G}^{\mathrm{sab}} : G \mapsto G[p^{\infty}]} & \mathrm{Def}_{(\mathscr{G}_0^{\mathrm{sab}},\lambda_0^{\mathrm{ab}}[p^{\infty}])}(S) \\
\downarrow & & \downarrow \\
\mathrm{Def}_{(A_0,\lambda_0^{\mathrm{ab}})}(S) & \xrightarrow{\mathscr{G}^{\mathrm{ab}} : A \mapsto A[p^{\infty}]} & \mathrm{Def}_{(\mathscr{G}_0^{\mathrm{ab}},\lambda_0^{\mathrm{ab}}[p^{\infty}])}(S).
\end{array}$$

Corollary 1.1.3.2. *All the horizontal arrows above are equivalences of categories.*

Proof. That the functors are full and faithful is immediate from (1.1.3.1). Note that a formal p-divisible group over R, that is, a p-divisible group over $\mathrm{Spf}\, R$, corresponds to a unique p-divisible group over $S = \mathrm{Spec}\, R$ by [dJ95, 2.4.4]. Therefore, by standard arguments, and the essential surjectivity of the functors in *loc. cit.*, objects on the right hand side of the arrows can be realized as p-divisible groups corresponding to the *formal* counterparts of objects on the left hand side. That is, over any quotient of R where p and I are nilpotent, they arise as p-divisible groups of honest deformations of M_0 (or G_0 or A_0, as the case may be). Any polarized formal abelian scheme $(\hat{A}, \hat{\lambda})$ over $\hat{S} = \mathrm{Spf}\, R$

(here, we are using the (I,p)-adic topology on R) deforming (A_0,λ_0) can be (uniquely) algebraized into an honest abelian scheme (A,λ) over R. This is because $\hat{\lambda}$ gives us an ample line bundle over $\hat{A}$, and so we can apply formal GAGA [EGAIII, 5.4.5]. We can then bootstrap algebraicity upwards through the left hand side of the diagram using the argument in [FC90, §II.1]: this says that any extension

$$0 \to \hat{T} \to \hat{G} \to \hat{A} \to 0$$

of formal group schemes over R deforming G_0 can be uniquely algebraized to an extension

$$0 \to T \to G \to A \to 0,$$

once $\hat{A}$ has been algebraized to A. Moreover, suppose that we have a formal 1-motif $\hat{M} = [Y \xrightarrow{\hat{u}} \hat{G}]$ over R, where $\hat{G}$ can be algebraized to G. To show that $\hat{M}$ can be algebraized, we only have to check that $H^0(\hat{S},\hat{G}) = H^0(S,G)$. This follows from [EGAIII, 5.4.1]. □

1.2 Log 1-motives

1.2.1

We recall some basic definitions from logarithmic geometry. References include [Kat89] and [Niz08].

Definition 1.2.1.1. A **log scheme** is a tuple $(S, \mathrm{M}_S, \alpha)$, where S is a scheme, M_S is an étale sheaf of monoids over S, and $\alpha : \mathrm{M}_S \to \mathscr{O}_S$ is a map of monoids (with $\mathscr{O}_S$ being a monoid under multiplication) such that the induced map

$$\alpha^{-1}(\mathscr{O}_S^\times) \to \mathscr{O}_S^\times$$

is an isomorphism. We will often omit α and sometimes even M_S from this notation, if this additional data is clear from context.

Maps between log schemes are defined in the obvious way. For a map $f : (S, \mathrm{M}_S) \to (T, \mathrm{M}_T)$, we will denote the induced map of sheaves of monoids $f^{-1}\,\mathrm{M}_T \to \mathrm{M}_S$ by $f^\sharp$.

Definition 1.2.1.2. A monoid P is:

1. **cancellative** if the map $P \hookrightarrow P^{\mathrm{gp}}$ into its group envelope is injective.
2. **fine** if it is cancellative and finitely generated.
3. **fine saturated** or **fs** if it is fine and if, for every $x \in P^{\mathrm{gp}}$, $x^n \in P$, for some $n \in \mathbb{Z}_{>0}$ if and only if $x \in P$.
4. **sharp** if $P^\times = \{1\}$; here, $P^\times$ is the sub-group of invertible elements in P.

A map of monoids $f : P \to Q$ is **continuous** if, for any $p \in P$, $f(p)$ is invertible in Q if and only if p is already invertible in P.

Remark 1.2.1.3. A monoid P is fine if the monoid ring $\mathbb{Z}[P]$ is finitely generated over $\mathbb{Z}$ and a domain. It is fs if the monoid ring $\mathbb{Z}[P]$ is in addition normal.

Definition 1.2.1.4. For any adjective '?' that can be applied to monoids, we will say a log scheme (S, M_S) is '?' if, for every geometric point $\bar{s} \to S$, the monoid $\mathrm{M}_{S,\bar{s}}/\mathscr{O}_{S,\bar{s}}^{\times}$ is '?'.

Definition 1.2.1.5. A **log ring** is a pair (R, M_R), where $(\operatorname{Spec} R, \mathrm{M}_R)$ is a log scheme. If R is an algebra over a ring A, then we will say that R is a **log A-algebra.**

Example 1.2.1.6. Any discrete valuation ring $\mathscr{O}$ can canonically be endowed with the structure of a log ring via the map $\mathscr{O} \setminus \{0\} \to \mathscr{O}$. Whenever we speak of such a ring as a log ring, we will mean for it to be endowed with this canonical log structure.

Example 1.2.1.7. More generally, if S is any scheme and $D \subset S$ is an effective Cartier divisor with complement $j : U = S \setminus D \hookrightarrow S$, then we can equip S with the log structure $j_*\mathscr{O}_U^{\times} \to \mathscr{O}_S$. This is the log structure **associated with the divisor** D.

Example 1.2.1.8. To any sharp, fs monoid P and any ring R, we can associate the log ring R_P. Its underlying ring is R; we have $\mathrm{M}_{R_P} = \mathbb{G}_m \times \oplus P$, and $\alpha : \mathrm{M}_{R_P} \to R$ is the map taking $P \setminus \{1\}$ to 0.

Let (S, M_S) be an fs log scheme. We have the functor $\mathbb{G}_m^{\log}$ on fs log schemes over (S, M_S) given by

$$\mathbb{G}_m^{\log} : (T, \mathrm{M}_T) \to \Gamma(T, \mathrm{M}_T^{\mathrm{gp}}).$$

For an appropriate topology (called the **Kummer log flat topology**; cf. [Niz08, 2.13]) on the category of fs log schemes over (S, M_S) refining the fppf topology on S, $\mathbb{G}_m^{\log}$ is a sheaf of abelian groups [Niz08, 2.22].

1.2.2

We will now fix an fs log scheme (S, M_S) for the rest of the section, unless otherwise notified. For any torus T over S with character group X we have the associated **log torus** $T^{\log} = \underline{\mathrm{Hom}}(X, \mathbb{G}_m^{\log})$ as a sheaf in the Kummer log flat topology. We have a short exact sequence

$$1 \to T \to T^{\log} \to \underline{\mathrm{Hom}}(X, \mathbb{G}_m^{\log}/\mathbb{G}_m) \to 1.$$

Suppose that J is a semi-abelian scheme over S that is an extension

$$0 \to T \to J \to B \to 0$$

of an abelian scheme B by a torus T. Pushing this extension forward along the inclusion $T \hookrightarrow T^{\log}$ gives us an extension of Kummer log flat sheaves

$$0 \to T^{\log} \to J^{\log} \to B \to 0$$

so that we have a short exact sequence

$$0 \to J \to J^{\log} \to \underline{\mathrm{Hom}}(X, \mathbb{G}_m^{\log}/\mathbb{G}_m) \to 0.$$

For example, if our log scheme is just $\mathrm{Spec}\,\mathscr{O}$ for a discrete valuation ring $\mathscr{O}$ with its canonical log structure, then $J^{\log}(\mathscr{O}) = J(\mathrm{Fr}(\mathscr{O}))$, where $\mathrm{Fr}(\mathscr{O})$ is the fraction field of $\mathscr{O}$, and the short exact sequence above, evaluated at $\mathscr{O}$, gives us:

$$0 \to J(\mathscr{O}) \to J(\mathrm{Fr}(\mathscr{O})) \to \mathrm{Hom}(X, \mathbb{Z}) \to 0,$$

where we fix a $\mathbb{Z}$-valuation on $\mathscr{O}$ to identify $\mathrm{Fr}(\mathscr{O})^\times/\mathscr{O}^\times$ with $\mathbb{Z}$.

Definition 1.2.2.1. A **log 1-motif** over (S, M_S) is a complex $[Y \xrightarrow{u} J^{\log}]$ in degrees $-1, 0$ of Kummer log flat sheaves of abelian groups over (S, M_S), where:

1. J is a semi-abelian scheme that is an extension

$$0 \to T \to J \to B \to 0$$

of an abelian scheme B by an iso-trivial torus T, and $J^{\log}$ is the associated sheaf described above.

2. Y is a sheaf of free abelian groups locally constant in the finite étale topology.

Recall that a torus T over S is **iso-trivial** if it is locally trivial in the finite étale topology.

Definition 1.2.2.2. To every log 1-motif $Q = [Y \xrightarrow{u} J^{\log}]$ we can associate the **monodromy map**

$$N_Q : Y \to \underline{\mathrm{Hom}}(X, \mathbb{G}_m^{\log}/\mathbb{G}_m),$$

induced from the surjection $J^{\log} \to \underline{\mathrm{Hom}}(X, \mathbb{G}_m^{\log}/\mathbb{G}_m)$.

Let J be a semi-abelian scheme as above, viewed as an extension

$$0 \to T \to J \xrightarrow{\pi} B \to 0$$

of an abelian scheme B by an iso-trivial torus T with character group X. This is classified by a homomorphism $c^\vee : X \to B^\vee$. For every $x \in X$, let J_x be the extension of B by $\mathbb{G}_m$ obtained by pushing J forward along $x : T \to \mathbb{G}_m$. Suppose that we have a homomorphism $c : Y \to B$, where Y is a free abelian group, classifying another semi-abelian extension

$$0 \to T^\vee \to J^\vee \xrightarrow{\pi^\vee} B^\vee \to 0,$$

where $T^\vee$ is the torus with character group Y. Associated with $x \in X$ and $y \in Y$, we have the $\mathbb{G}_m$-torsor $I_{y,x} = \pi_x^{-1}(c(y))$, where $\pi_x : J_x \to B$ is the quotient map.

We can package the $\mathbb{G}_m$-torsors $I_{y,x}$ into a $\mathbb{G}_m$-torsor I over $Y \times X$: this is nothing but the pull-back under the map $c \times c^\vee : Y \times X \to B \times B^\vee$ of the inverse Poincaré bundle

$\mathcal{P}_B^{-1}$ on $B \times B^\vee$. Then I has the structure of $\mathbb{G}_m$**-bi-extension** over $Y \times X$ (cf. [Mum69]). Concretely, this means that we have, for $(y,x),(y',x') \in Y \times X$, a canonical isomorphism:

$$\eta_{(y,x),(y',x')} : I_{y+y',x+x'} \xrightarrow{\simeq} I_{y,x} \otimes I_{y,x'} \otimes I_{y',x} \otimes I_{y',x'},$$

of $\mathbb{G}_m$-torsors, and these canonical isomorphisms satisfy the requisite associativity and commutativity constraints. The natural map $\mathbb{G}_m \to \mathbb{G}_m^{\log}$ induces a $\mathbb{G}_m^{\log}$-bi-extension $I^{\log}$ over $Y \times X$.

At the same time, we can also consider the pull-back $I^\vee$ of the inverse Poincaré bundle $\mathcal{P}_{B^\vee}^{-1}$ on $B^\vee \times B^{\vee\vee} = B^\vee \times B$ to $Y \times X$ under the map

$$Y \times X \xrightarrow{s} X \times Y \xrightarrow{c^\vee \times c} B^\vee \times B.$$

Here $s : Y \times X \to X \times Y$ is the 'flip' isomorphism $(y,x) \mapsto (x,y)$. This is again a $\mathbb{G}_m$-bi-extension of $Y \times X$. Concretely, for a section $(y,x) \in Y \times X$, $I^\vee_{x,y}$ is the $\mathbb{G}_m$-torsor $(\pi_y^\vee)^{-1}(c^\vee(x))$, where $\pi_y : J_y^\vee \to B^\vee$ is the natural surjection. Here, $J_y^\vee$ is the push-forward of $J^\vee$ under the character $y : T^\vee \to \mathbb{G}_m$.

Lemma 1.2.2.3. *Let the notation be as above, and, for each section $(y,x) \in Y \times X$, let $\bar{I}_{y,x}$ be the $\mathbb{G}_m^{\log}/\mathbb{G}_m$-torsor induced from $I^{\log}_{y,x}$ under the surjection $\mathbb{G}_m^{\log} \to \mathbb{G}_m^{\log}/\mathbb{G}_m$.*

1. *Giving a lift $u : Y \to J^{\log}$ of c is equivalent to giving a trivialization*

$$\tau : \mathbf{1}^{\log}_{Y \times X} \xrightarrow{\simeq} I^{\log}$$

of $\mathbb{G}_m^{\log}$-bi-extensions over $Y \times X$. Concretely, this amounts to giving trivializations $\tau(y,x) \in I^{\log}_{y,x}(S)$ of $\mathbb{G}_m^{\log}$-torsors such that

$$\tau(y+y',x+x') = \tau(y,x)\tau(y,x')\tau(y',x)\tau(y',x'),$$

for all sections $(y,x),(y',x') \in Y \times X$. Here, we make sense of the identity using the canonical isomorphism $\eta_{(y,x),(y',x')}$.

2. *The $\mathbb{G}_m$-bi-extensions I and $I^\vee$ over $Y \times X$ are canonically isomorphic.*

3. *Fix a lift $u : Y \to J^{\log}$ of c giving rise to a log 1-motif $Q = [Y \xrightarrow{u} J^{\log}]$. Let $\tau(y,x) \in I^{\log}_{y,x}(S)$ be the associated compatible trivializations as in (2), and let $\bar{\tau}(y,x)$ be their images in $\bar{I}_{y,x}$. Then in $\Gamma\left(S, \mathbb{G}_m^{\log}/\mathbb{G}_m\right)$ we have the equality*

$$\bar{\tau}(y,x) = N_Q(y)(x),$$

where N_Q is the monodromy pairing.

Proof. (1) is a direct check from the definitions. As for (2): this follows from the fact that the Poincaré bundles on $B^\vee \times B^{\vee\vee}$ and $B \times B^\vee$ are identified under the isomorphisms

$$B^\vee \times B^{\vee\vee} \xrightarrow{\simeq} B^\vee \times B \xrightarrow{s} B \times B^\vee,$$

where s is the 'flip' isomorphism.

Finally, for (3), observe that $\bar{I}_{y,x}$, being induced from $I_{y,x}$ via the trivial map $\mathbb{G}_m \xrightarrow{0} \mathbb{G}_m^{\log}/\mathbb{G}_m$ is canonically trivialized, and so we can identify it with $\mathbb{G}_m^{\log}/\mathbb{G}_m$. So it makes sense to view $\bar{\tau}(y,x)$ as an element of $\Gamma\left(S, \mathbb{G}_m^{\log}/\mathbb{G}_m\right)$. Now the claimed equality is immediate. □

Corollary 1.2.2.4. *Consider the functor that associates with each log 1-motif $Q = [Y \xrightarrow{u} J^{\log}]$ over S the tuple $(B, Y, X, c, c^\vee, \tau)$, where B is the maximal abelian scheme quotient of J, X is the character group of the maximal torus T of J, $c : Y \to B$ is the map induced from u, $c^\vee : X \to B^\vee$ is the classifying map for J, and τ a trivialization of the $\mathbb{G}_m^{\log}$-bi-extension $\left((c \times c^\vee)^* \mathcal{P}_B^{-1}\right)^{\log}$ as in (1.2.2.3)(1) above. This functor is an equivalence of categories between the category of log 1-motifs Q over S and the category of tuples $(B, Y, X, c, c^\vee, \tau)$, where*

- *B is an abelian scheme over S.*
- *Y and X are sheaves of finite free abelian groups over S locally trivial in the finite étale topology.*
- *$c : Y \to B$ and $c^\vee : X \to B^\vee$ are homomorphisms of sheaves of groups over S.*
- *$\tau : \mathbf{1}_{Y \times X}^{\log} \xrightarrow{\simeq} I^{\log} = \left((c \times c^\vee)^* \mathcal{P}_B^{-1}\right)^{\log}$ is a trivialization of $\mathbb{G}_m^{\log}$-bi-extensions over $Y \times X$.*

Proof. This is immediate. □

Definition 1.2.2.5. Let $Q = [Y \xrightarrow{u} J^{\log}]$ be a log 1-motif corresponding to the tuple $(B, Y, X, c, c^\vee, \tau)$; then the **dual log 1-motif** $Q^\vee = [X \xrightarrow{u} (J^\vee)^{\log}]$ is the log 1-motif corresponding to the tuple $(B^\vee, X, Y, c^\vee, c, \tau)$.

Remark 1.2.2.6. Suppose $Q = [Y \xrightarrow{u} T^{\log}]$, where T is the torus as above; this simply corresponds to a map

$$u : Y \to \mathrm{Hom}(X, \mathbb{G}_m^{\log}).$$

The dual $Q^\vee$ is the log 1-motif corresponding to the map

$$u^\vee : X \to \mathrm{Hom}(Y, \mathbb{G}_m^{log})$$
$$u^\vee(x)(y) = u(y)(x).$$

Definition 1.2.2.7. A **polarization** $\lambda : Q \to Q^\vee$ is a diagram

$$\begin{array}{ccc} Y & \xrightarrow{u} & J^{\log} \\ {\scriptstyle \lambda^{\text{ét}}}\downarrow & & \downarrow{\scriptstyle \lambda^{\text{sab}}} \\ X & \xrightarrow[u^\vee]{} & (J^\vee)^{\log}, \end{array}$$

where $\lambda^{\text{ét}}$ is injective with finite co-kernel, and λ^{sab} is a map of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & J & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle \lambda^{\text{mult}}}\downarrow & & {\scriptstyle \lambda^{\text{sab}}}\downarrow & & {\scriptstyle \lambda^{\text{ab}}}\downarrow & & \\ 0 & \longrightarrow & T^\vee & \longrightarrow & J^\vee & \longrightarrow & B^\vee & \longrightarrow & 0, \end{array}$$

with λ^{ab} a polarization on B and λ^{mult} the isogeny $\text{Hom}(X, \mathbb{G}_m) \xrightarrow{\lambda^{\text{ét},*}} \text{Hom}(Y, \mathbb{G}_m)$.

The **degree** of a polarization λ is the natural number $\deg(\lambda^{\text{ab}})(\sharp \operatorname{coker}(\lambda^{\text{ét}}))^2$. A polarization λ is **prime-to-N**, for some $N \in \mathbb{Z}_{\geq 0}$, if the degree of λ is prime to N.

Suppose now that we have a log 1-motif Q over S corresponding to a tuple $(B, Y, X, c, c^\vee, \tau)$. As usual, let $I = (c \times c^\vee)^* \mathcal{P}_B^{-1}$ be the associated $\mathbb{G}_m$-bi-extension over $Y \times X$. Suppose also that we have an injective map of sheaves of groups $\lambda^{\text{ét}} : Y \to X$ with finite co-kernel, and a polarization $\lambda^{\text{ab}} : B \to B^\vee$ such that the diagram:

$$\begin{array}{ccc} Y & \xrightarrow{c} & B \\ {\scriptstyle \lambda^{\text{ét}}}\downarrow & & \downarrow{\scriptstyle \lambda^{\text{ab}}} \\ X & \xrightarrow[c^\vee]{} & B^\vee, \end{array} \tag{1.2.2.7.1}$$

commutes. Let $s : B \times B \xrightarrow{\simeq} B \times B$ be the flip isomorphism; then, since λ^{ab} is a polarization, it is in particular symmetric, and we have a canonical isomorphism $s^*(1 \times \lambda^{\text{ab}})^* \mathcal{P}_B \xrightarrow{\simeq} (1 \times \lambda^{\text{ab}})^* \mathcal{P}_B$. This means that $(1 \times \lambda^{\text{ab}})^* \mathcal{P}_B^{-1}$ is a **symmetric $\mathbb{G}_m$-bi-extension** over $B \times B$ (cf. [Mum69]). In particular,

$$(1 \times \lambda^{\text{ét}})^* I = (c \times c)^* (1 \times \lambda^{\text{ab}})^* \mathcal{P}_B^{-1}$$

is a symmetric $\mathbb{G}_m$-bi-extension of $Y \times Y$. Concretely, this means that, for all sections

$(y, y') \in Y \times Y$, we have canonical identifications

$$I_{y,\lambda^{\text{ét}}(y')} \xrightarrow{\simeq} I_{y',\lambda^{\text{ét}}(y)}. \tag{1.2.2.7.2}$$

It is a direct check from the definitions that the data of λ^{ab} and $\lambda^{\text{ét}}$ can be extended to a polarization λ of Q if and only if the trivialization $\tau : \mathbf{1}^{\log}_{Y\times X} \xrightarrow{\simeq} I^{\log}$ of $\mathbb{G}_m^{\log}$-bi-extensions of $Y \times X$ induces a *symmetric* trivialization $\mathbf{1}^{\log}_{Y\times Y} \xrightarrow{\simeq} (1 \times \lambda^{\text{ét}})^* I^{\log}$ of symmetric $\mathbb{G}_m^{\log}$-bi-extensions of $Y \times Y$. Concretely, this means that, for all sections $(y, y') \in Y \times Y$, the trivializations $\tau(y, \lambda^{\text{ét}}(y'))$ and $\tau(y', \lambda^{\text{ét}}(y))$ of the two $\mathbb{G}_m^{\log}$-torsors $I^{\log}_{y,\lambda^{\text{ét}}(y')}$ and $I^{\log}_{y',\lambda^{\text{ét}}(y')}$, respectively, match up under the isomorphism between them induced from (1.2.2.7.2).

So we obtain:

Proposition 1.2.2.8. *There is an equivalence of categories between polarized log* 1*-motifs* (Q, λ) *over* S *and tuples* $(B, Y, X, c, c^\vee, \lambda^{\text{ab}}, \lambda^{\text{ét}}, \tau)$*, where* $(B, Y, X, c, c^\vee, \tau)$ *is a tuple as in (1.2.2.4), and:*

- $\lambda^{\text{ab}} : B \to B^\vee$ *is a polarization and* $\lambda^{\text{ét}} : Y \to X$ *is an injective map with finite co-kernel such that the diagram (1.2.2.7.1) commutes.*
- $(1 \times \lambda^{\text{ét}})^*\tau$ *gives a trivialization*

$$\mathbf{1}^{\log}_{Y\times Y} \xrightarrow{\simeq} \left((c \times c^\vee\lambda^{\text{ét}})^* \mathcal{P}_B^{-1}\right)^{\log}$$

of symmetric $\mathbb{G}_m^{\log}$*-bi-extensions of* $Y \times Y$ *over* S*.*

□

1.2.3

We will now discuss level structures on log 1-motifs over S. We will hew closely, modulo the appropriate translations, to [FC90, §IV.6]. Fix $N \in \mathbb{Z}_{>0}$, and let (Q, λ) be a polarized log 1-motif over S corresponding to a tuple $(B, Y, X, c, c^\vee, \lambda^{\text{ab}}, \lambda^{\text{ét}}, \tau)$ as above. We will suppose that λ is prime-to-N. We then have induced perfect pairings

$$\begin{aligned} e_{\lambda^{\text{ab}}} : B[N] \times B[N] &\to \mu_N \\ e_{\lambda^{\text{ab}}}(b, b') &= e_B(b, \lambda^{\text{ab}}(b')); \end{aligned}$$

$$\begin{aligned} e_{\lambda^{\text{ét}}} : Y/NY \times T[N] &\to \mu_N \\ e_{\lambda^{\text{ét}}}(y, t) &= \lambda^{\text{ét}}(y)(t). \end{aligned}$$

Here, μ_N is the finite flat group scheme over S of N^{th}-roots of unity, and

$$e_B : B[N] \times B^\vee[N] \to \mu_N$$

is the Weil pairing.

Suppose $r = \operatorname{rank} Y = \operatorname{rank} X$, and let $g \in \mathbb{Z}_{\geq 0}$ be such that $g - r = \dim_S B$ is the relative dimension of B over S.

Definition 1.2.3.1. Let $\Lambda_{N,g-r}$ be a free $\mathbb{Z}/N\mathbb{Z}$-module of rank $2(g-r)$ equipped with a symplectic pairing into $\mathbb{Z}/N\mathbb{Z}$. A **principal symplectic level N structure of type** $\Lambda_{N,g-r}$ on $(B, \lambda^{\mathrm{ab}})$ is an isomorphism

$$\varphi_N^{\mathrm{ab}} : \Lambda_{N,g-r} \xrightarrow{\simeq} B[N]$$

of sheaves of abelian groups over S, which carries the symplectic form on $\Lambda_{N,g-r}$ to a $(\mathbb{Z}/N\mathbb{Z})^\times$-multiple of $e_{\lambda^{\mathrm{ab}}}$. By this, we mean that there is an isomorphism of sheaves of groups

$$\nu(\varphi_N^{\mathrm{ab}}) : \mu_N \xrightarrow{\simeq} \mathbb{Z}/N\mathbb{Z}$$

such that $\nu(\varphi_N^{\mathrm{ab}}) \circ e_{\lambda^{\mathrm{ab}}} \circ \varphi_N^{\mathrm{ab}}$ is equal to the symplectic pairing on $\Lambda_{N,g-r}$.

We will usually suppress the adjectives 'principal symplectic' and refer to this simply as a **level N structure of type** $\Lambda_{N,g-r}$ on $(B, \lambda^{\mathrm{ab}})$.

Let $\Lambda_{N,g}$ be a free $\mathbb{Z}/N\mathbb{Z}$-module of rank $2g$ equipped with a symplectic pairing into $\mathbb{Z}/N\mathbb{Z}$, and let $\Sigma_{N,g,r} \subset \Lambda_{N,g}$ be a free isotropic sub-module of rank r such that the quotient $\Lambda_{N,g}/\Sigma_{N,g,r}$ is again free over $\mathbb{Z}/N\mathbb{Z}$. Let $\Psi_{N,g,r} = (\Sigma_{N,g,r})^\perp \subset \Lambda_{N,g}$ be the radical of $\Sigma_{N,g,r}$. We then have a perfect pairing

$$\Lambda_{N,g}/\Psi_{N,g,r} \times \Sigma_{N,g,r} \to \mathbb{Z}/N\mathbb{Z} \tag{1.2.3.1.1}$$

induced by the symplectic pairing on $\Lambda_{N,g}$. The sub-quotient $\Lambda_{N,g-r} = \Psi_{N,g,r}/\Sigma_{N,g,r}$ will inherit a symplectic pairing from $\Lambda_{N,g}$. Let us denote by $\mathfrak{C}$ the pair $(\Lambda_{N,g}, \Sigma_{N,g,r})$.

Definition 1.2.3.2. A **principal symplectic level N structure of type $\mathfrak{C}$** on the tuple $(B, Y, X, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}})$ is a tuple $(\varphi_N^{\mathrm{ab}}, \varphi_N^{\text{ét}}, \varphi_N^{\mathrm{mult}})$ where:

1. φ_N^{ab} is a level N structure on $(B, \lambda^{\mathrm{ab}})$ of type $\Lambda_{N,g-r}$ as in (1.2.3.1), with an associated isomorphism of sheaves of groups $\nu(\varphi_N^{\mathrm{ab}}) : \mu_N \xrightarrow{\simeq} \mathbb{Z}/N\mathbb{Z}$.

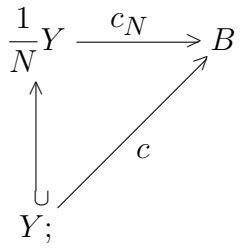
2. $$\varphi_N^{\mathrm{mult}} : \Sigma_{N,g,r} \xrightarrow{\simeq} T[N]; \text{ and}$$
$$\varphi_N^{\text{ét}} : \Lambda_{N,g}/\Psi_{N,g,r} \xrightarrow{\simeq} Y/NY;$$
are isomorphisms of sheaves of groups such that
$$\nu(\varphi_N^{\mathrm{ab}}) \circ e_{\lambda^{\text{ét}}} \circ (\varphi^{\text{ét}} \times \varphi^{\mathrm{mult}}) : \Lambda_{N,g}/\Psi_{N,g,r} \times \Sigma_{N,g,r} \to \mathbb{Z}/N\mathbb{Z}$$
is equal to the pairing in (1.2.3.1.1).

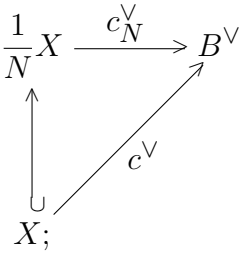
Again, we will usually omit the adjectives 'principal symplectic'.

Definition 1.2.3.3. A **principal symplectic level N structure of type $\mathfrak{C}$** on (Q,λ) is a tuple $(\varphi_N^{\mathrm{ab}},\varphi_N^{\text{ét}},\varphi_N^{\mathrm{mult}},c_N,c_N^\vee,\tau_N,\delta)$, where:

1. $(\varphi_N^{\mathrm{ab}},\varphi_N^{\text{ét}},\varphi_N^{\mathrm{mult}})$ are a level N structure of type $\mathfrak{C}$ on $(B,Y,X,\lambda^{\mathrm{ab}},\lambda^{\text{ét}})$.

2. $c_N:\frac{1}{N}Y\to B$ and $c_N^\vee:\frac{1}{N}X\to B^\vee$ are maps such that the diagrams





   commute.

3. Let I_N be the $\mathbb{G}_m$-bi-extension $(c_N\times c^\vee)^*\mathcal{P}_B^{-1}$ over $\frac{1}{N}Y\times X$. Then, τ_N is a trivialization
$$\tau_N:\mathbf{1}^{\log}_{\frac{1}{N}Y\times X}\xrightarrow{\simeq} I_N^{\log}$$
   of $\mathbb{G}_m^{\log}$-bi-extensions over $\frac{1}{N}Y\times X$ restricting to the trivialization τ of $I^{\log}$ over $Y\times X$.

4. $$\delta:\Sigma_{N,g,r}\oplus\Lambda_{N,g-r}\oplus(\Lambda_{N,g}/\Psi_{N,g,r})\xrightarrow{\simeq}\Lambda_{N,g}$$
   is a symplectic splitting of the filtration
$$0\subset\Sigma_{N,g,r}\subset\Psi_{N,g,r}\subset\Lambda_{N,g}.$$

1.2.4

Definition 1.2.4.1. Let E be a $\mathbb{G}_m$-torsor over S, and let $E^{\log}$ be the associated $\mathbb{G}_m^{\log}$-torsor. The induced $\mathbb{G}_m^{\log}/\mathbb{G}_m$-torsor $\overline{E}$ is canonically trivialized and can therefore be identified with $\mathbb{G}_m^{\log}/\mathbb{G}_m$. We say that an étale local section e of $E^{\log}$ is **positive**, if, for every geometric point $\bar{s}\to S$, the image of e in $\mathrm{M}^{\mathrm{gp}}_{S,\bar{s}}/\mathscr{O}^\times_{S,\bar{s}}$ lies in $\left(\mathrm{M}_{S,\bar{s}}/\mathscr{O}^\times_{S,\bar{s}}\right)\setminus\{1\}$.

Let (Q, λ) be a polarized log 1-motif corresponding to a tuple $(B, Y, X, c, c^\vee, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau)$. We will say that (Q, λ) is **positive** if, for every geometric point $\bar{s} \to S$, and all sections $y \in Y_{\bar{s}}$, the section $\tau(y, \lambda^{\text{ét}}(y))$ of $I^{\log}_{y,\lambda^{\text{ét}}(y)}$ is positive.

Suppose now that $S = \operatorname{Spec} R$, where R is a complete local Noetherian normal ring, and suppose that M_R is defined by a divisor $D \subset R$. Let $U \subset \operatorname{Spec} R$ be the complement of D. Let $\mathrm{DEG}_{\mathrm{pol}}$ be the category of positive polarized log 1-motifs (Q, λ) over S. Equivalently, $\mathrm{DEG}_{\mathrm{pol}}$ is the category of tuples $(B, Y, X, c, c^\vee, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau)$ as in (1.2.2.8)(2) satisfying the positivity property in (1.2.4.1). Let $\mathrm{DD}_{\mathrm{pol}}$ be the category of polarized abelian varieties (A, λ) over U that extend to semi-abelian schemes over $\operatorname{Spec} R$.

Proposition 1.2.4.2. *With the hypotheses as above, the categories* $\mathrm{DEG}_{\mathrm{pol}}$ *and* $\mathrm{DD}_{\mathrm{pol}}$ *are naturally equivalent.*

Proof. The proof can be found in [FC90, Ch. III]. □

1.3 Log F-crystals

1.3.1

Let $S = \operatorname{Spec} R$ be an affine scheme in which p is nilpotent. Following [BBM82], we have the exact contra-variant **Dieudonné crystal** functor

$$\mathbb{D} : \Big(p\text{-divisible groups over } S\Big) \longrightarrow \Big(\text{Dieudonné crystals over } S\Big).$$

We will not give a precise definition of a Dieudonné crystal, for which cf. [dJ95, 2.3.2]. However, we can give a very concrete description of a Dieudonné crystal over S in the following situation: Suppose that we have:

- A formally smooth $\mathbb{Z}_p$-algebra $\tilde{R}$ isomorphic to $W[|x_1, \ldots, x_n|]$, where $W = W(k)$ is the ring of Witt vectors with coefficients in a perfect extension $k/\mathbb{F}_p$;
- A lift $\varphi_{\tilde{R}} : \tilde{R} \to \tilde{R}$ of the p-power Frobenius map on $\tilde{R}/p\tilde{R}$; and
- A surjection $\tilde{R} \to R$ with kernel I, so that $p^n \tilde{R} \subset I$, for some $n \in \mathbb{Z}_{>0}$.

Let $D_{\tilde{R}}(I)$ be the divided power envelope of I in R, and let D_R be its p-adic completion; for every $a \in I$, let $a^{[n]} \in D_R$ be the n^{th} divided power of a. Let $\hat{\Omega}^1_{\tilde{R}/\mathbb{Z}_p}$ be the module of continuous differentials of $\tilde{R}$ over $\mathbb{Z}_p$; this is a finite free $\tilde{R}$-module. There is a natural connection $\nabla : D_R \to D_R \hat{\otimes} \hat{\Omega}^1_{\tilde{R}/\mathbb{Z}_p}$ such that $\nabla(a^{[n]}) = a^{[n-1]} \hat{\otimes} da$, for all $a \in I$. Since $p\tilde{R}$ canonically admits divided powers, we have a canonical identification $D_{\tilde{R}}(I) = D_{\tilde{R}}(I + pR)$. In particular, the Frobenius lift $\varphi_{\tilde{R}}$ extends to a Frobenius lift φ_{D_R} over D_R. We will make the following:

Assumption 1.3.1.1. D_R is flat over $\mathbb{Z}_p$.

It will hold in all situations that we consider, which will take one of the following two forms:

- $R = \tilde{R}/p^n\tilde{R}$, for some $n \in \mathbb{Z}_{>0}$, so that $D_R = \tilde{R}$.
- $R = \mathscr{O}_K/p^n\mathscr{O}_K$, for some $n \in \mathbb{Z}_{>0}$ and some finite extension $K/\mathbb{Q}_p$. We can choose $\tilde{R} = W(k)[|u|]$ and $I \subset \tilde{R}$ to be the ideal generated by p^n and an Eisenstein polynomial $E(u)$ in $W(k)[u]$ corresponding to a uniformizer in K. One checks that, in this case, D_R embeds into $K_0[|u|]$ and is thus flat over $\mathbb{Z}_p$.

Giving a Dieudonné crystal over $S = \operatorname{Spec} R$ is equivalent to giving (cf. [dJ95, §2.3]) a **Dieudonné module** over D_R; that is, a tuple (M, φ_M, ∇_M), where:

1. M is a finite free D_R-module.
2. φ_M is a D_R-linear map
$$\varphi_M : \varphi_{D_R}^* M = M \otimes_{D_R,\varphi} D_R \to M$$
whose image contains pM.
3. ∇_M is an integrable topologically quasi-nilpotent connection
$$\nabla_M : M \to M \hat{\otimes} \hat{\Omega}^1_{\tilde{R}/\mathbb{Z}_p},$$
compatible with the natural connection on D_R, for which φ_M is a parallel map. The topological quasi-nilpotence means that, for every derivation $\xi \in \operatorname{Hom}_{\tilde{R}}(\hat{\Omega}^1_{\tilde{R}/W}, \tilde{R})$, there exists $n \in \mathbb{Z}_{>0}$ such that $\nabla(\xi)^n(M) \subset pM$.

Note that, since D_R is p-torsion free by hypothesis, we need not separately require, as in [dJ95, 2.3.4], the existence of a map $V_M : M \to \varphi_{D_R}^* M$ such that $\varphi_M V_M = p$; it will be uniquely determined by φ_M.

Theorem 1.3.1.2 (de Jong)**.** *Suppose $I = pR$, so that $R = \tilde{R}/p\tilde{R}$ and $D_R = \tilde{R}$. Then the functor*

$$\mathbb{D} : \begin{pmatrix} \textit{p-divisible groups} \\ \textit{over } S \end{pmatrix} \to \begin{pmatrix} \textit{Dieudonné crystals} \\ \textit{over } S \end{pmatrix} = \begin{pmatrix} \textit{Dieudonné modules} \\ \textit{over } \tilde{R} \end{pmatrix}$$

is an equivalence of categories.

Proof. This follows from [dJ95, Main Theorem 1]. □

1.3.2

Let $R, \tilde{R}, I, D_R$ be as above. There are a few p-divisible groups $\mathscr{G}$ over S, for which we can easily give an explicit description of $\mathbb{D}(\mathscr{G})$ as a Dieudonné module over D_R. We have (cf. [dJ95, 4.3.1][2])

$$\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p) = (D_R, \mathbf{1}_{D_R}, \nabla)$$
$$\mathbb{D}(\mu_{p^\infty}) = (D_R, p\mathbf{1}_{D_R}, \nabla).$$

We will denote these Dieudonné modules by $\underline{D_R}$, the **trivial Dieudonné module** and $\underline{D_R}(1)$, the **Tate twist**, respectively.

Suppose now that $L = [Y \xrightarrow{u} T]$ is a 1-motif over S, where T is a split torus with character group X; let $\mathscr{G} = L[p^\infty]$. In this case as well we can quite explicitly describe $\mathbb{D}(L) = \mathbb{D}(\mathscr{G})$ as a Dieudonné module over D_R. We will go about it in a slightly roundabout way so as to motivate an analogous definition for log 1-motives that we will make soon. We begin with the short exact sequence:

$$1 \to \mathrm{Hom}(X, 1+I) \to T(\tilde{R}) \xrightarrow{\pi} T(R) \to 1.$$

Pulling this short exact sequence back along the map $u : Y \to T(R)$ gives us an extension

$$1 \to \mathrm{Hom}(X, 1+I) \to E_u \to Y \to 0.$$

Explicitly,

$$E_u = \{(f, y) \in T(\tilde{R}) \oplus Y : \pi(f) = u(y)\}.$$

We have a map

$$\varphi_{E_u} : E_u \to E_u$$
$$(f, y) \mapsto (\varphi(f), py).$$

Choose a lift $\tilde{u} : Y \to T(\tilde{R})$ of u; this gives us an isomorphism

$$\alpha_{\tilde{u}} : \mathrm{Hom}(X, 1+I) \oplus Y \xrightarrow{\simeq} E_u$$
$$(f, y) \mapsto (f\tilde{u}(y), y).$$

It is easy to see that, under this isomorphism, the map φ_{E_u} pulls back to

$$\varphi_{\tilde{u}} : \mathrm{Hom}(X, 1+I) \oplus Y \to \mathrm{Hom}(X, 1+I) \oplus Y$$
$$(f, y) \mapsto (\varphi(\tilde{u}(y))\tilde{u}(y)^{-p}\varphi(f), py).$$

---

2. note however that de Jong is using the covariant Dieudonné functor

Consider the map of groups:

$$\log : 1 + I \to D_R$$
$$a \mapsto \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!a^{[n]}.$$

Pushing E_u forward along $\log : \mathrm{Hom}(X, 1+I) \to \mathrm{Hom}(X, D_R)$, we obtain an extension

$$0 \to \mathrm{Hom}(X, D_R) \to F_u \to Y \to 0.$$

Finally, let M_u be the push-forward of $F_u \otimes_{\mathbb{Z}} D_R$ under the multiplication map

$$m : \mathrm{Hom}(X, D_R) \otimes_{\mathbb{Z}} D_R \to \mathrm{Hom}(X, D_R).$$

It is an extension

$$0 \to \mathrm{Hom}(X, D_R) \to M_u \to Y \otimes D_R \to 0.$$

The map φ_{E_u} gives rise to

$$\varphi_{M_u} : \varphi_{D_R}^* M_u \to M_u,$$

and, under the splitting

$$\alpha_{\tilde{u}} : \mathrm{Hom}(X, D_R) \oplus (Y \otimes D_R) \xrightarrow{\simeq} M_u,$$

φ_{M_u} pulls back to the map

$$\varphi_{\tilde{u}} : (h, y \otimes 1) \mapsto (\varphi(h) + \Phi_{\tilde{u}}(y), py \otimes 1).$$

Here $\Phi_{\tilde{u}} : Y \to \mathrm{Hom}(X, \tilde{R})$ is given by $y \mapsto \log(\varphi(\tilde{u}(y))\tilde{u}(y)^{-p})$; this logarithm lies in $\mathrm{Hom}(X, \tilde{R})$, since $\varphi(\tilde{u}(y))\tilde{u}(y)^{-p}$ lies in $\mathrm{Hom}(X, 1 + p\tilde{R})$.

From now on, identify (M_u, φ_{M_u}) with $(\mathrm{Hom}(X, D_R) \oplus (Y \otimes D_R), \varphi_{\tilde{u}})$. For each $y \in Y$, set $\omega_{\tilde{u}}(y) = \mathrm{dlog}(\tilde{u}(y)) \in \mathrm{Hom}(X, \hat{\Omega}^1_{\tilde{R}/\mathbb{Z}_p})$, and let $\nabla_{M_u} : M_u \to M_u \hat{\otimes} \hat{\Omega}^1_{\tilde{R}/\mathbb{Z}_p}$ be the connection that restricts to the trivial connection on $\mathrm{Hom}(X, D_R)$ and takes $y \otimes 1$ to $\omega_{\tilde{u}}(y)$. We see immediately that $(M_u, \varphi_{M_u}, \nabla_{M_u})$ is a Dieudonné crystal over D_R.

Definition 1.3.2.1. For any Dieudonné module (M, φ_M, ∇_M) over D_R, its **Cartier dual**, denoted $(M^\vee, \varphi_{M^\vee}, \nabla_{M^\vee})$ will again be a Dieudonné module over D_R. We have $M^\vee = \mathrm{Hom}_{D_R}(M, D_R)$; $\varphi_{M^\vee}$ will be the dual of the unique map $V_M : M \to \varphi_{D_R}^* M$ satisfying $\varphi_M V_M = p$; and $\nabla_{M^\vee}$ will just be the dual log connection:

$$\nabla_{M^\vee}(m^\vee)(m) = -m^\vee(\nabla_M(m)).$$

There is a natural identification of (M, φ_M, ∇_M) with its double Cartier dual.

Lemma 1.3.2.2. *As a Dieudonné module over D_R, $\mathbb{D}(L)$ is isomorphic to the Cartier dual $(M_u^\vee, \varphi_{M_u^\vee}, \nabla_{M_u^\vee})$ of $(M_u, \varphi_{M_u}, \nabla_{M_u})$.*

Proof. This follows from [dJ95, §4.3]. As pointed out before, de Jong is using the co-variant Dieudonné functor, while we are using the contra-variant version. The difference between the two functors is Cartier duality. This explains why we need the Cartier dual in our statement. □

1.3.3

Suppose now that R has an fs log structure $\mathrm{M}_R \to R$ such that $\mathrm{M}_R/R^\times = P$, so that P is a constant, sharp, fs monoid. Following [KT03, §4.7], we will construct a functor

$$\mathbb{D} : \begin{pmatrix}\text{log 1-motifs}\\ \text{over } R\end{pmatrix} \longrightarrow \begin{pmatrix}\text{log Dieudonné crystals}\\ \text{over } R\end{pmatrix}$$

extending the composition

$$\begin{pmatrix}\text{1-motifs}\\ \text{over } R\end{pmatrix} \xrightarrow{L\mapsto L[p^\infty]} \begin{pmatrix}p\text{-divisible groups}\\ \text{over } R\end{pmatrix} \xrightarrow{\mathbb{D}} \begin{pmatrix}\text{log Dieudonné crystals}\\ \text{over } R\end{pmatrix}.$$

First, suppose that we have a 1-motif $L = [Y \xrightarrow{u} T^{\log}]$, where T is a split torus over R with character group X. By construction, $\mathbb{D}(L)$ will be an extension

$$0 \to \mathrm{Hom}(Y, \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)) \to \mathbb{D}(L) \to \mathbb{D}(T) \to 0.$$

For any fs log algebra (B_0, M_{B_0}) over (R, M_R), and any log PD-thickening (B, M_B) of (B_0, M_{B_0}) defined by a nilpotent PD-ideal $J \subset B$, we will define $\mathbb{D}(L)(B)$ in the following way: Start with the short exact sequence

$$0 \to \mathrm{Hom}(X, 1+J) \to T^{\log}(B) \to T^{\log}(B_0) \to 0,$$

where T over B is again the split torus with character group X, and so $T^{\log}(B) = \mathrm{Hom}(X, \mathbb{G}^{\log}_{m,B})$. We can pull it back along the map $u : Y \to T^{\log}(B_0)$ to get an extension

$$0 \to \mathrm{Hom}(X, 1+J) \to E_u(B) \to Y \to 0.$$

Since J has divided powers, we can define

$$\begin{aligned}\log : 1+J &\to B\\ a &\mapsto \sum_{n=1}^{\infty} (-1)^n (n-1)! a^{[n]}.\end{aligned}$$

Pushing $E_u(B)$ forward along $\log : \mathrm{Hom}(X, 1+J) \to \mathrm{Hom}(X, B)$ we obtain an extension

$$0 \to \mathrm{Hom}(X, B) \to F_u(B) \to Y \to 0.$$

If we now push forward the tensor product $F_u(B) \otimes_{\mathbb{Z}} B$ along the multiplication map $\mathrm{Hom}(X, B) \otimes_{\mathbb{Z}} B \to \mathrm{Hom}(X, B)$, we get an extension of finite free B-modules

$$0 \to \mathrm{Hom}(X, B) \to M_u(B) \to Y \otimes B \to 0.$$

Define $\mathbb{D}(L)(B) = \mathrm{Hom}_B(M_u, B)$.

Our construction so far gives us a log crystal $\mathbb{D}(L)$ over (R, M_R). It can be naturally endowed with the structure of a log Dieudonné crystal over (R, M_R). In the interest of expediency, we exhibit this as follows: Suppose that $\tilde{R}$ has an fs log structure $\mathrm{M}_{\tilde{R}}$ such that

- $(\tilde{R}, \mathrm{M}_{\tilde{R}})$ is isomorphic to $W[|t_1, \ldots, t_n|]$ with the log structure determined by the divisor cut out by $t_1 t_2 \cdots t_r$, with $0 \leq r \leq n$. In particular, it is smooth and log smooth over $\mathbb{Z}_p$.
- The Frobenius lift $\varphi_{\tilde{R}}$ can be extended compatibly to $\mathrm{M}_{\tilde{R}}$ so that it induces the p-power map on $\mathrm{M}_{\tilde{R}/p\tilde{R}}$.
- The map $(\tilde{R}, \mathrm{M}_{\tilde{R}}) \to (R, \mathrm{M}_R)$ is strict, so that the map $\mathrm{M}_{\tilde{R}} / \tilde{R}^{\times} \to \mathrm{M}_R / R^{\times}$ is an isomorphism.

Just as in the case of Dieudonné crystals above, it follows from the theory of [Kat89, §6] that giving a log crystal over (R, M_R) is equivalent to giving a pair (M, ∇_M) where

1. M is a finite free D_R-module.
2. ∇_M is an integrable topologically quasi-nilpotent connection

$$\nabla_M : M \to M \hat{\otimes} \hat{\Omega}^{1,\log}_{\tilde{R}/\mathbb{Z}_p},$$

compatible with the natural connection on D_R.

Moreover, giving a log Dieudonné crystal over (R, M_R) is equivalent to giving a tuple (M, φ_M, ∇_M), where (M, ∇_M) are as above, and φ_M is a D_R-linear, ∇_M-parallel map

$$\varphi_M : \varphi^*_{D_R} M = M \otimes_{D_R, \varphi} D_R \to M$$

whose image contains pM.

Let us briefly explain some of the notation. Here $\hat{\Omega}^{1,\log}_{\tilde{R}/\mathbb{Z}_p}$ is the module of **continuous logarithmic differentials** on $(\tilde{R}, \mathrm{M}_{\tilde{R}})$ (cf. [Kat89, §1.7]). Let $q_1, \ldots, q_n \in \mathrm{M}_{\tilde{R}}$ be elements such that the elements $\mathrm{dlog}(q_1), \ldots, \mathrm{dlog}(q_n)$ form a basis for $\hat{\Omega}^{1,\log}_{\tilde{R}/\mathbb{Z}_p}$, and let $\partial_1^{\log}, \ldots, \partial_n^{\log}$ in $\mathrm{Hom}_{\tilde{R}}(\hat{\Omega}^{1,\log}_{\tilde{R}/\mathbb{Z}_p}, R)$ form the dual basis. Then topological quasi-nilpotence of ∇_M is

equivalent to requiring that, for any any $m \in M$, we can find $r_1, \ldots, r_k, s_1, \ldots, s_k \in \mathbb{N}$ such that (cf. [Kat89, 6.2(iii)])

$$\prod_{1 \leq i \leq n, 1 \leq j \leq k} \left(\nabla(\partial_i^{\log}) - r_j \right)^{s_j} (m) \in pM.$$

Since we already have the structure of a log crystal on $\mathbb{D}(L)$, all that remains to do is to endow $M = \lim_n \mathbb{D}(L)(D_R/p^n D_R)$ with a parallel map φ_M as above. For this, we observe that M is obtained in a very simple way. As always, we start with the short exact sequence

$$0 \to \mathrm{Hom}(X, 1+I) \to T^{\log}(\tilde{R}) \to T^{\log}(R) \to 0.$$

We pull it back along the map $u : Y \to T^{\log}(R)$, push the result forward along $\log : \mathrm{Hom}(X, 1+I) \to \mathrm{Hom}(X, D_R)$, tensor the result with D_R, and push forward what we get along the multiplication $\mathrm{Hom}(X, D_R) \otimes D_R \to D_R$. Finally, we take D_R-linear duals, and this gives us M. The map φ_M is simply the one induced from the endomorphism

$$\varphi : T^{\log}(\tilde{R}) = \mathrm{Hom}(X, \mathrm{M}_{\tilde{R}}^{\mathrm{gp}}) \to \mathrm{Hom}(X, \mathrm{M}_{\tilde{R}}^{\mathrm{gp}}) = T^{\log}(\tilde{R})$$

arising from the Frobenius lift $\varphi_{\tilde{R}} : \mathrm{M}_{\tilde{R}}^{\mathrm{gp}} \to \mathrm{M}_{\tilde{R}}^{\mathrm{gp}}$.

Lemma 1.3.3.1. *1. The assignment $L \mapsto \mathbb{D}(L)$ defines a functor from the category of log 1-motives over (R, M_R) of the form $[Y \xrightarrow{u} T^{\log}]$ with T a split torus over R to the category of log Dieudonné crystals over (R, M_R).*

2. If we consider a 1-motif $L = [Y \xrightarrow{u} T]$ over R as a log 1-motif $L^{log} = [Y \xrightarrow{u} T^{\log}]$, then $\mathbb{D}(L^{\log})$ is naturally isomorphic to the Dieudonné crystal $\mathbb{D}(L[p^\infty])$ over R.

3. Suppose we have two log 1-motifs $L_1 = [Y \xrightarrow{u_1} T^{\log}]$ and $L_2 = [Y \xrightarrow{u_2} T^{\log}]$, and set $L = [Y \xrightarrow{u_1+u_2} T^{\log}]$; then the log Dieudonné crystal $\mathbb{D}(L)$ is the Baer sum of $\mathbb{D}(L_1)$ and $\mathbb{D}(L_2)$ in the category of extensions of $\mathbb{D}(T)$ by $\mathrm{Hom}(Y, \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p))$.

Proof. The functoriality in (1) and the assertion in (3) both follow directly from the construction. The second assertion follows from (1.3.2.2). □

Now we consider a general log 1-motif $L = [Y \xrightarrow{u} G^{\log}]$ over (R, M_R), where G is an extension

$$0 \to T \to G \to A \to 0,$$

where A is an abelian scheme over R and T is a split torus over R with character group X. Associated to this we have the map

$$N_L : Y \to G^{\log}(R)/G(R) = T^{\log}(R)/T(R) \xrightarrow{\simeq} \mathrm{Hom}(X, \mathrm{M}_R^{\mathrm{gp}}/R^\times).$$

Choose some lift $u' : Y \to T^{\log}(R)$ of N_L: this gives us a log 1-motif $L' = [Y \xrightarrow{u'} T^{\log}]$, which we will conflate with the log 1-motif $[Y \xrightarrow{u'} G^{\log}]$ obtained from the inclusion $T^{\log} \hookrightarrow$

$G^{\log}$. Let $u'' = u - u'$: this maps Y into $G \subset G^{\log}$ and gives us a classical 1-motif $[Y \xrightarrow{u''} G]$ over R. We have already constructed $\mathbb{D}(L')$ as an extension

$$0 \to \mathrm{Hom}(Y, \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)) \to \mathbb{D}(L') \to \mathbb{D}(T[p^\infty]) \to 0.$$

We can pull this back along the surjection $\mathbb{D}(G[p^\infty]) \to \mathbb{D}(T[p^\infty])$ and think of $\mathbb{D}(L')$ as an extension

$$0 \to \mathrm{Hom}(Y, \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)) \to \mathbb{D}(L') \to \mathbb{D}(G[p^\infty]) \to 0.$$

To L'' we can associate the Dieudonné crystal $\mathbb{D}(L'') = \mathbb{D}(L''[p^\infty])$: this is also an extension of $\mathbb{D}(G[p^\infty])$ by $\mathrm{Hom}(Y, \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p))$. We will take $\mathbb{D}(L)$ to be the Baer sum of $\mathbb{D}(L')$ and $\mathbb{D}(L'')$ in the category of such extensions.

Note that we made a choice of lift u' of N_L to make our construction. If we fix a chart $\alpha : \mathrm{M}_R/R^\times \to \mathrm{M}_R$, we can choose this lift compatibly for all log 1-motifs over (R, M_R) and so we see that $\mathbb{D}$ in fact gives us a functor. But in fact the isomorphism class of $\mathbb{D}(L)$ does not depend on the lift u'. We have:

Lemma 1.3.3.2. *The extension class of $\mathbb{D}(L)$ does not depend on the choice of lift u'.*

Proof. Suppose we have two lifts u'_1 and u'_2 giving us two decompositions

$$u = u'_1 + u''_1 = u'_2 + u''_2$$

Let L'_1, L''_1, L'_2, L''_2 be the corresponding log 1-motives; L''_1 and L''_2 are classical 1-motives, and, by (1.1.2.4), the Baer difference of $\mathbb{D}(L''_1)$ and $\mathbb{D}(L''_2)$ corresponds to the Dieudonné crystal associated to the 1-motif $[Y \xrightarrow{u''_1 - u''_2} G]$. But this last 1-motif is equal to $[Y \xrightarrow{u'_2 - u'_1} G]$, whose Dieudonné crystal is the Baer difference of $\mathbb{D}(L'_2)$ and $\mathbb{D}(L'_1)$ by (1.3.3.1). This means precisely that the Baer sum of $\mathbb{D}(L'_1)$ and $\mathbb{D}(L''_1)$ equals the Baer sum of $\mathbb{D}(L'_2)$ and $\mathbb{D}(L''_2)$, as was to be shown. □

Suppose that we have a map $f : (R, \mathrm{M}_R) \to (R', \mathrm{M}_{R'})$ of log algebras, and suppose that $(R', \mathrm{M}_{R'})$ satisfies the same hypotheses that (R, M_R) does. Then we have a diagram of functors:

$$\begin{array}{ccc} \begin{pmatrix} \text{log 1-motifs} \\ \text{over } (R, \mathrm{M}_R) \end{pmatrix} & \xrightarrow{\quad f^* \quad} & \begin{pmatrix} \text{log 1-motifs} \\ \text{over } (R', \mathrm{M}_{R'}) \end{pmatrix} \\ \mathbb{D} \big\downarrow & & \mathbb{D} \big\downarrow \\ \begin{pmatrix} \text{log Dieudonné crystals} \\ \text{over } (R, \mathrm{M}_R) \end{pmatrix} & \xrightarrow{f^*} & \begin{pmatrix} \text{log Dieudonné crystals} \\ \text{over } (R', \mathrm{M}_{R'}) \end{pmatrix} \end{array}$$

Here, we have chosen charts $\mathrm{M}_R/R^\times \to \mathrm{M}_R$ and $\mathrm{M}_{R'}/(R')^\times \to \mathrm{M}_{R'}$ in order to be able to define the vertical functors. From our construction, and the compatibility under pull-back of the classical Dieudonné functor, we have:

Lemma 1.3.3.3. *The diagram commutes.*

□

1.3.4

Suppose that we have a polarized log 1-motif (L, λ) over S, with $L = [Y \xrightarrow{u} J^{\log}]$, such that the associated sheaves of free abelian groups Y and X are both in fact constant. We then have the associated Dieudonné module $\mathbb{D}(L)(D_R)$ over D_R also equipped with a polarization ψ. Fix some chart $\alpha : \mathrm{M}_R/R^\times \to \mathrm{M}_{\tilde{R}}$, and suppose that we have a Frobenius lift $\varphi_{\tilde{R}}$ such that $\varphi_{\tilde{R}}(\alpha(m)) = \alpha(m)^p$, for all $m \in \mathrm{M}_R/R^\times$. Let T be the torus with character group X. This gives us a lift $\tilde{u}_\alpha^{\log} : Y \to T^{\log}(\tilde{R})$ of the monodromy $N_L : Y \to \mathrm{Hom}(X, \mathrm{M}_R/R^\times)$ and thus a lift $u_\alpha^{\log} : Y \to T^{\log}(R)$ as well, giving a log 1-motif L_α. Let $u' = u - u_\alpha^{\log} : Y \to J$: this gives us a 1-motif L' over R.

Given our choice of Frobenius lift $\varphi_{\tilde{R}}$ and our construction of $\mathbb{D}(L_\alpha)$ using the lift $\tilde{u}_\alpha^{\log}$ above (1.3.3.1), it is easy to check that the underlying φ-module of $\mathbb{D}(L_\alpha)(D_R)$ is isomorphic to the direct sum $\mathrm{Hom}(Y, D_R) \oplus \mathbb{D}(T)(D_R)$, and thus is the trivial extension of $\mathbb{D}(T)(D_R)$ by $\mathrm{Hom}(Y, D_R)$. By construction, the φ-module underlying $\mathbb{D}(L)(D_R)$ is the Baer sum of $\mathbb{D}(L')(D_R)$ and the extension of $\mathbb{D}(J)(D_R)$ by $\mathrm{Hom}(Y, D_R)$ induced from $\mathbb{D}(L_\alpha)(D_R)$. Since the latter extension is trivial, we find that $\mathbb{D}(L)(D_R)$ and $\mathbb{D}(L')(D_R)$ are isomorphic as φ-modules over D_R. This identification gives rise to an identification $\mathbb{D}(L)(R) = \mathbb{D}(L')(R)$, and the Hodge filtration on $\mathbb{D}(L')(R)$ (cf. [Kis10, §1.4]) gives rise to a direct summand $\mathrm{Fil}^1 \mathbb{D}(L)(R) \subset \mathbb{D}(L)(R)$. Let $I \subset D_R$ also denote the PD-ideal that is the kernel of the map $D_R \to R$, and for each $n \in \mathbb{Z}_{>0}$, let $I^{[n]}$ be the n^{th}-divided power of I. Let

$$\hat{D}_R = \varprojlim_n D_R/I^{[n]}$$

be the PD-completion of D_R.

Lemma 1.3.4.1. *Lifting (L, λ) to a polarized log 1-motif over $\hat{D}_R$ is equivalent to lifting the Hodge filtration $\mathrm{Fil}^1 \mathbb{D}(L)(R)$ to a direct summand $\mathrm{Fil}^1 \mathbb{D}(L)(\hat{D}_R)$ of $\mathbb{D}(L)(\hat{D}_R) = \mathbb{D}(L)(D_R) \otimes_{D_R} \hat{D}_R$ that is isotropic with respect to ψ.*

Proof. We will only prove one direction of the equivalence, since that is what we will need in the future. The other implication, in any case, is easier. Suppose that we have a lift of the filtration; since we already have the lift $\tilde{u}_\alpha^{\log}$ of $u_\alpha^{\log}$, it suffices to find a lift of the 1-motif L' to $\hat{D}_R$ corresponding the lift of the filtration. By classical Grothendieck-Messing theory (cf. [Mes72]), lifting the filtration on $\mathbb{D}(L')(R)$ to one on $\mathbb{D}(L')(\hat{D}_R)$ corresponds to a deformation $\mathscr{G}$ of the p-divisible group $L'[p^\infty]$ over $\hat{D}_R$. Saying that the lift of the filtration is isotropic is equivalent to saying that the polarization $\lambda'[p^\infty]$ on $L'[p^\infty]$ also lifts to a polarization of $\mathscr{G}$. By (1.1.3.2), this gives us a deformation of (L', λ') over $\hat{D}_R$, and thus a deformation of (L, λ) over $\hat{D}_R$. □

1.4 Dieudonné theory over formally smooth rings

1.4.1

Let W be the ring of Witt vectors over some perfect extension $k/\mathbb{F}_p$.

Definition 1.4.1.1. An **augmented W-algebra** is a pair (R, J_R), where

- R is isomorphic to $W[|t_1, \ldots, t_n|]$.
- J_R is the kernel of an augmentation map $\iota_R : R \to W$.

Maps between augmented W-algebras are defined in the obvious fashion.

Definition 1.4.1.2. A **contracting Frobenius lift** on an augmented W-algebra (R, J_R) is a *faithfully flat* lift φ_R of the p-power Frobenius on R/pR such that $\varphi_R(J_R) \subset J_R^2$

In this situation, we will sometimes say that φ_R is J_R-contracting.

Remark 1.4.1.3. Note that $(W, (0))$ is an augmented W-algebra admitting a unique contracting Frobenius lift. It is, by definition, the final object in the appropriate category, and we will refer to it simply as W.

Lemma 1.4.1.4. *Let (R, J_R) be an augmented W-algebra, equipped with a contracting Frobenius lift φ_R. Then the image of the induced endomorphism φ_R^* of $\hat{\Omega}^1_{R/W}$ lies inside $J_R\hat{\Omega}^1_{R/W}$.*

Proof. $\hat{\Omega}^1_{R/W}$ is generated by elements of the form da, for $a \in J_R$. Since $\varphi_R(a) \in J_R^2$, it follows that

$$\varphi^*(da) = d(\varphi(a)) \in J_R\hat{\Omega}^1_{R/W}.$$

The lemma is an immediate consequence. □

Definition 1.4.1.5. Fix an augmented W-algebra (R, J_R) equipped with a contracting Frobenius lift φ_R. A **filtered Dieudonné module over** (R, φ_R) is a tuple

$$\underline{M} = (M, \varphi_M, \mathrm{Fil}^1 M, \nabla_M)$$

where

- M is a finite free R-module.
- φ_M is an R-linear, ∇_M-parallel map

$$\varphi_M : \varphi_R^* M = M \otimes_{R,\varphi} R \to M$$

whose image contains pM.

- $\mathrm{Fil}^1 M \subset M$ is a direct summand satisfying

$$\varphi_M\left(\varphi_R^*(\mathrm{Fil}^1 M + pM)\right) = pM.$$

- ∇_M is an integrable topologically quasi-nilpotent connection

$$\nabla_M : M \to M\hat{\otimes}\hat{\Omega}^{1,\log}_{R/W},$$

  for which φ_M is a parallel map.

We will often say that $\underline{M}$ is a filtered Dieudonné module over R. The category of filtered Dieudonné modules over R will be denoted $\mathcal{MF}_{[0,1]}(R)$.

Remark 1.4.1.6. An object in $\mathcal{MF}_{[0,1]}(W)$ is simply a 3-tuple $(M_0, \varphi_{M_0}, \mathrm{Fil}^1 M_0)$ that satisfies conditions (1) to (3) above. For every filtered Dieudonné module $\underline{M}$ over (R, J_R, φ_R), we have the induced filtered Dieudonné module $\underline{M}_0$ over W obtained by reducing modulo J_R.

Proposition 1.4.1.7. *There is an equivalence of categories*

$$\begin{pmatrix} p\text{-divisible groups} \\ \text{over } R \end{pmatrix} \xrightarrow{\simeq} \mathcal{MF}_{[0,1]}(R).$$

Proof. This is an immediate consequence of [Fal89, Theorem 7.2]. As observed in [Moo98, §4.1], it also follows from (1.3.1.2) and Grothendieck-Messing theory [Mes72]. □

Definition 1.4.1.8. The **trivial filtered Dieudonné module** over (R, J_R, φ_R), denoted $\underline{R}$, is the tuple $(R, \mathbf{1}_R, (0), d)$. The **Tate twist**, denoted $\underline{R}(1)$, is the tuple $(R, p\mathbf{1}_R, R, d)$.

1.4.2

The functoriality of these Dieudonné modules is a little involved, since the Frobenius lift φ_R is not canonical. But things are clear if we view them as *crystals* instead. Suppose $(R', J_{R'})$ is another augmented W-algebra equipped with a contracting Frobenius lift $\varphi_{R'}$. If we have a morphism $f : (R, J_R) \to (R', J_{R'})$, then we obtain a functor $f^* : \mathcal{MF}_{[0,1]}(R) \to \mathcal{MF}_{[0,1]}(R')$ in the following fashion:

Given an object $(M, \varphi_M, \mathrm{Fil}^1 M, \nabla_M)$ in $\mathcal{MF}_{[0,1]}(R)$, $f^*(M, \mathrm{Fil}^1 M, \nabla_M)$ will be obtained by the usual base change from R to R'. Then we observe that there is a canonical isomorphism

$$\epsilon_{f,M} : \varphi_{R'}^* f^* M \xrightarrow{\sim} f^* \varphi_R^* M$$

induced by the connection ∇_M. This is essentially given by parallel transport, after one notes that $\varphi_{R'} \circ f$ and $f \circ \varphi_R$ agree modulo p. For more details, see [Moo98, 4.3.3]. Now, one can define φ_{f^*M} as the composition

$$\varphi_{R'}^* f^* M \xrightarrow{\epsilon_{f,M}} f^* \varphi_R^* M \xrightarrow{f^*\varphi_M} f^* M.$$

In particular, the filtered Dieudonné module categories associated to two different contracting Frobenius lifts φ_R and φ'_R are canonically equivalent.

Definition 1.4.2.1. For any $\underline{M}$ in $\mathcal{MF}_{[0,1]}(R)$, the associated **Kodaira-Spencer map** for M is a map

$$\mathrm{KS}_M : \mathrm{Der}_W(R) \to \mathrm{Hom}\left(\mathrm{Fil}^1 M, \frac{M}{\mathrm{Fil}^1 M}\right)$$

where $\mathrm{Der}_W(R) = \mathrm{Hom}_R(\hat{\Omega}^1_{R/W}, R)$ is the module of continuous W-derivations of R. It is obtained from the R-linear map

$$\mathrm{Fil}^1 M \to \left(M/\mathrm{Fil}^1 M\right) \otimes_R \hat{\Omega}^1_{R/W}$$

induced by the connection ∇_M, which we will, abusing notation, also call KS_M.

We say that an object $\underline{M}$ in $\mathcal{MF}_{[0,1]}(R)$ is **versal** if KS_M is a surjection.

Lemma 1.4.2.2. *Let $\underline{M}$ be a filtered Dieudonné module over (R, J_R, φ_R). Let $R_1 = R/J_R^2$, let $\underline{M}_1$ be the induced filtered Dieudonné module over $(R_1, J_R/J_R^2, \varphi_R)$, and let $\underline{M}_0$ be the filtered Dieudonné module over W induced from $\underline{M}$.*

1. *There is a canonical isomorphism of tuples*

$$(M_0 \otimes_W R_1, \varphi_{M_0} \otimes 1, 1 \otimes d) \xrightarrow[\simeq]{A_M} (M_1, \varphi_{M_1}, \nabla_{M_1}),$$

   reducing to the identity modulo J_R/J_R^2.

2. *The composition*

$$\mathrm{Fil}^1 M_1 \subset M_1 \xrightarrow{A_M^{-1}} M_0 \otimes_W R_1 \to \frac{M_0}{\mathrm{Fil}^1 M_0} \otimes_W R_1 \xrightarrow{1 \otimes d} \frac{M_0}{\mathrm{Fil}^1 M_0} \otimes_W (\hat{\Omega}^1_{R/W} \otimes_R W).$$

   is naturally identified with the reduction of KS_M modulo J_R.

3. *$\mathrm{KS}_M \otimes R/J_R$ can be identified with the negative of the map*

$$\begin{aligned}\Theta_M : T_R \coloneqq \mathrm{Hom}_W(J_R/J_R^2, W) &\to \mathrm{Hom}_W\left(\mathrm{Fil}^1 M_0, M_0/\mathrm{Fil}^1 M_0\right) \\ f &\mapsto \left(m \mapsto f\left(A_M(m) (\mathrm{mod}\, \mathrm{Fil}^1 M_1)\right)\right).\end{aligned}$$

   Here, we are using the fact that the image of $A_M(m)$ in $M_1/\mathrm{Fil}^1 M_1$ lies in

$$J_R(M_1/\mathrm{Fil}^1 M_1) = (M_0/\mathrm{Fil}^1 M_0) \otimes_W (J_R/J_R^2).$$

4. *$\underline{M}$ is versal if and only if Θ_M is surjective.*

Proof. Let $\tilde{M} = p^{-1}\operatorname{Fil}^1 M + M \subset p^{-1}M$; then φ_M induces an isomorphism

$$\varphi_M : \varphi_R^* \tilde{M} \xrightarrow{\simeq} M.$$

Also, since $\varphi_R(J_R) \subset J_R^2$, the Frobenius lift $\varphi_{R_1} : R_1 \to R_1$ factors as

$$R_1 \to W \xrightarrow{\varphi_W} W \hookrightarrow R_1,$$

and so $\varphi_{R_1}^* \tilde{M}_1 = \varphi_W^* \tilde{M}_0 \otimes_W R_1$. Define A_M so that the following diagram commutes:

$$\begin{array}{ccc} \varphi_W^* \tilde{M}_0 \otimes_W R_1 & = & \varphi_{R_1}^* \tilde{M}_1 \\ \uparrow{\scriptstyle \varphi_{M_0}^{-1} \otimes 1} & & \downarrow{\scriptstyle \varphi_{M_1}} \\ M_0 \otimes_W R_1 & \xrightarrow{A_M} & M_1 \end{array}$$

Using the fact that φ_{M_1} is parallel for ∇_{M_1}, we can easily check that A_M satisfies the conditions stated in the lemma.

Statement (2) follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} M_1 & \xrightarrow{\nabla_{M_1}} & M_1 \otimes_{R_1} \hat{\Omega}^1_{R_1/W} & & \\ \downarrow{\scriptstyle A_M^{-1}} & & \downarrow{\scriptstyle A_M^{-1} \otimes 1} & \searrow & \\ M_0 \otimes_W R_1 & \xrightarrow{1 \otimes d} & M_0 \otimes_W \hat{\Omega}^1_{R_1/W} & & \dfrac{M_1}{\operatorname{Fil}^1 M_1} \otimes_{R_1} \hat{\Omega}^1_{R_1/W} \\ \downarrow & & \downarrow & & \downarrow \\ \dfrac{M_0}{\operatorname{Fil}^1 M_0} \otimes_W R_1 & \xrightarrow{1 \otimes d} & \dfrac{M_0}{\operatorname{Fil}^1 M_0} \otimes_W \hat{\Omega}^1_{R_1/W} & \longrightarrow & \dfrac{M_0}{\operatorname{Fil}^1 M_0} \otimes_W (\hat{\Omega}^1_{R_1/W} \otimes_{R_1} W). \end{array}$$

Assertion (3) now follows immediately from the fact that the natural map

$$J_R/J_R^2 \xrightarrow{d} \hat{\Omega}^1_{R_1/W} \otimes_{R_1} W$$

is an isomorphism when R is formally smooth over W.

(4) is just an application of Nakayama's lemma. □

The next result is basically [Moo98, 4.4], which is itself an elaboration of the argument in [Fal99, Theorem 10].

Proposition 1.4.2.3. *Suppose $\underline{M}_0 = (M_0, \varphi_{M_0}, \mathrm{Fil}^1 M_0)$ is an object in $\mathcal{MF}_{[0,1]}(W)$, and suppose that there exists a formally smooth augmented W-algebra (R, J_R) with a contracting Frobenius lift φ_R, and a versal object $\underline{M}$ in $\mathcal{MF}_{[0,1]}(R)$ equipped with an identification $\iota_R^* \underline{M} = \underline{M}_0$. Suppose also that we have another augmented W-algebra $(R', J_{R'})$ equipped with a contracting Frobenius lift $\varphi_{R'}$, and a tuple $(M', \varphi_{M'}, \mathrm{Fil}^1 M')$ over R' such that:*

1. *$(M', \varphi_{M'}, \mathrm{Fil}^1 M')$ satisfies conditions (1) to (3) of (1.4.1.5) with respect to R'.*
2. *We have an isomorphism*

$$\tau_0 : (M_0, \varphi_{M_0}, \mathrm{Fil}^1 M_0) \xrightarrow{\simeq} \iota_{R'}^*(M', \varphi_{M'}, \mathrm{Fil}^1 M').$$

Then there exists a map $f : (R, J_R) \to (R', J_{R'})$ of augmented W-algebras and an isomorphism

$$\tau : f^*(M, \varphi_M, \mathrm{Fil}^1 M) \xrightarrow{\simeq} (M', \varphi_{M'}, \mathrm{Fil}^1 M')$$

lifting τ_0. In particular, there is a topologically quasi-nilpotent flat connection $\nabla_{M'}$ that completes the tuple $(M', \varphi_{M'}, \mathrm{Fil}^1 M')$ to a filtered Dieudonné module over R'.

Proof. Let $R'_n = R'/J_{R'}^{n+1}$, $J'_n = J_{R'}R'_n$, $\varphi'_n = \varphi_{R'} (\mathrm{mod}\ J_{R'}^{n+1})$, $M'_n = M' \otimes_{R'} R'_n$, and $\varphi'_n = \varphi_{M'} (\mathrm{mod}\ J_{R'}^{n+1})$. We will build f and τ by inductively constructing a coherent sequence $\{(f_n, \tau_n)\}_{n \geq 0}$, where

$$f_n : (R, J_R) \to (R'_n, J'_n),$$

and

$$\tau_n : f_n^*(M, \varphi_M, \mathrm{Fil}^1 M) \xrightarrow{\simeq} (M'_n, \varphi'_n, \mathrm{Fil}^1 M'_n).$$

We have (f_0, τ_0) given to us by hypothesis. So our problem is to construct (f_{n+1}, τ_{n+1}) once we are given (f_n, τ_n).

To do this, pick any lift $\tilde{f}_n : (R, J_R) \to (R'_{n+1}, J'_{n+1})$ of f_n: this is possible since R is formally smooth. The space of such lifts is naturally a torsor under

$$T_R \otimes_W J_{n+1}^{`n+1} \coloneqq \mathrm{Hom}_W(J_R/J_R^2, W) \otimes_W J_{n+1}^{`n+1}.$$

We can also choose a lift

$$\tilde{\tau}_n : \tilde{f}_n^*(M, \mathrm{Fil}^1 M) \xrightarrow{\simeq} (M'_{n+1}, \mathrm{Fil}^1 M'_{n+1})$$

of τ_n.

Since $\varphi'_{n+1}((J'_{n+1})^{n+1}) = 0$, there is a unique map $\sigma_n : R'_n \to R'_{n+1}$ such that φ'_{n+1} factors as

$$R'_{n+1} \to R'_n \xrightarrow{\sigma_n} R'_{n+1}.$$

The map $\varphi'_{n+1} \circ \tilde{f}_n$ factors as

$$R \xrightarrow{f_n} R'_n \xrightarrow{\sigma_n} R'_{n+1},$$

and is thus independent of the choice of lift $\tilde{f}_n$. It is therefore harmless to write $\varphi'_{n+1} \circ f_{n+1}$ for this map, and to write $\varphi'^*_{n+1} f^*_{n+1} M$ for the pull-back of M along it. Similarly, the map $\tilde{f}_n \circ \varphi_R$ is independent of the choice of lift $\tilde{f}_n$, and so we obtain meaning for the symbols $f_{n+1} \circ \varphi_R$ and $f^*_{n+1}\varphi^*_R M$. We can easily check that the parallel transport isomorphism

$$\varphi'^*_{n+1} f^*_{n+1} M \xrightarrow{\epsilon_{\tilde{f}_n, M}} f^*_{n+1}\varphi^*_R M$$

between these former ambiguities is also independent of the choice of lift; so we will call it $\epsilon_{f_{n+1},M}$. In the same vein, we have a unique isomorphism $\varphi^{`*}_{n+1}(\tau_{n+1})$ between $\varphi'^*_{n+1} f^*_{n+1} M$ and $\varphi'^*_{n+1} M'_{n+1}$ lifting $\varphi'^*_n(\tau_n)$.

The Frobenius $\varphi_{\tilde{f}_n, \tilde{\tau}_n}$ induced on M'_{n+1} via $\tilde{f}_n$ and $\tilde{\tau}_n$ fits in the following diagram:

$$\begin{array}{ccc}
\varphi'^*_{n+1} f^*_{n+1} M & \xrightarrow{\epsilon_{f_{n+1},M}} & f^*_{n+1}\varphi^*_R M \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{f}^*_n \varphi_M} \\
\varphi'^*_{n+1}(\tau_{n+1}) & & \tilde{f}^*_n M \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{\tau}_n} \\
\varphi'^*_{n+1} M'_{n+1} & \xrightarrow{\varphi_{\tilde{f}_n, \tilde{\tau}_n}} & M'_{n+1}
\end{array}$$

For any other lift $\tilde{f}'_n$ of f_n, for $\tilde{M}$ as in the proof of (1.4.2.2), we have the map

$$\tilde{f}^*_n M \xrightarrow{\tilde{f}^*_n \varphi^{-1}_M} f^*_{n+1}\varphi^*_R \tilde{M} \xrightarrow{\tilde{f}'^{`*}_n \varphi_M} \tilde{f}'^{`*}_n M,$$

which reduces to the identity modulo $J^{`n+1}_{n+1}$, and thus induces a map

$$A_{\tilde{f}_n, \tilde{f}'_n} : \mathrm{Fil}^1 M_0 \to (M_0/\mathrm{Fil}^1 M_0) \otimes_W J^{`n+1}_{n+1},$$

when restricted to $\tilde{f}^*_n \mathrm{Fil}^1 M$. The difference between $\tilde{f}_n$ and $\tilde{f}'_n$ is an element of $T_R \otimes_W J^{`n+1}_{n+1}$ and $A_{\tilde{f}_n, \tilde{f}'_n}$ is, up to sign, simply the image of this element in

$$\mathrm{Hom}_W \left(\mathrm{Fil}^1 M_0, M_0/\mathrm{Fil}^1 M_0\right) \otimes_W J^{`n+1}_{n+1}$$

under the map Θ_M of (1.4.2.2).

Let $\theta : M'_{n+1} \xrightarrow{\simeq} M'_{n+1}$ be such that $\varphi'_{n+1} = \theta \circ \varphi_{\tilde{f}_n, \tilde{\tau}_n}$. Then $\theta^{-1} \circ \tilde{\tau}_n$ also induces a

map

$$A : \mathrm{Fil}^1 M_0 \to (M_0/\mathrm{Fil}^1 M_0) \otimes_W J_{n+1}^{`n+1}.$$

Using the versality of $\underline{M}$ and (4) of (1.4.2.2), we can choose a lift f_{n+1} such that $A_{\tilde{f}_n, f_{n+1}} = A$. For this choice, the unique map τ_{n+1} for which the diagram below commutes respects filtrations.

$$\begin{array}{ccccc} f_{n+1}^*\varphi_R^* M & \xrightarrow{f_{n+1}^*\varphi_M} & f_{n+1}^* M & \xrightarrow{\tau_{n+1}} & M'_{n+1} \\ & \searrow^{\tilde{f}_n^*\varphi_M} & & & \downarrow \theta \\ & & \tilde{f}_n^* M & \xrightarrow[\tilde{\tau}_n]{} & M'_{n+1}. \end{array}$$

The pair (f_{n+1}, τ_{n+1}) now does the job for us. □

Corollary 1.4.2.4. *The category of tuples $(M, \varphi_M, \mathrm{Fil}^1 M)$ over (R, J_R) satisfying conditions (1) to (3) of (1.4.1.5) is equivalent to $\mathcal{MF}_{[0,1]}(R)$.*

Proof. There is a natural forgetful functor from $\mathcal{MF}_{[0,1]}(R)$ to the category of such tuples, and the proposition above tells us that it is essentially surjective provided we allow the following

Assumption 1.4.2.5. For every $\underline{M}_0$ in $\mathcal{MF}_{[0,1]}(W)$, there is an augmented W-algebra $(R', J_{R'})$ and a versal object $\underline{M}'$ in $\mathcal{MF}_{[0,1]}(R')$ such that $\iota_{R'}^* \underline{M}' = \underline{M}_0$.

Admitting this for the moment, it only remains to show that the forgetful functor is fully faithful. So suppose that we have two objects $\underline{M}$ and $\underline{M}'$ in $\mathcal{MF}_{[0,1]}(R)$, and suppose that we have a map $f : M \to M'$ of R-modules such that $f(\mathrm{Fil}^1 M) \subset \mathrm{Fil}^1 M'$ and $\varphi_{M'} \varphi_R^*(f) = f F_M$. We would like to show that f also respects the connections on both sides. Consider

$$\delta = (f \otimes 1) \circ \nabla_M - \nabla_{M'} \circ f \in \mathrm{Hom}_R(M, M') \otimes_R \hat{\Omega}^1_{R/W} :$$

Let $\mathrm{Ad}(\varphi)(\delta)$ be the image of δ under the composition

$$\begin{aligned} \mathrm{Hom}(M, M') \otimes \hat{\Omega}^1_{R/W} &\xrightarrow{\varphi_R^* \otimes \varphi_R^*} \mathrm{Hom}(\varphi_R^* M, \varphi_R^* M') \otimes \hat{\Omega}^1_{R/W} \\ &\xrightarrow{\varphi_{M'}(\cdot)\varphi_M^{-1} \otimes 1} \mathrm{Hom}(M, M') \left[\frac{1}{p}\right] \otimes \hat{\Omega}^1_{R/W}. \end{aligned}$$

Then, by our hypotheses on f,

$$\delta = \mathrm{Ad}(\varphi)(\delta) \in \mathrm{Hom}(M, M') \left[\frac{1}{p}\right] \otimes \hat{\Omega}^1_{R/W},$$

Since $\hat{\Omega}^1_{R/W}$ is without p-torsion, we can use (1.4.1.4) to conclude that we have

$$\delta = \mathrm{Ad}(\varphi)(\delta) \in \mathrm{Hom}(M, M') \otimes_R J_R \hat{\Omega}^1_{R/W}.$$

Repeating this process, we find

$$\delta \in \bigcap_{n \geq 1} \mathrm{Hom}(M, M') \otimes_R J_R^n \hat{\Omega}^1_{R/W} = 0.$$

As for the still unproven assumption (1.4.2.5) above, it is best viewed as a result in the deformation theory of p-divisible groups: the deformation functor for any p-divisible group over a perfect field is representable and formally smooth. We refer to [Fal99] or [dJ95] for further details. □

1.4.3

Let P be a sharp, fs monoid, and consider the log algebra W_P (cf. 1.2.1.8). If $P \neq 1$, we will have many choices for a log Frobenius lift on W_P, but the set of such choices is naturally a torsor under the group $\mathrm{Hom}(P^{\mathrm{gp}}, 1 + pW)$. Indeed, suppose that we have two Frobenius lifts φ and φ' over W_P, and consider their difference $f : m \mapsto \varphi(m)\varphi'(m)^{-1} \in \mathrm{M}^{\mathrm{gp}}_{W_P}$. We have a short exact sequence:

$$0 \to W^\times \to \mathrm{M}^{\mathrm{gp}}_{W_P} \to P^{\mathrm{gp}} \to 0.$$

f restricts to the identity on $W^\times$, induces the identity on P^{gp} and gives rise to an element of $\mathrm{Hom}(P^{\mathrm{gp}}, 1 + pW)$.

Notice that we also have the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & 1 + pW & \longrightarrow & \mathrm{M}^{\mathrm{gp}}_{W_P} & \longrightarrow & \mathrm{M}^{\mathrm{gp}}_{k_P} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & 1 + pW & \longrightarrow & W^\times & \longrightarrow & k^\times & \longrightarrow & 1
\end{array}$$

The short exact sequence at the bottom of the diagram is canonically split by the Teichmuüller lift. Sections of the short exact sequence on top inducing the Teichmüller splitting on the sequence at the bottom also form a torsor under $\mathrm{Hom}(P^{\mathrm{gp}}, 1 + pW)$. We now have:

Lemma 1.4.3.1. *There is a bijection of* $\mathrm{Hom}(P^{\mathrm{gp}}, 1+pW)$*-torsors:*

$$\Big(\text{Frobenius lifts on } W_P\Big) \leftrightarrow \begin{pmatrix} \text{Splittings of the short exact sequence} \\ 1 \to 1+pW \to \mathrm{M}^{\mathrm{gp}}_{W_P} \to \mathrm{M}^{\mathrm{gp}}_{k_P} \to 1 \\ \text{lifting the Teichmüller splitting of the sequence} \\ 1 \to 1+pW \to W^\times \to k^\times \to 1 \end{pmatrix}$$

Proof. Fix a Frobenius lift φ on W_P, and consider the map

$$\begin{aligned} \Phi : \mathrm{M}^{\mathrm{gp}}_{W_P} &\to 1+pW \\ m &\mapsto \varphi(m)m^{-p}. \end{aligned}$$

Its restriction to $W^\times$ has kernel $k^\times$ and it is simple to check that its restriction to $1+pW$ is bijective. The kernel of Φ will be the section of (1.4.3) corresponding to the Frobenius lift φ, and we have a splitting

$$\mathrm{M}^{\mathrm{gp}}_{W_P} \to \mathrm{M}^{\mathrm{gp}}_{k_P} \oplus (1+pW), \tag{1.4.3.1.1}$$

where the projection onto the second summand is $\Big(\Phi|_{1+pW}\Big)^{-1} \circ \Phi$.

Conversely, suppose that we have a splitting as above compatible with the Teichmüller splitting on $W^\times$. The corresponding Frobenius lift is now the direct sum of the natural Frobenius maps on each of the summands. □

Even though W_P is not log smooth over $\mathbb{Z}_p$, we can make sense of a log Dieudonné module over it.

Definition 1.4.3.2. Fix a Frobenius lift φ_W on W_P. A **log Dieudonné module** over W_P is a tuple (M, φ_M, N_M), where

- M is a finite free W-module.
- $\varphi_M : \varphi_W^* M \to M$ is an injective map whose image contains pM.
- $N_M : P^{\mathrm{gp},\vee} \to \mathrm{End}(M)$ is a map satisfying

  $$N_M(f)\varphi_M = p\varphi_M \varphi_W^*\Big(N_M(f)\Big),$$

  for all $f \in P^{\mathrm{gp},\vee}$, the dual group for P^{gp}.

Remark 1.4.3.3. If we were less pedantic, and identified φ_{M_0} with the φ_W-semi-linear map induced by it, and if we set $N = N_{\underline{M}}(f)$ and $\varphi = \varphi_M$, then this condition would be the more legible and familiar

$$N\varphi = p\varphi N.$$

Remark 1.4.3.4. Why do we keep track of the Frobenius lift in this definition, even though the required properties of the tuple (M, φ_M, N_M) appear to have nothing to do with it? To answer this, we need to observe that there is one (perhaps the only) important way in which we obtain log Dieudonné modules over W_P: We begin with an augmented W-algebra (R, J_R) equipped with an fs log structure $\mathrm{M}_R \to R$, and a log Frobenius lift φ_R such that $\varphi_R(J_R) \subset J_R$. Suppose that $\mathrm{M}_R/R^\times = P$ and that the induced log structure on $W = R/J_R$ makes it a log W-algebra isomorphic to W_P. Then we have an induced Frobenius lift φ_W on W_P. Now, any log Dieudonné module (M, φ_M, ∇_M) over R will give rise under reduction modulo J_R to a log Dieudonné module over W_P and with respect to this particular Frobenius lift.

Remark 1.4.3.5. Functoriality between the categories of log Dieudonné modules over W_P for different Frobenius lifts is determined by the requirement that it be compatible with the functoriality for log Dieudonné modules (that is, log F-crystals) over log formally smooth W-algebras), and the reduction functor described in (1.4.3.4). To describe this, suppose that we have a Frobenius lift φ_P on W_P. Let Q be any other sharp, fs monoid. Fix any Frobenius lift φ_Q on W_Q, and let $f : W_P \to W_Q$ be any map of log W-algebras that is the identity on W. This amounts to giving a map $f^\sharp : \mathrm{M}_{W_P} \to \mathrm{M}_{W_Q}$ of monoids. Let (M, φ_M, N_M) be a log Dieudonné module over W_P. Then the pull-back $f^*(M, \varphi_M, N_M)$ over W_Q is described as follows. We have $f^*M = M$, and

$$N_{f^*M} : Q^{\mathrm{gp},\vee} \xrightarrow{f^\sharp} P^{\mathrm{gp},\vee} \to \mathrm{End}(M).$$

Describing φ_{f^*M} is only a little more involved. Consider the map

$$\begin{aligned} \Phi_f : \mathrm{M}_{W_P}^{\mathrm{gp}} &\to 1 + pW \\ m &\mapsto \varphi_Q(f^\sharp(m)) f^\sharp(\varphi_P(m))^{-1}. \end{aligned}$$

This factors through P^{gp}, and applying the p-adic logarithm gives us the map $\log(\Phi_f) : P^{\mathrm{gp}} \to W$. We can think of this as an element of $P^{\mathrm{gp},\vee} \otimes W$; evaluating N_M on this element gives us $N_M(\log \Phi_f) \in \mathrm{End}(M)$. We then have:

$$\varphi_{f^*M} = \varphi_M \circ \left(1 + N_M(\log \Phi_f)\right).$$

CHAPTER 2
p-ADIC HODGE THEORY FOR DEGENERATING ABELIAN VARIETIES

2.1 Splittings of filtrations

2.1.1

Let A be a commutative ring, and M a finite free A-module.

Definition 2.1.1.1. A **decreasing exhaustive filtration** $F^\bullet M$ on M is a collection $\{F^i M\}_{i\in\mathbb{Z}}$ of finite free A-sub-modules of M such that:

1. For all $i \in \mathbb{Z}$, $F^{i+1}M \subset F^i M$.
2. For all $i \in \mathbb{Z}$, the A-module $\mathrm{gr}_F^i M = \frac{F^i M}{F^{i+1}M}$ is again finite free.
3. There exists $k \in \mathbb{Z}$ such that $F^k M = M$.

We will usually suppress the adjectives and refer to such a gadget simply as a **filtration**.

Remark 2.1.1.2. One way to obtain filtrations on M is via a co-character $\mu : \mathbb{G}_m \to \mathrm{GL}(M)$: this defines a grading $M = \oplus_{i\in\mathbb{Z}} M^i$, where

$$M^i = \{m \in M : \mu(z)m = z^i m \text{ for all } z \in \mathbb{G}_m\}.$$

Given such a grading of M we have the associated decreasing filtration given by

$$F^i M = \oplus_{j\geq i} M^i.$$

In this situation, we will say that $F^i M$ is **split by the co-character** μ.

Let $G \subset \mathrm{GL}(M)$ a closed, connected, reductive sub-group. Suppose that M is equipped with a decreasing filtration $F^\bullet M$. Let $P_F \subset G$ be the sub-group that stabilizes this filtration, and let $U_F \subset P_F$ be the sub-group that acts trivially on $\mathrm{gr}_F^\bullet M$. Then we have the following:

Lemma 2.1.1.3. [Kis10, Lemma 1.1.1] *The following are equivalent:*

- *$F^\bullet M$ can be split by a co-character $\mu : \mathbb{G}_m \to G$.*
- *P_F is a parabolic sub-group of G with unipotent radical U_F, and the grading on $\mathrm{gr}_F^\bullet M$ is induced by a co-character $\mu : \mathbb{G}_m \to P_F/U_F$.*

Definition 2.1.1.4. When the equivalent conditions of (2.1.1.3) hold, we will say that the filtration $F^\bullet M$ is G**-split**.

More generally, for any flat, closed sub-group $H \subset G$, we will say that $F^\bullet M$ is H**-split** if it is G-split and we can choose a splitting co-character μ that factors through H.

Following [Kis10], for G not necessarily connected, we will say that $F^\bullet M$ is G-split if it is G°-split. Here G° is the connected component of G.

Let $F^\bullet M$ and $W^\bullet M$ be two G-split filtrations of M, let P_F and P_W be the corresponding parabolic sub-groups of G given to us by (2.1.1.3), and let L_F and L_W be their respective maximal reductive quotients. We have closed embeddings $L_? \hookrightarrow \mathrm{GL}(\mathrm{gr}^\bullet_? M)$ for $? = F, W$. Let $\overline{F}^\bullet$ be the filtration on $\mathrm{gr}^\bullet_W M$ induced by $F^\bullet$.

Lemma 2.1.1.5. *Let the notation be as above.*

1. *Suppose that we have a short exact sequence*

$$0 \to N' \to N \xrightarrow{\pi} N'' \to 0$$

of A-modules. Let $I \subset M$ be an A-sub-module, and suppose that we have a direct sum decomposition $N = N_1 \oplus N_2$ of N inducing direct sum decompositions

$$I = (I \cap N_1) \oplus (I \cap N_2); N' = (N' \cap N_1) \oplus (N' \cap N_2).$$

Then $N_1 \subset I$ if and only if $N' \cap N_1 \subset N' \cap I$ and $\pi(N_1) \subset \pi(I)$

2. *Suppose that we have a co-character $\mu : \mathbb{G}_m \to P_F \cap P_W$ and let $F^\bullet_\mu M$ be the associated filtration split by μ. Suppose that the filtration $\overline{F}^\bullet_\mu$ on $\mathrm{gr}^\bullet_W M$ induced by $F^\bullet_\mu M$ is equal to $\overline{F}^\bullet$. Then $F^i M = F^i_\mu M$, for all $i \in \mathbb{Z}$. In particular, μ splits $F^\bullet M$.*

Proof. Let us begin with (1): the only if part of it is immediate. So suppose $N' \cap N_1 \subset N' \cap I$ and $\pi(N') \subset \pi(I)$. Choose an element $n \in N_1$. By hypothesis, there is an element $m = m_1 + m_2 \in I$, with $m_i \in I \cap N_i$ $(i = 1, 2)$, such that $\pi(n) = \pi(m)$. To show that n lies in I, it is now enough to show that $m_1 - n \in I$. To see this, simply note that we have $n' = m - n = (m_1 - n) + m_2 \in N'$, where

$$m_1 - n \in N' \cap N_1 \subset N' \cap I.$$

We now prove (2). Let $\oplus_{i \in \mathbb{Z}} M^i$ be the grading on M induced by μ. The assumption that μ factors through $P_F \cap P_W \subset G$ implies that, for each $j \in \mathbb{Z}$, we have induced gradings:

$$W^j M = \oplus_{i \in \mathbb{Z}} W^j M \cap M^i;$$

$$F^j M = \oplus_{i \in \mathbb{Z}} F^j M \cap M^i.$$

To show that $F^jM = F^j_\mu M$, for all $j \in \mathbb{Z}$, it is enough to show that $M^i \subset F^jM$ for all $i, j \in \mathbb{Z}$ such that $i \geq j$. In fact, since $W^\bullet M$ is exhaustive, it is enough to show that, for all integers $i, j, k \in \mathbb{Z}$ such that $i \geq j$, we have

$$M^i \cap W^kM \subset F^j \cap W^kM.$$

For fixed i and j, this can be shown by descending induction on k. Suppose that the assertion is true for i, j, k as above. Then the assertion for $i, j, k-1$ will follow from applying (1), with

$$N = W^{k-1}M; N' = W^kM; N'' = \mathrm{gr}^{k-1}_W M; J = W^{k-1}M \cap F^jM;$$

$$N_1 = W^{k-1}M \cap M^i; N_2 = W^{k-1}M \cap \left(\oplus_{l \neq i} M^l\right).$$

□

Let $F^\bullet M$, $W^\bullet M$, P_F, P_W, L_W and $\overline{F}^\bullet$ be as above.

Lemma 2.1.1.6. *Suppose that $A = k$ is a perfect field, and that $\overline{F}^\bullet$ is L_W-split. Then $F^\bullet M$ is P_W-split.*

Proof. Let $\overline{P}_F \subset L_W$ be the image of $P_F \cap P_W$: this is the parabolic sub-group of L_W corresponding to the L_W-split filtration $\overline{F}^\bullet$. First, assume that k is algebraically closed. Then, by [Bor91, IV.14.13], we can find a maximal torus $T \subset P_F \cap P_W$. This T maps isomorphically to a maximal torus $\overline{T} \subset \overline{P}_F$.

Let $\overline{U}_F$ be the unipotent radical of $\overline{P}_F$. Then, the space of co-characters of L_W splitting $\overline{F}^\bullet$ is a torsor under $\overline{U}_F$. Moreover, by [Bor91, III.10.6], all maximal tori of $\overline{P}_F$ are conjugate to each other under $\overline{U}_F$. Putting these two statements together, we see that we can choose our co-character $\overline{\mu}$ splitting $\overline{F}^\bullet$ such that it factors through $\overline{T}$. We can lift this uniquely to a co-character $\mu : \mathbb{G}_m \to T \hookrightarrow P_F \cap P_W$, and it follows from (2.1.1.5)(2) that μ splits $F^\bullet M$.

Now let us consider the general case where k is any perfect field. Take the functor Q on k-algebras given by:

$$Q(R) = \left\{\text{Co-characters } \mu : \mathbb{G}_m \otimes_k R \to P_W \otimes_k R \text{ splitting } F^\bullet M \otimes_k R\right\} \quad ,$$

for any k-algebra R. In general, this functor is a pseudo-torsor under the unipotent group $U_F \cap P_W$. The proof above for k algebraically closed shows that it is in fact a $U_F \cap P_W$-torsor. Since $U_F \cap P_W$ is connected unipotent and k is perfect, any $U_F \cap P_W$-torsor over k is trivial. In particular, $Q(k)$ is non-empty, and we have our result. □

Remark 2.1.1.7. Since $U_F \cap P_W$ is a sub-group of the unipotent radical U_F of a parabolic sub-group P_F of a reductive group G, this result is valid without the assumption that k is perfect, but we will not need this more general statement.

2.1.2

Suppose that K is a field with $\operatorname{char} K = 0$, and let $\mathbf{C}$ be a neutral K-linear Tannakian category with fiber functor $\omega : \mathbf{C} \to \mathrm{Vect}_K$. Suppose that we have a 1-dimensional object $T \in \mathrm{Obj}(\mathbf{C})$; fix an isomorphism $\lambda : K \xrightarrow{\simeq} \omega(T)$. For any object L of $\mathbf{C}$, and any $k \in \mathbb{Z}_{\geq 0}$, denote by $L(k)$ the tensor product $L \otimes T^{\otimes k}$. We will also denote the K-vector space $\omega(L(k))$ by $\omega(L)(k)$. The choice of λ gives us an identification $\lambda_L^k : \omega(L) \xrightarrow{\simeq} \omega(L)(k)$, for any object L of $\mathbf{C}$.

Fix $D \in \mathrm{Obj}(\mathbf{C})$; let $\mathbf{C}_D$ be the Tannakian sub-category of $\mathbf{C}$ generated by D, and let $\mathbf{C}_{D,T}$ be the Tannakian sub-category of $\mathbf{C}$ generated by D and T. Let ω_D (resp. $\omega_{D,T}$) be the restriction of ω to $\mathbf{C}_D$ (resp. $\mathbf{C}_{D,T}$). Then $H = \mathrm{Aut}^{\otimes}(\omega_{D,T})$ is a closed sub-group of $\mathrm{GL}(V \oplus \omega(T))$, where $V = \omega(D)$. If T is isomorphic to an object in $\mathbf{C}_D$, then we can view H as a closed sub-group of $\mathrm{GL}(V)$. Let $N : D \to D(1)$ be a morphism such that the composition

$$N_\lambda : V \xrightarrow{\omega(N)} V(1) \xrightarrow{\lambda_D^{-1}} V$$

is a nilpotent endomorphism of V. Note that $N_\lambda \in \mathrm{End}(V)$ determines a map

$$\begin{aligned} f_\lambda : \mathbb{G}_a &\to \mathrm{GL}(V) \\ a &\mapsto \exp(aN_\lambda). \end{aligned}$$

The associated differential

$$\mathrm{Lie}(f_\lambda) : \mathrm{Lie}(\mathbb{G}_a) \to \mathrm{End}(V)$$

is independent of the choice of λ up to multiplication by an element of $K^\times$. So we have a well-defined Lie sub-algebra $\mathfrak{N} \subset \mathrm{End}(V)$: this is the image of $\mathrm{Lie}(f_\lambda)$, for any choice of λ.

In [Del80, 1.6.1] (cf. also [Del80, 1.6.4]), we find a construction of the unique ascending filtration $M_\bullet V$ on V such that:

- $N(M_i V) \subset M_{i-2}V(1)$;
- N^k induces an isomorphism $\mathrm{gr}_k^M M \xrightarrow{\simeq} \mathrm{gr}_{-k}^M V(k)$.

This is the **Jacobson-Morosov filtration** on V associated with the morphism N. It is clear that for any $a \in K^\times$, $M_\bullet V$ will also satisfy the properties above with respect to the morphism aN. In particular, $M_\bullet V$ only depends on the Lie sub-algebra $\mathfrak{N} \subset \mathrm{End}(M)$. We can convert it into a descending filtration $W^\bullet V$ by setting $W^i V = M_{-i} V$. In this case N^k will induce an isomorphism $\mathrm{gr}_W^{-k} V \xrightarrow{\simeq} \mathrm{gr}_W^k V(k)$.

Lemma 2.1.2.1. *Maintain the notation as above. Let $G \subset \mathrm{GL}(V)$ be a closed, reductive sub-group.*

1. *The Lie sub-algebra $\mathfrak{N} \subset \mathrm{End}(V)$ is stabilized by H. In particular, H stabilizes the filtration $W^\bullet V$.*
2. *Suppose that $\mathfrak{N} \subset \mathrm{Lie}(G)$; then the filtration $W^\bullet V$ is G-split.*

3. *Let $P_W \subset G$ be the parabolic sub-group stabilizing $W^\bullet V$. Suppose that G contains the image of H in $\mathrm{GL}(V)$, and suppose also that we have an exact tensor filtration $F^\bullet$ on ω_D (cf.* [SR72, IV.2.1.1]*); then $F^\bullet V$ is P_W-split.*

Proof. Since $H = \mathrm{Aut}^\otimes(\omega_{D,T})$ and T is an object of $\mathbf{C}_{D,T}$, for any K-algebra R and any element $h \in H(R)$, we have an associated automorphism h_T of $\omega(T) \otimes_K R$. If we denote by $h(1)$ the automorphism $h \otimes h_T$ of $V(1) \otimes R = V \otimes \omega(T) \otimes R$, then the diagram

$$\begin{array}{ccc} V \otimes_K R & \xrightarrow{\omega(N)\otimes 1} & V(1) \otimes_K R \\ \Big\downarrow h & & \Big\downarrow h(1) \\ V \otimes_K R & \xrightarrow{\omega N \otimes 1} & V(1) \otimes_K R \end{array}$$

commutes. Moreover, for any choice $\lambda : K \xrightarrow{\simeq} \omega(T)$, the automorphism $\lambda_D^{-1} h(1) \lambda_D$ of $V \otimes_K R$ is a scalar multiple of h. Since $N_\lambda = \lambda_D^{-1} \omega(N)$, we find that $h(N_\lambda \otimes 1)h^{-1}$ is a scalar multiple of $N_\lambda \otimes 1$. In other words, $\mathfrak{N}$ is stabilized by H, and we have shown the first part of assertion (1). The second part is now immediate.

Assertion (2) follows from [SR72, IV.2.5.3].

For (3), let $\overline{H} \subset \mathrm{GL}(V)$ be the image of H in $\mathrm{GL}(V)$. Then $\overline{H}$ is simply the group $\mathrm{Aut}^\otimes(\omega_D)$, and $F^\bullet V$ is $\overline{H}$-split by [SR72, Theorem IV.2.4]. The second part of (1) shows that $\overline{H} \subset P_W$, and so the filtration $F^\bullet V$ must necessarily be P_W-split. □

Note on Notation 2.1.2.2. Suppose R is a commutative ring and suppose that $\mathbf{C}$ is an R-linear tensor category that is a faithful tensor sub-category of Mod_R, the category of R-modules. Suppose in addition that $\mathbf{C}$ is closed under taking duals, symmetric and exterior powers in Mod_R. Then, for any object $D \in \mathrm{Obj}(\mathbf{C})$, we will denote by $D^\otimes$ the direct sum of the tensor, symmetric and exterior powers of D and its dual.

2.2 p-adic Hodge theory

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Let $W = W(k)$ be the ring of Witt vectors with coefficients in k equipped with its Frobenius lift φ_W, and let $K_0 = W\left[\frac{1}{p}\right] \subset K$ be the maximal absolutely unramified sub-field. A **Galois representation** will be a continuous finite dimensional $\mathbb{Q}_p$-representation of $\mathrm{Gal}(\overline{K}/K)$.

2.2.1

We refer the reader to Fontaine's article [Fon94a] for the definition of the p-adic period rings B_{cris}, B_{st} and B_{dR}. We will simply note:

- B_{dR} is a filtered K-algebra equipped with a $\mathrm{Gal}(\overline{K}/K)$-action; $B_{\mathrm{cris}} \subset B_{\mathrm{st}}$ are K_0-algebras equipped compatibly with a φ_W-semi-linear endomorphism φ and a $\mathrm{Gal}(\overline{K}/K)$-action; and B_{st} is additionally endowed with a B_{cris}-linear derivation N that is defined up to a $\mathbb{Q}$-multiple, and depends on a choice of p-adic valuation on K.
- We have $B_{\mathrm{dR}}^{\mathrm{Gal}(\overline{K}/K)} = K$, $B_{\mathrm{st}}^{\mathrm{Gal}(\overline{K}/K)} = B_{\mathrm{cris}}^{\mathrm{Gal}(\overline{K}/K)} = K_0$.
- There exists a natural Galois-equivariant embedding $B_{\mathrm{cris}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}$, inducing a filtration on $B_{\mathrm{cris}} \otimes_{K_0} K$.

Remark 2.2.1.1. While there is no canonical embedding of $B_{\mathrm{st}} \otimes_{K_0} K$ into B_{dR}, this lack can be ameliorated in the following way: Fix a p-adic valuation ν on K. Set

$$K^{\log} = \frac{K\left[l_\alpha : \alpha \in K^\times\right]}{\left(l_{\alpha\beta} - l_\alpha - l_\beta, \text{ for } \alpha, \beta \in K^\times; l_\alpha = \log(\alpha), \text{ for } \alpha \in 1 + \pi\mathscr{O}_K.\right)}.$$

This is a K-algebra, and giving a section $c : K^{\log} \to K$ corresponds precisely to giving a branch of the p-adic logarithm over K.

$K^{\log}$ can be equipped with the K-derivation N given by:

$$N : l_\alpha \to -\nu(\alpha) l_\alpha.$$

We can define a similar ring $\overline{K}^{\log}$ for $\overline{K}$, and it is easy to see that the natural map

$$\overline{K} \otimes_K K^{\log} \to \overline{K}^{\log}$$

is an isomorphism.

There is also the universal logarithm:

$$\begin{aligned} \log : \overline{K}^\times &\to \overline{K}^{\log} \\ \alpha &\mapsto l_\alpha. \end{aligned}$$

We claim that there is a canonical embedding $(B_{\mathrm{st}} \otimes_{K_0} K^{\log})^{N=0} \hookrightarrow B_{\mathrm{dR}}$, where we are taking the invariants of the diagonal operator $N \otimes 1 + 1 \otimes N$ on the right hand side (note that this operator is independent of the choice of valuation ν). To construct this embedding, in [Fon94a, 4.2.2], we simply have to replace the choice of logarithm $\log : \overline{K}^\times \to \overline{K}$ over K with the universal one into $\overline{K}^{\log} = \overline{K} \otimes_K K^{\log}$. In particular, any branch of the p-adic logarithm over K corresponds to a map $c : K^{\log} \to K$ of K-algebras and determines an embedding $\iota_c : B_{\mathrm{st}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}$.

Following the remark, we give the following very slight modification of the definition of a filtered (φ, N)-module:

Definition 2.2.1.2. A filtered (φ, N)-module D over K is a vector space D over K_0, equipped with a Frobenius semi-linear operator φ, a linear operator N and a filtration on the K-vector space $\left(D \otimes_{K_0} K^{\log}\right)^{N=0}$. This data satisfies: $N\varphi = p\varphi N$.

Remark 2.2.1.3. For every choice of logarithm $c : K^{\log} \to K$, we obtain a filtration on $D \otimes_{K_0} K$, and thus a filtered (φ, N)-module in the usual sense. The notion of weak admissibility is preserved and reflected under this operation, so we can speak of a weakly admissible (φ, N)-module in our sense as well.

Fix the derivation N on B_{st} that corresponds to the choice of valuation ν such that $\nu(\pi) = 1$, for some (hence any) uniformizer π of K. We can now define Fontaine's (covariant) functor D_{st} from Galois representations to filtered (φ, N)-modules over K (see [Fon94b]) by the formula

$$D_{\mathrm{st}}(V) = \mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K)}(\mathbb{Q}_p, B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V).$$

If $D_{\mathrm{dR}}(V) = \mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K)}(\mathbb{Q}_p, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is the corresponding filtered K-vector space, then we have a natural map

$$\left(D_{\mathrm{st}}(V) \otimes_{K_0} K^{\log}\right)^{N=0} \hookrightarrow D_{\mathrm{dR}}(V),$$

which respects filtrations on both sides.

One can also similarly define the corresponding crystalline functor D_{cris} to filtered φ-modules over K.

2.2.2

Let V be a semi-stable Galois representation with Hodge-Tate weights in $\{0, 1\}$[1]; let $D = D_{\mathrm{st}}(V)$ be the associated weakly admissible filtered (φ, N)-module. Fix a uniformizer $\pi \in K$ and fix the choice of logarithm taking π to 0. This endows $D_K = D \otimes_{K_0} K$ with a filtration $\mathrm{Fil}^\bullet D_K$, and so we can also think of D as a filtered φ-module. Some of the discussion below can also be found in [Pau04].

Let $\mathbb{Q}_p(1)$ be the 1-dimensional Galois representation corresponding to the p-adic cyclotomic character $\chi : \mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p^\times$, and set $\mathbb{Q}_p(-1) = \mathrm{Hom}(\mathbb{Q}_p(1), \mathbb{Q}_p)$. Let $K_0(1) = D_{\mathrm{cris}}(\mathbb{Q}_p(-1))$ be the associated 1-dimensional weakly admissible filtered φ-module over K. Choose a generator ϵ for $\mathbb{Q}_p(1)$: this amounts to choosing a generator compatible system of p-power roots of unity in $\overline{K}$. Associated with this is a cylotomic period $t \in B_{\mathrm{cris}}$ (cf. [Fon94a, 2.3.4]); the element $e = t \otimes \epsilon^{-1}$ is a canonical basis element for $K_0(1)$, and we have $\varphi(e) = pe$. For any weakly admissible filtered (φ, N)-module E and any integer $r \in \mathbb{Z}$, set $E(r) = E \otimes K_0(1)^{\otimes r}$, where for $r < 0$ $K_0(1)^{\otimes r}$ is defined to be $\mathrm{Hom}(K_0(1)^{\otimes -r}, K_0)$.

1. we use the convention where the Tate twist $\mathbb{Q}_p(1)$ has Hodge-Tate weight -1.

The nilpotent endomorphism $N : D \to D$ satisfies the condition $N\varphi = p\varphi N$, and the choice of basis e for $K_0(1)$ allows us to view N as a map $N : D \to D(1)$ of weakly admissible modules.

Let K_0 be the trivial filtered φ-module over K. Set:

$$V_{\mathrm{st}}(D) = \mathrm{Hom}_{\varphi,N,\mathrm{Fil}}(K_0, B_{\mathrm{st}} \otimes_{K_0} D).$$

Here the sub-script represents the structures that are supposed to be preserved. Since V is semi-stable, $V_{\mathrm{st}}(D)$ is a finite dimensional Galois representation, and we have a natural identification of Galois representations $V = V_{\mathrm{st}}(D)$. Applying the functor V_{st} to the map N, we obtain a map $V_{\mathrm{st}}(N) : V \to V_{\mathrm{st}}(D(1)) = V \otimes V_{\mathrm{st}}(K_0(1)) = V(-1)$ of Galois representations. We will denote this map again by N; this should not be a source of confusion.

We can now apply the theory of (2.1.2). First, in the notation of *loc. cit.*, we take $\mathbf{C}$ to be the base change over K_0 of the $\mathbb{Q}_p$-linear (non-neutral) Tannakian category of weakly admissible filtered (φ, N)-modules, equipped with the forgetful fiber functor ω to Vect_{K_0}. We pick D to be our object in $\mathbf{C}$, we take T to be $K_0(1)$, and we take N to be the map $N : D \to D(1)$. We then have the associated ascending Jacobson-Morosov filtration $M_\bullet D$ on D (here we are conflating D with the K_0-vector space underlying it). Up to shift, it agrees with the three-step filtration

$$0 = W_{-1}D \subset W_0 D \subset W_1 D \subset W_2 D = D,$$

where $W_0 D = \mathrm{im}\, N$ and $W_1 D = \ker N$.

Next, we can take $\mathbf{C}$ to be the $\mathbb{Q}_p$-linear Tannakian category of continuous $\mathbb{Q}_p$-representations of $\mathrm{Gal}(\overline{K}/K)$ and ω to be the forgetful functor to $\mathrm{Vect}_{\mathbb{Q}_p}$. We will take D to be V, T to be the inverse Tate twist $\mathbb{Q}_p(-1)$, and N to be the map $N : V \to V(-1)$. We have the associated Jacobson-Morosov filtration $M_\bullet V$. We will use the shifted three-step filtration $W_\bullet V$ satisfying $W_i V = M_{i-1} V$, so that we again have a three-step filtration

$$0 = W_{-1}V \subset W_0 V \subset W_1 V \subset W_2 V = V.$$

Again, $W_1 V = \ker N$ and $W_0 V(-1) = \mathrm{im}\, N$.

Lemma 2.2.2.1. *Let the notation be as above.*

1. *The filtration $W_\bullet D$ of D is a filtration by weakly admissible filtered (φ, N)-sub-modules.*

2. *The filtration $W_\bullet V$ is a filtration by $\mathrm{Gal}(\overline{K}/K)$-sub-representations. The filtration $D_{\mathrm{st}}(W_\bullet(V))$ on $D = D_{\mathrm{st}}(V)$ is identified with $W_\bullet D$.*

3. *$W_1 V$ is crystalline, as are all the associated graded terms $\mathrm{gr}_i^W V$. Moreover, $W_0 V$ is potentially unramified, and $\mathrm{gr}_2^W V$ is potentially a Tate twist by $\mathbb{Q}_p(-1)$ of an unramified representation.*

4. *Suppose that we have a closed, reductive sub-group $G_{K_0} \subset \mathrm{GL}(D)$ that is the pointwise stabilizer of a collection of φ-invariant tensors $\{s_\alpha\} \subset \mathrm{Fil}^0(D^\otimes)^{N=0}$. Then the filtration $W_\bullet D$ is G_{K_0}-split.*

5. *With the hypotheses as in (1), let $P_W \subset G_{K_0}$ be the parabolic sub-group stabilizing $W_\bullet D$; then the two-step filtration $\mathrm{Fil}^\bullet D_K$ is $P_W \otimes_{K_0} K$-split.*

6. *Suppose that we have a closed, reductive sub-group $G_{\mathbb{Q}_p} \subset \mathrm{GL}(V)$ that is the pointwise stabilizer of a collection of $\mathrm{Gal}(\overline{K}/K)$-invariant tensors $\{s_\alpha\} \subset V^\otimes$. Then the filtration $W_\bullet V$ is $G_{\mathbb{Q}_p}$-split.*

Proof. Assertions (1) and (2) are seen from the explicit descriptions of $W_\bullet D$ and $W_\bullet V$ given above.

The statements in (3) about $W_i V$ and $\mathrm{gr}_i^W V$ are immediately translated into the following on the weakly admissible module side: N is trivial on $W_1 D$, and on all the associated graded terms $\mathrm{gr}_i^W D$. Moreover, $\varphi|_{W_0 D}$ is an isomorphism. Both these assertions are easily checked.

As for (4), the hypotheses on G_{K_0} ensure that $\mathrm{Aut}^\otimes(\omega_D) \subset G_{K_0}$. Since $N(s_\alpha) = 0$, we also see that N lies in $\mathrm{Lie}(G_{K_0})$. The conclusion follows from (2.1.2.1)(2).

(5) is more or less immediate from (2.1.2.1)(3): we only have to note that, to apply it directly, we would need to take our Tannakian category $\mathbf{C}$ to be the base change over K of the category of weakly admissible (φ, N)-modules, D to be the corresponding object of $\mathbf{C}$, now viewed as a K-linear category, ω to be the forgetful fiber functor to Vect_K, and $F^\bullet$ to be the exact tensor filtration on ω_D induced by $\mathrm{Fil}^\bullet D_K$.

(6) follows from (2.1.2.1)(2) for reasons analogous to those found in the proof of (4). □

2.2.3

Maintain the notation as above. Set

$$V_{\mathrm{cris}}(D) = \mathrm{Hom}_{\varphi,\mathrm{Fil}}(K_0, B_{\mathrm{cris}} \otimes_{K_0} D).$$

For a general weakly admissible filtered (φ, N)-module D, $V_{\mathrm{cris}}(D)$ need not even be finite dimensional[2], but in our special situation, it is finite dimensional with dimension equal to $\dim D$. In fact, we can say more: Fix a compatible system $(\pi_n)_{n\geq 0} = (\sqrt[p^{n+1}]{\pi})_{n\geq 0}$ of p-power roots of π in $\overline{K}$. Set $K_\infty = \bigcup_{n\geq 1} K(\pi_n) \subset \overline{K}$, and let $\mathrm{Gal}(\overline{K}/K_\infty)$ be the absolute Galois group of K_∞. Let $l_\pi \in B_{\mathrm{st}}$ be the element $\log \frac{[\pi]}{\pi}$ considered in [Bre02, §3.5]: it arises from the choice of a coherent system of p-power roots of π made above, and transcendentally generates B_{st} over B_{cris}. Sending l_π to 0 gives us a projection $B_{\mathrm{st}} \to B_{\mathrm{cris}}$, which then gives us a map $\lambda_V : V \to V_{\mathrm{cris}}(D)$. Identify V with V_π via λ_V. Let $\rho, \rho_\pi : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$ be the continuous homomorphisms corresponding to V and V_π, respectively.

---

2. cf. [CF00, Théorème 4.3]

Proposition 2.2.3.1. *With the notation and definitions as above:*

1. *Let D_π be the filtered φ-module obtained from D by 'forgetting N' and via our choice of logarithm. Then D_π is again weakly admissible, and the map*

$$\lambda_V : V \to V_\pi \coloneqq V_{\mathrm{cris}}(D) = V_{\mathrm{cris}}(D_\pi)$$

   is a $\mathrm{Gal}(\overline{K}/K_\infty)$-equivariant isomorphism. If V is in fact crystalline, then λ_V is simply the canonical identification $V = V_{\mathrm{cris}}(D_{\mathrm{cris}}(V))$.

2. *We have*

$$\rho = (\mathbf{1}_V + t_p \otimes N) \circ \rho_\pi,$$

   Here, $t_p : \mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p(1)$ is the 1-cocycle defined in [Bre02, 3.5.4]*, so that we have $gl_u = l_\pi + t_p(g)t$, for all $g \in \mathrm{Gal}(\overline{K}/K)$; and $t \in B_{\mathrm{cris}}$ is Fontaine's cyclotomic period mentioned in the previous sub-section.*

3. *The following diagram commutes:*

$$\begin{array}{ccc} D \otimes B_{\mathrm{st}} & \xrightarrow{\simeq} & V \otimes B_{\mathrm{st}} \\ \Big\downarrow{\scriptstyle 1 + l_\pi N} & & \Big\downarrow{\scriptstyle \lambda_V \otimes B_{\mathrm{st}}} \\ D \otimes B_{\mathrm{st}} & \xrightarrow{\simeq} & V_\pi \otimes B_{\mathrm{st}} \end{array}$$

   Here, the horizontal maps are Fontaine's canonical comparison isomorphisms.

Proof. For (1), see [Bre02, 3.5.1,3.5.3]. Note, however, that Breuil uses the contravariant Fontaine functors, while we have employed their covariant counterparts. Assertion (2) follows from [Bre02, 3.5.4].

For (3), let Set $c(g) = t_p(g^{-1})$: this gives a $\mathbb{Z}_p(-1)$ valued 1-cocycle. Let E be the 2-dimensional Galois representation given by

$$g \mapsto \begin{pmatrix} 1 & c(g) \\ 0 & \chi(g)^{-1} \end{pmatrix},$$

in a basis $\{e_1, e_2\}$ for E.

This is semi-stable with Hodge-Tate weights in $\{0, 1\}$. Using (2) and the functoriality of λ_V, to prove (3), it is enough to show that the diagram in (3) commutes for the representation E. For this, we can do an explicit computation. We check:

- $D_{\mathrm{st}}(E)$ is spanned by $f_1 = 1 \otimes e_1$ and $f_2 = l_\pi \otimes e_1 + t \otimes e_2$, and N maps f_2 to f_1.
- $E_\pi = V_{\mathrm{cris}}(D_{\mathrm{st}}(E))$ is spanned by $1 \otimes f_1$ and $t^{-1} \otimes f_2$;
- λ_E maps e_1 to $1 \otimes f_1$ and e_2 to $t^{-1} \otimes f_2$.

Putting all this together, it is easy to see that the map on the left hand side of the square is indeed $1 + l_u N$. □

We will denote by $\mathrm{Rep}^{\mathrm{cris},\circ}_{\mathrm{Gal}(\overline{K}/K)}$ the category of Galois-stable $\mathbb{Z}_p$-lattices in crystalline $\mathrm{Gal}(\overline{K}/K)$-representations. $\mathrm{Rep}^{\circ}_{\mathrm{Gal}(\overline{K}/K_\infty)}$ will denote the category of $\mathrm{Gal}(\overline{K}/K_\infty)$-stable $\mathbb{Z}_p$-lattices in $\mathrm{Gal}(\overline{K}/K_\infty)$-representations.

Lemma 2.2.3.2. *The restriction functor from* $\mathrm{Rep}^{\mathrm{cris},\circ}_{\mathrm{Gal}(\overline{K}/K)}$ *to* $\mathrm{Rep}^{\circ}_{\mathrm{Gal}(\overline{K}/K_\infty)}$ *is fully faithful*

Proof. This is [Kis09, 2.1.4]. □

Suppose now that we have a Galois-stable lattice $\Lambda \subset V$. The filtration $W_\bullet V$ on V intersects with Λ to give rise to a filtration $W_\bullet \Lambda$ on Λ. Via the isomorphism $\lambda_V : V \to V_\pi$ above, we obtain a $\mathrm{Gal}(\overline{K}/K_\infty)$-stable lattice $\Lambda_\pi = \lambda_V(\Lambda) \subset V_\pi$.

Corollary 2.2.3.3. *Maintain the notation as above.*

1. *The map $N : V \to V(-1)$ restricts to a map $N : \Lambda \to \Lambda(-1)$.*
2. *The $\mathbb{Z}_p$-lattice $\Lambda_\pi \subset V_\pi$ is Galois-stable, and the map N gives rise to a Galois-equivariant map $N : \Lambda_\pi \to \Lambda_\pi(-1)$ via the isomorphism λ_V.*
3. *The filtration $W_\bullet \Lambda$ is Galois-stable and is taken to a Galois-stable filtration $W_\bullet \Lambda_\pi$ of Λ_π.*

Proof. Both (1) and (2) follow from [Bre02, 3.5.5]. For (3) the description in terms of of $W_\bullet \Lambda$ above in terms of the Galois-equivariant operator N shows that it is Galois-stable. Since the isomorphism λ_V is $\mathrm{Gal}(\overline{K}/K_\infty)$-equivariant, for each i, the map

$$W_i \Lambda \hookrightarrow \Lambda \xrightarrow[\simeq]{\lambda_V} \Lambda_\pi \tag{2.2.3.3.1}$$

is again $\mathrm{Gal}(\overline{K}/K_\infty)$-equivariant. For $i < 2$, by (2.2.2.1)(3), $W_i \Lambda$ is a Galois-stable $\mathbb{Z}_p$-lattice in a crystalline representation. It now follows from (2.2.3.2) that, for $i < 2$, the map in (2.2.3.3.1) is in fact $\mathrm{Gal}(\overline{K}/K)$-equivariant. This means precisely that the filtration $W_\bullet \Lambda_\pi$ is Galois-stable as well. □

2.2.4

Let $\bar{k}$ be the residue field of $\overline{K}$; for any extension l/k within $\bar{k}$, set $\mathfrak{S}(l) = W(l)[|u|]$. Let $\mathfrak{S} = \mathfrak{S}(k)$, and let $E(u) \in \mathfrak{S}$ be the monic Eisenstein polynomial associated with the uniformizer π. We equip $\mathfrak{S}$ with the lift $\varphi_{\mathfrak{S}}$ of the p-power Frobenius on $\mathfrak{S}/p\mathfrak{S}$ given by:

$$\varphi_{\mathfrak{S}}|_W = \varphi_W$$
$$\varphi_{\mathfrak{S}}(u) = u^p.$$

Definition 2.2.4.1. A **φ-module** over $\mathfrak{S}$ is a finite free $\mathfrak{S}$-module $\mathfrak{M}$ endowed with an $\mathfrak{S}$-linear isomorphism

$$\varphi_{\mathfrak{M}} : \varphi_{\mathfrak{S}}^{*}\mathfrak{M}\left[E(u)^{-1}\right] \xrightarrow{\simeq} \mathfrak{M}\left[E(u)^{-1}\right]$$

The category of φ-modules over $\mathfrak{S}$ will be denoted $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. If the map $\varphi_{\mathfrak{M}}$ actually arises from a map $\varphi_{\mathfrak{S}}^{*}\mathfrak{M} \to \mathfrak{M}$ whose co-kernel is killed by $E(u)^r$, for some natural number r, we will say that $\mathfrak{M}$ has **E-height** r.

A **(φ, N)-module** over $\mathfrak{S}$ is a φ-module $\mathfrak{M}$ equipped with an endomorphism N of $\mathfrak{M}/u\mathfrak{M}\left[\frac{1}{p}\right]$ satisfying $N\varphi = p\varphi N$, where φ is the φ_W-semi-linear endomorphism of $\mathfrak{M}/u\mathfrak{M}$ induced from $\varphi_{\mathfrak{M}}$. The category of (φ, N)-modules over $\mathfrak{S}$ will be denoted $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$.

A **Barsotti-Tate module** over $\mathfrak{S}$ is a φ-module of E-height 1. The category of Barsotti-Tate modules over $\mathfrak{S}$ will be denoted $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$.

There is a fully faithful exact tensor functor $\mathfrak{M} : \mathrm{Rep}^{\mathrm{st}}_{\mathrm{Gal}(\overline{K}/K)} \to \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ from the category of semi-stable Galois representations to the isogeny category of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, so that, for any $V \in \mathrm{Rep}^{\mathrm{st}}_{\mathrm{Gal}(\overline{K}/K)}$, we have canonical isomorphisms:

$$\frac{\mathfrak{M}(V)}{u\mathfrak{M}(V)}\left[\frac{1}{p}\right] \xrightarrow{\simeq} D_{\mathrm{st}}(V);$$
$$\frac{\varphi_{\mathfrak{S}}^{*}\mathfrak{M}(V)}{E(u)\varphi_{\mathfrak{S}}^{*}\mathfrak{M}(V)}\left[\frac{1}{p}\right] \xrightarrow{\simeq} D_{\mathrm{dR}}(V).$$

The first isomorphism is equivariant with respect to φ and N, and the second respects filtrations, where the filtration on the left hand side is induced from:

$$\mathrm{Fil}^{i}\, \varphi_{\mathfrak{S}}^{*}\mathfrak{M}(V) = \varphi^{-1}_{\mathfrak{M}(V)}\left(E(u)^{i}\mathfrak{M}(V)\right).$$

Moreover, for any natural number r, Galois representations with Hodge-Tate weights in $[0, r]$ are taken to φ-modules of E-height r.

All this follows from [Kis06, 1.3.15], which shows the above with the category of weakly admissible filtered (φ, N)-modules over K replacing the category of semi-stable Galois representations. But these two categories are equivalent via the functor D_{st}. [3]

By [Kis10, 1.2.1], we also have a fully faithful exact tensor functor

$$\mathfrak{M}^{\circ} : \mathrm{Rep}^{\mathrm{cris},\circ}_{\mathrm{Gal}(\overline{K}/K)} \to \mathrm{Mod}^{\varphi}_{/\mathfrak{S}},$$

3. In *loc. cit.*, Kisin also restricts himself to the situation where the Hodge-Tate weights are all non-negative, but this can be worked around using Tate twists. See proof of [Kis10, 1.2.1]

for which the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{cris},\circ}_{\mathrm{Gal}(\overline{K}/K)} & \xrightarrow{\mathfrak{M}^\circ} & \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \\ \downarrow & & \downarrow \\ \mathrm{Rep}^{\mathrm{st}}_{\mathrm{Gal}(\overline{K}/K)} & \xrightarrow{\mathfrak{M}} & \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p. \end{array}$$

Let Λ be a Galois-stable $\mathbb{Z}_p$-lattice in a semi-stable $\mathrm{Gal}(\overline{K}/K)$-representation V with Hodge-Tate weights in $\{0,1\}$. Let $D = D_{\mathrm{st}}(V)$ be the associated weakly admissible filtered (φ, N)-module over K. Let V_π be as in (2.2.3.1), equipped with a $\mathrm{Gal}(\overline{K}/K_\infty)$-equivariant isomorphism $\lambda_V : V \xrightarrow{\simeq} V_\pi$. Let Λ_π be the image of Λ under this isomorphism, as in (2.2.3.3). There is a unique isogeny representative of $\mathfrak{M}(V)$ that is isomorphic to $\mathfrak{M}^\circ(\Lambda_\pi)$ in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$: we denote this by $\mathfrak{M}(\Lambda)$. In particular, we have natural identifications

$$\mathfrak{M}(\Lambda)/u\mathfrak{M}(\Lambda)\left[\frac{1}{p}\right] = D_{\mathrm{cris}}(V_\pi) = D;$$

$$\frac{\varphi^*_{\mathfrak{S}}\mathfrak{M}(\Lambda)}{E(u)\varphi^*_{\mathfrak{S}}\mathfrak{M}(\Lambda)}\left[\frac{1}{p}\right] = D_{\mathrm{dR}}(V).$$

Moreover, the filtration $W_\bullet\Lambda_\pi$ (cf. (2.2.3.3)(3)) gives rise to a filtration $W_\bullet\mathfrak{M}(\Lambda)$ on $\mathfrak{M}(\Lambda)$ under the functor $\mathfrak{M}^\circ$.

Suppose that we have Galois-invariant tensors $\{s_\alpha\} \subset \Lambda^\otimes$ (see (2.1.2.2)) such that their pointwise stabilizer is a *reductive* sub-group $G_{\mathbb{Z}_p} \subset \mathrm{GL}(\Lambda)$. We can think of these tensors as $\mathrm{Gal}(\overline{K}/K)$-equivariant maps $\mathbf{1} \to \Lambda^\otimes$, where $\mathbf{1}$ is the trivial representation $\mathbb{Z}_p$. By abuse of notation, let $\mathbf{1}$ again denote the trivial filtered φ-module over K. Then, by the tensor-functoriality of D_{st}, we obtain sections $s_{\alpha,\mathrm{st}} : \mathbf{1} \to D^\otimes$.

Proposition 2.2.4.2. *With the notation as above, we have φ-invariant tensors $\{s_{\alpha,\mathfrak{S}}\} \subset \mathfrak{M}(\Lambda)^\otimes$ such that:*

1. *The natural identification*
$$\frac{\mathfrak{M}(\Lambda)}{u\mathfrak{M}(\Lambda)}\left[\frac{1}{p}\right] = D;$$
takes $\{s_{\alpha,\mathfrak{S}}\}$ to $\{s_{\alpha,\mathrm{st}}\}$.

2. *There exists an isomorphism*
$$\Lambda \otimes_{\mathbb{Z}_p} \mathfrak{S}(\bar{k}) \xrightarrow{\simeq} \mathfrak{M}(\Lambda) \otimes_{\mathfrak{S}} \mathfrak{S}(\bar{k})$$
under which the tensors $\{s_\alpha \otimes 1\}$ are taken to $\{s_{\alpha,\mathfrak{S}} \otimes 1\}$.

3. *There exists a map* $N_{\mathfrak{S}} : \mathfrak{M}(\Lambda) \to \mathfrak{M}(\Lambda)$ *satisfying*

$$N_{\mathfrak{S}}\varphi_{\mathfrak{M}(\Lambda)} = \frac{pE(u)}{E(0)}\varphi_{\mathfrak{M}(\Lambda)}N_{\mathfrak{S}},$$

and reducing modulo u *to the nilpotent operator* N *on* D.

Proof. If we think of s_α as a Galois-equivariant map $s_\alpha : \mathbf{1} \to \Lambda^{\otimes}$, we obtain, using λ_V, a $\mathrm{Gal}(\overline{K}/K_\infty)$-equivariant map $s_{\alpha,\pi} : \mathbf{1} \to \Lambda_\pi^{\otimes}$. By (2.2.3.2), The restriction functor from $\mathrm{Rep}^{\mathrm{cris},\circ}_{\mathrm{Gal}(\overline{K}/K)}$ to $\mathrm{Rep}^{\circ}_{\mathrm{Gal}(\overline{K}/K_\infty)}$ is fully faithful. Therefore, $s_{\alpha,\pi}$ in fact determines a Galois-invariant tensor in $\Lambda_\pi^{\otimes}$. Via the functor $\mathfrak{M}$ we then obtain φ-invariant tensors $s_{\alpha,\mathfrak{S}} \in \mathfrak{M}(\Lambda)^{\otimes}$.

Let $\{s_{\alpha,\pi,\mathrm{cris}}\} \subset D^{\otimes}$ be the tensors obtained from the reduction of $\{s_{\alpha,\mathfrak{S}}\}$ modulo (u). To prove (1), it suffices to show that these tensors agree with $\{s_{\alpha,\mathrm{st}}\}$. We see from (2.2.3.1)(3) that, in $D^{\otimes} \otimes B_{\mathrm{cris}}$,

$$s_{\alpha,\pi,\mathrm{cris}} \otimes 1 = \exp(l_\pi N)(s_{\alpha,\mathrm{st}} \otimes 1).$$

But $N(s_{\alpha,\mathrm{st}}) = 0$, and so $s_{\alpha,\mathrm{st}}$ is indeed equal to $s_{\alpha,\pi,\mathrm{cris}}$, as required.

For (2), it suffices to prove the statement with Λ replaced by Λ_π. But this is a consequence of [Kis10, 1.3.4]. Note that this is the place where reductivity of $G_{\mathbb{Z}_p}$ is crucial.

Finally, in (3), $N_{\mathfrak{S}}$ arises from the map $N : \Lambda_\pi \to \Lambda_\pi(-1)$ in (2.2.3.3)(2) via the functor $\mathfrak{M}$. We just have to observe that the underlying $\mathfrak{S}$-module for $\mathfrak{M}(\mathbb{Z}_p(-1))$ is simply $\mathfrak{S}$ with φ being multiplication by $\frac{pE(u)}{E(0)}$. See the proof of [Kis10, 1.2.1]. □

Corollary 2.2.4.3. *Let* $M_0 = \mathfrak{M}(\Lambda)/u\mathfrak{M}(\Lambda)$, *so that* $M_0\left[\frac{1}{p}\right] = D$. *Then:*

1. *The tensors* $\{s_{\alpha,\mathrm{st}}\}$ *lie in* $M_0^{\otimes}$.
2. *The filtration* $W_\bullet\Lambda$ *is* $G_{\mathbb{Z}_p}$*-split.*
3. *There is an isomorphism*

$$\Lambda \otimes_{\mathbb{Z}_p} W(\bar{k}) \xrightarrow{\simeq} M_0 \otimes_W W(\bar{k})$$

which takes $\{s_\alpha \otimes 1\}$ *to* $\{s_{\alpha,\mathrm{st}} \otimes 1\}$.

4. *The pointwise stabilizer* $G_W \subset \mathrm{GL}(M_0)$ *of the tensors* $\{s_{\alpha,\mathrm{st}}\}$ *is a pure inner form of* $G_{\mathbb{Z}_p} \otimes W$, *and is in particular reductive.*

Proof. (1) and (3) are immediate from the proposition above. For (2), by [Kis10, 1.1.4], it is enough to show that $W_\bullet V$ is $G_{\mathbb{Q}_p}$-split, where $G_{\mathbb{Q}_p}$ is the generic fiber of $G_{\mathbb{Z}_p}$. This follows from (2.2.2.1)(5). For (4), we simply have to observe that G_W is a twist of $G_{\mathbb{Z}_p} \otimes W$ by the $G_{\mathbb{Z}_p} \otimes W$-torsor Q'. □

2.2.5

For any finite extension L/K_0 with residue field l, denote by $L_0 = W(l)\left[\frac{1}{p}\right]$ the maximal absolutely unramified sub-extension of L. Choose some uniformizer π_L for L, and let $E_L(u) \in W(l)[|u|]$ be the monic Eisenstein corresponding to π_L. Let S_L be the p-adic completion of the divided power envelope of $\mathscr{O}_L$ in $W(l)[u]$. More explicitly:

$$S_L = \left\{ \sum_i a_i \frac{u^i}{q(i)!} \in L_0[|u|] : a_i \in W(l),\ \lim_{i\to\infty} a_i = 0 \right\}.$$

Here $q(i) = \lfloor \frac{i}{e} \rfloor$, where e is the ramification index of L. See [Bre00, 2.1.1]. S_L is equipped with the log structure M_{S_L} corresponding to the divisor defined by u and also a Frobenius lift φ taking u to u^p.

Let $S = S_K$ be the W-algebra associated to K and π. We will treat S as an $\mathfrak{S}$-algebra via the map $u \mapsto u$: this is clearly compatible with the Frobenius lifts on S and $\mathfrak{S}$.

With the notation from before, let $\mathcal{M}(\Lambda) = \varphi^*_{\mathfrak{S}}\mathfrak{M}(\Lambda) \otimes_{\mathfrak{S}} S$. Since $\varphi_S(E(u)) = pa$, for $a \in S^\times$, the induced map

$$\varphi_{\mathcal{M}(\Lambda)} : \varphi^*_S \mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda)$$

has its cokernel killed by p.

Lemma 2.2.5.1. *Let $\Lambda_\pi \in \mathrm{Rep}^{\mathrm{cris},\circ}_{\mathrm{Gal}(\overline{K}/K)}$ be the crystalline lattice associated with Λ as in the proof of (2.2.4.2), and suppose that $\Lambda_\pi = T_p(\mathscr{G})^\vee$, for a p-divisible group $\mathscr{G}$ over $\mathscr{O}_K$. Let $\mathbb{D}(\mathscr{G})$ be the contra-variant Dieudonné F-crystal over $\mathscr{O}_K$ associated with $\mathscr{G}$.*

1. *There is a natural isomorphism of S-modules*

$$\mathbb{D}(\mathscr{G})(S) \xrightarrow{\simeq} \mathcal{M}(\Lambda)$$

taking $\varphi_{\mathbb{D}(\mathscr{G})(S)}$ to $\varphi_{\mathcal{M}(\Lambda)}$, the former arising from the F-crystal structure on $\mathbb{D}(\mathscr{G})$.

2. *There is a natural isomorphism*

$$\mathbb{D}(\mathscr{G}_0)(W) \xrightarrow{\simeq} \varphi^*_W M_0,$$

of Dieudonné modules over W, where $\mathscr{G}_0$ is the reduction of $\mathscr{G}$ to k.

3. *There is a natural logarithmic connection*

$$\nabla_{\mathcal{M}(\Lambda)} : \mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda) \otimes_{W[u]} W[u]\,\mathrm{dlog}(u),$$

that is compatible with $\varphi_{\mathcal{M}(\Lambda)}$, and whose residue is the endomorphism $\varphi^*_W N$ of $\varphi^*_W M_0$

4. There is a natural isomorphism

$$\mathcal{M}(\Lambda)/E(u)\mathcal{M}(\Lambda)\left[\frac{1}{p}\right] \xrightarrow{\simeq} D_{\mathrm{dR}}(\Lambda),$$

respecting the Hodge filtration on both sides. Here, $\mathcal{M}(\Lambda)/E(u)\mathcal{M}$ is equipped with the filtration $\mathrm{Fil}^1\,(\varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M})\otimes_{\mathfrak{S}} S$ *(cf. 2.2.4.2).*

Proof. By construction, $\mathcal{M}(\Lambda)$ can be identified with the S-module $\mathcal{M}(\mathfrak{M}(\mathscr{G}))$ in [Kis10, 1.4.2][4]. So the first assertion follows from *loc. cit.* The second assertion is an immediate consequence of the first.

For the third, we first remark that there is a connection

$$\nabla_{\mathscr{G}} : \mathbb{D}(\mathscr{G})(S) \to \mathbb{D}(\mathscr{G})(S)\otimes_{W[u]} W[u]du,$$

arising from the fact that $\mathbb{D}(\mathscr{G})$ is a crystal over $\mathscr{O}_K$. This gives rise via the isomorphism in (1) to a connection

$$\nabla_{\pi} : \mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda)\otimes_{W[u]} W[u]du.$$

Let $N_{\mathfrak{S}} : \mathfrak{M}(\Lambda) \to \mathfrak{M}(\Lambda)$ be the endomorphism associated with the map $N : \Lambda_\pi \to \Lambda_\pi(-1)$ as in (2.2.4.2). Let $N : S \to S$ be the derivation taking u to $-u$, and set

$$N_{\mathcal{M}(\Lambda)} = -\varphi^*_{\mathfrak{S}} N_{\mathfrak{S}}\otimes 1 + 1\otimes N,$$

as a derivation of $\mathcal{M}(\Lambda) = \varphi^*_{\mathfrak{S}}\mathfrak{M}(\Lambda)\otimes_{\mathfrak{S}} S$. Then

$$\nabla_{\mathcal{M}(\Lambda)} = \nabla_\pi - N_{\mathcal{M}(\Lambda)}\otimes \mathrm{dlog}(u)$$

is the connection we are looking for.

Finally, for (4), we can simply appeal to (2.2.4.2)(1) and the definition of $\mathcal{M}(\Lambda)$. □

Remark 2.2.5.2.

- We can always find a p-divisible group $\mathscr{G}$ such that $\Lambda_\pi = T_p(\mathscr{G})^\vee$. This follows from the fact that every crystalline representation with Hodge-Tate weights in $\{0,1\}$ arises from the Tate module of a p-divisible group over $\mathscr{O}_K$. See [Kis06, 2.2.6]. Any two such p-divisible groups will be isomorphic by the full-faithfulness of the functor T_p (Tate's theorem).

- For a different choice of the branch of the p-adic logarithm (see (2.2.1), the p-divisible group attached to the corresponding crystalline representation will not in general be isomorphic to $\mathscr{G}$. However, for $i \leq 1$, the p-divisible group $W_i\mathscr{G}$ associated with $W_i\Lambda$, and, for all i, the p-divisible groups $\mathrm{gr}^W_i\,\mathscr{G}$ associated with $\mathrm{gr}^W_i\,\Lambda$ are unambiguously determined, independently of the choice of logarithm.

4. The definition of $\mathfrak{M}(\mathscr{G})$ is a little off in *loc. cit.*: it should be $\mathfrak{M}(\mathscr{G}) \coloneqq \mathfrak{M}(T_p(\mathscr{G})^\vee)$.

- Λ arises from a log p-divisible group $\mathscr{G}^{\log}$ over $\mathscr{O}_K$, and the S-module $\mathfrak{M}(\Lambda)$ simply corresponds to the log Dieudonné F-crystal associated with $\mathscr{G}^{\log}$ via logarithmic Dieudonné theory; cf. [Katb].

2.2.6

Let R_e be the ring of functions on the rigid analytic open disk of radius $p^{-\frac{p-1}{e}}$, and let us fix a co-ordinate u on this disk; then R_e admits a log structure $\mathrm{M}_{R_e} = R_e^{\times} \oplus u^{\mathbb{N}}$. We can embed S into R_e via $u \mapsto u$: this clearly respects log structures. Set

$$R_e^{\log} = \frac{R_e\left[l_\alpha : \alpha \in \mathrm{M}_{R_e}^{\mathrm{gp}}\right]}{\left(l_{\alpha\beta} - l_\alpha - l_\beta, \text{ for } \alpha, \beta \in \mathrm{M}_{R_e}^{\mathrm{gp}}; l_\alpha = \log(\alpha), \text{ whenever } |\alpha - 1| < 1.\right)}.$$

Here, by $|\alpha - 1| < 1$, we mean that $|\alpha(x) - 1| < 1$, for all x in the rigid analytic open disk of radius $p^{-\frac{p-1}{e}}$.

$R_e^{\log}$ can be equipped with a natural logarithmic connection $\nabla : l_\alpha \mapsto -1 \otimes \mathrm{dlog}(\alpha)$, and a semi-linear map φ lifting φ_S given by $\varphi(l_\alpha) = p l_\alpha$. Set

$$\Psi(\Lambda) = \left(\mathcal{M}(\Lambda) \otimes_S R_e^{\log}\right)^{\nabla = 0},$$

where we endow $\mathcal{M}(\Lambda) \otimes_S R_e^{\log}$ with the tensor-product connection. This is naturally a (φ, N)-module over K_0. By [Vol03, Theorem 9], the inclusion $\Psi(\Lambda) \hookrightarrow \mathcal{M}(\Lambda) \otimes_S R_e^{\log}$ induces a (φ, ∇)-equivariant isomorphism

$$\Psi(\Lambda) \otimes_{K_0} R_e^{\log} \xrightarrow{\simeq} \mathcal{M}(\Lambda) \otimes_S R_e^{\log}. \tag{2.2.6.0.1}$$

The natural surjection $R_e \to K$ sending u to π can be extended to a surjection $R_e^{\log} \to K$ by sending l_u to 0, and reducing the isomorphism (2.2.6.0.1) along this surjection gives us an isomorphism:

$$\beta_{\mathrm{dR}} : \Psi(\Lambda) \otimes_{K_0} K \xrightarrow{\simeq} \mathcal{M}(\Lambda) \otimes_S K \xrightarrow{\simeq} D_{\mathrm{dR}}(\Lambda), \tag{2.2.6.0.2}$$

where the last isomorphism follows from (2.2.5.1)(4).

We can similarly extend the natural surjection $R_e \to K_0$ sending u to 0 to $R_e^{\log} \to K_0$ sending l_u to 0. Reducing (2.2.6.0.1) along this, gives us a (φ, N)-equivariant map

$$\beta_{\mathrm{st}} : \Psi(\Lambda) \xrightarrow{\simeq} \mathcal{M}(\Lambda) \otimes_S K_0 \xrightarrow{\simeq} \varphi_W^* D_{\mathrm{st}}(\Lambda) \xrightarrow[\simeq]{\varphi_D} D_{\mathrm{st}}(\Lambda). \tag{2.2.6.0.3}$$

Here, the second isomorphism follows from (2.2.4.2)(1).

Lemma 2.2.6.1. *The diagonal isomorphism in the diagram below is the one induced from the embedding $B_{\mathrm{st}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}$ given by the choice of logarithm taking π to 0:*

$$\begin{array}{ccc} \Psi(\Lambda) \otimes_{K_0} K & \xrightarrow[\simeq]{\beta_{\mathrm{dR}}} & D_{\mathrm{dR}}(\Lambda) \\ \beta_{\mathrm{st}} \otimes 1 \downarrow \simeq & \nearrow \simeq & \\ D_{\mathrm{st}}(\Lambda) \otimes_{K_0} K & & \end{array}$$

Proof. This follows from the argument in [Kis06, 1.2.8]. □

2.3 The log F-crystal associated with a semi-stable abelian variety

2.3.1

Let $K/\mathbb{Q}_p$ be a finite extension with residue field k. Let A be a polarizable semi-stable abelian variety over K extending to a semi-abelian scheme G' over $\mathscr{O}_K$. By (1.2.4.2), after finite unramified base-change, if necessary, we can find a positive, log 1-motif $[Y \xrightarrow{\iota} J^{\log}]$ over $\mathscr{O}_K$, where Y is a J is an extension

$$0 \to T \to J \to B \to 0$$

of an abelian scheme B over $\mathscr{O}_K$ and T is a split torus over $\mathscr{O}_K$ with character group a free abelian group X.

By the theory in [Ray71], we have an isomorphism

$$J^{\mathrm{an}}/\iota(Y) \xrightarrow{\simeq} A^{\mathrm{an}}, \tag{2.3.1.0.1}$$

of rigid analytic varieties over K. We also have the **monodromy map** $N_A = N_L : Y \to \mathrm{Hom}(X, \mathbb{Z})$ for A (cf. 1.2.2.2). A choice of polarization λ on A determines among other things a map $\lambda^{\text{ét}} : Y \to X$ such that $(y, y') \mapsto N_A(\lambda^{\text{ét}}(y))$ induces a positive definite symmetric bilinear form on $Y \otimes \mathbb{Q}$ (cf. 1.2.4.1). In particular, $\mathrm{rk}\,\mathrm{im}\, N_A = \mathrm{rk}\, X$.

Suppose that we fix an algebraic closure $\overline{K}/K$, and that we set, for any algebraic group H over K,

$$T_p(H) = \varprojlim H[p^n](\overline{K});$$

$$H^1(H, \mathbb{Z}_p) = H^1_{\text{ét}}(H_{\overline{K}}, \mathbb{Z}_p).$$

Then, from the uniformation (2.3.1.0.1), we obtain a short exact sequence of $\mathrm{Gal}(\overline{K}/K)$-representations:

$$0 \to T_p(J) \to T_p(A) \to Y \otimes \mathbb{Z}_p \to 0.$$

Dualizing it gives us

$$0 \to \mathrm{Hom}(Y, \mathbb{Z}_p) \to H^1(A, \mathbb{Z}_p) \to H^1(J, \mathbb{Z}_p) \to 0.$$

Also, $H^1(J, \mathbb{Z}_p)$ sits in a further short exact sequence

$$0 \to H^1(B, \mathbb{Z}_p) \to H^1(J, \mathbb{Z}_p) \to X \otimes \mathbb{Z}_p(-1) \to 0.$$

Putting all this together gives us an ascending three-step **weight filtration** $W_\bullet H^1(A, \mathbb{Z}_p)$ on $H^1(A, \mathbb{Z}_p)$, with $W_{-1}H^1(A, \mathbb{Z}_p) = 0$; $\mathrm{gr}_0^W H^1(A, \mathbb{Z}_p) = \mathrm{Hom}(Y, \mathbb{Z}_p)$; $\mathrm{gr}_1^W H^1(A, \mathbb{Z}_p) = H^1(B, \mathbb{Z}_p)$; and $\mathrm{gr}_2^W H^1(A, \mathbb{Z}_p) = X \otimes \mathbb{Z}_p(-1)$.

Consider now the de Rham cohomology $H^1_{\mathrm{dR}}(A)$: this is a filtered K-vector space. For any integer i, let $K(-i)$ be the filtered one dimensional K-vector space with $\mathrm{Fil}^i = K$ and $\mathrm{Fil}^{i+1} = 0$. We will denote $K(0)$ simply by K. Then, by the same considerations as above, $H^1_{\mathrm{dR}}(A)$ also admits an ascending weight filtration $W_\bullet H^1_{\mathrm{dR}}(A)$ with $W_{-1}H^1_{\mathrm{dR}}(A) = 0$; $\mathrm{gr}_0^W H^1_{\mathrm{dR}}(A) = \mathrm{Hom}(Y, K)$; $\mathrm{gr}_1^W H^1_{\mathrm{dR}}(A) = H^1_{\mathrm{dR}}(B_K)$; and $\mathrm{gr}_2^W H^1_{\mathrm{dR}}(A) = H^1_{\mathrm{dR}}(T) = X \otimes K(-1)$ (cf. [CI99, §I.2]). Observe also that, after tensoring with K, the monodromy N_A induces a nilpotent endomorphism of $H^1_{\mathrm{dR}}(A)$ with $W_1H^1_{\mathrm{dR}}(A)$ as its kernel and with its image equal to $\mathrm{gr}_2^W H^1_{\mathrm{dR}}(A)$: we will call this operator $N_{A,\mathrm{dR}}$ (cf. [CI99, §I.2.1]).

Set

$$D_{\mathrm{st}}(A) \coloneqq \left(B_{\mathrm{st}} \otimes_{\mathbb{Z}_p} H^1(A, \mathbb{Z}_p)\right)^{\mathrm{Gal}(\overline{K}/K)}.$$

This is a weakly admissible filtered (φ, N)-module over K with Hodge-Tate weights in $\{0, 1\}$. Recall that, with our convention, this means that

$$D_{\mathrm{dR}}(A) = \left(D_{\mathrm{st}}(A) \otimes_{K_0} K^{\log}\right)^{N=0}$$

is endowed with a filtration. The weight filtration on $H^1(A, \mathbb{Z}_p)$ gives rise to a filtration $W_\bullet D_{\mathrm{st}}(A)$ by weakly admissible filtered (φ, N)-modules, with $W_{-1}D_{\mathrm{st}}(A) = 0$; $\mathrm{gr}_0^W D_{\mathrm{st}}(A) = \mathrm{Hom}(Y, K_0)$; $\mathrm{gr}_1^W D_{\mathrm{st}}(A) = D_{\mathrm{cris}}(B)$; and $\mathrm{gr}_2^W D_{\mathrm{st}}(A) = X \otimes K_0(-1)$.

Remark 2.3.1.1. As explained in [CI99, §II.5], $D_{\mathrm{st}}(A)$ admits a φ-equivariant splitting

$$D_{\mathrm{st}}(A) = \mathrm{Hom}(Y, K_0) \oplus D_{\mathrm{cris}}(B) \oplus D_{\mathrm{cris}}(T)$$

of the weight filtration. This is a consequence of the Riemann hypothesis for the reciprocal eigenvalues of the crystalline Frobenius (cf. [KM74]).

Proposition 2.3.1.2. *There is a natural isomorphism*

$$\eta_A : H^1_{\mathrm{dR}}(A) \xrightarrow{\simeq} D_{\mathrm{dR}}(A) = \left(D_{\mathrm{st}}(A) \otimes_{K_0} K^{\log}\right)^{N=0}$$

that:

1. *respects both Hodge and weight filtrations;*

2. *carries the nilpotent operator* $N_{A,\mathrm{dR}}$ *on the left hand side to the nilpotent operator* N_{st} *on the right hand side induced by the one on* $D_{\mathrm{st}}(A)$*;*

3. *induces the identity from* $\mathrm{Hom}(Y,K) = \mathrm{gr}_0^W H^1_{\mathrm{dR}}(A)$ *to* $\mathrm{Hom}(Y,K) = \mathrm{gr}_0^W D_{\mathrm{dR}}(A)$*, and from* $Y \otimes K = \mathrm{gr}_2^W H^1_{\mathrm{dR}}(A)$ *to* $Y \otimes K = \mathrm{gr}_2^W D_{\mathrm{dR}}(A)$*;*

4. *induces on the* gr_1^W *components the p-adic comparison isomorphism*

$$\eta_B : H^1_{\mathrm{dR}}(B) \xrightarrow{\simeq} D_{\mathrm{dR}}(B)$$

constructed in [Fon82, §6].

In particular, the weight filtration on $H^1_{\mathrm{dR}}(A)$ *is up to shift the pre-image under* η_A *of the Jacobson-Morosov filtration associated with the nilpotent operator* N_{st}*.*

Proof. The existence of η_A satisfying the numbered properties is shown in [CI99, II.6.2]. We remark that the operator N_{st} on the right hand side of the isomorphism is given by

$$N_{\mathrm{st}}\left(\sum_i d_i \otimes x_i\right) = \sum_i N(d_i) \otimes x_i = -\sum_i d_i \otimes N(x_i).$$

□

2.3.2

Let $\mathbb{D}(J)$, $\mathbb{D}(B)$ and $\mathbb{D}(T)$ be the contra-variant Dieudonné F-crystals over $\mathscr{O}_K$ associated with the p-divisible groups $J[p^\infty]$, $B[p^\infty]$, and $T[p^\infty]$, respectively. Fix a uniformizer $\pi \in K$ and let $E(u) \in W[u]$ be its monic Eisenstein polynomial. Let S be as in (2.2.5) associated to the uniformizer π. In the notation of that section, take $\Lambda = H^1(A, \mathbb{Z}_p)$: this is a Galois-stable $\mathbb{Z}_p$-lattice in the semi-stable representation $H^1(A, \mathbb{Q}_p)$, which has Hodge-Tate weights $\{0, 1\}$. By (2.2.5.1) and (2.2.4.2), we have a φ-module $\mathcal{M}(A) \coloneqq \mathcal{M}(\Lambda)$ over S equipped with natural (in A) isomorphisms of filtered K-vector spaces

$$\mathcal{M}(A)/E(u)\mathcal{M}(A)\left[\frac{1}{p}\right] \xrightarrow{\simeq} D_{\mathrm{dR}}(\Lambda).$$

Moreover, $\mathcal{M}(A)$ is equipped with a weight filtration $W_\bullet\mathcal{M}(A)$ such that $W_{-1}\mathcal{M}(A) = 0$; $\mathrm{gr}_0^W \mathcal{M}(A) = \mathrm{Hom}(Y, S)$; $\mathrm{gr}_1^W \mathcal{M}(A) \xrightarrow{\simeq} \mathbb{D}(A)(S)$; and $\mathrm{gr}_2^W \mathcal{M}(A) \xrightarrow{\simeq} \mathbb{D}(T)(S)$. The last two isomorphisms follow from [Kis10, 1.4.2].

Recall that $\mathcal{M}(A)$ is equipped with a logarithmic connection and a map

$$\varphi_{\mathcal{M}(A)} : \varphi_S^*\mathcal{M}(A) \to \mathcal{M}(A)$$

that is parallel for the connection. This gives rise to a logarithmic F-crystal over $\mathscr{O}_K$ (cf. [Vol03, §3.9]), and the weight filtration on $\mathcal{M}(A)$ gives rise to a weight filtration on this

log crystal. Observe that the map $\iota : Y \to J(K)$ gives rise to a log 1-motif $Q = [Y \xrightarrow{\iota} J^{\log}]$ over $\mathscr{O}_K$ (cf. 1.3), and by the theory of *loc. cit.* we can associate with it a log F-crystal $\mathbb{D}(A) = \mathbb{D}(Q)$ over $\mathscr{O}_K$. By its construction, $\mathbb{D}(A)$ sits in a short exact sequence:

$$0 \to \mathrm{Hom}(Y, \mathbf{1}) \to \mathbb{D}(A) \to \mathbb{D}(J) \to 0,$$

where $\mathbf{1}$ is the trivial F-crystal over $\mathscr{O}_K$, and this corresponds to a weight filtration $W_\bullet\mathbb{D}(A)$ on $\mathbb{D}(A)$ such that $W_{-1}\mathbb{D}(A) = 0$; $\mathrm{gr}_0^W \mathbb{D}(A) = \mathrm{Hom}(Y, \mathbf{1})$; $\mathrm{gr}_1^W \mathbb{D}(A) = \mathbb{D}(B)$; and $\mathrm{gr}_2^W \mathbb{D}(A) = \mathbb{D}(T)$.

Lemma 2.3.2.1. *Let D be a finite free $K_0[|u|]$-module equipped with a map*

$$\varphi_D : \varphi^* D \to D.$$

Let $D_0 = D/uD$. Suppose that we have two logarithmic connections

$$\nabla_1, \nabla_2 : D \to D \,\mathrm{dlog}(u),$$

and suppose that their residues $\mathrm{res}(\nabla_1)$ *and* $\mathrm{res}(\nabla_2)$ *are equal as endomorphisms of D_0. If φ_D is parallel for both ∇_1 and ∇_2, then $\nabla_1 = \nabla_2$.*

Proof. Consider $\theta = \nabla_1 - \nabla_2$: since ∇_1 and ∇_2 have the same residue, θ is an element of $\mathrm{Hom}(D, D) \otimes \hat{\Omega}^1_{K_0[|u|]/K_0}$. Since φ_D is parallel with respect to both ∇_1 and ∇_2, using the same argument as in the proof of (1.4.2.4), it follows that

$$\theta \in \bigcap_{n \geq 1} \mathrm{Hom}(D, D) \otimes u^n \hat{\Omega}^1_{K_0[|u|]/K_0} = 0.$$

□

Proposition 2.3.2.2. *$\mathbb{D}(A)$ is naturally isomorphic to the log F-crystal over $\mathscr{O}_K$ arising from $\mathcal{M}(A)$. This isomorphism preserves the weight filtrations on both sides.*

Proof. It suffices to construct an isomorphism

$$\mathbb{D}(A)(S) \xrightarrow{\simeq} \mathcal{M}(A),$$

respecting weight filtrations and equivariant with respect to φ and the logarithmic connections on both sides.

Consider the log 1-motif induced by the map $N_{A,\pi} : Y \xrightarrow{N_A} \mathrm{Hom}\,(X, \mathbb{Z}) \xrightarrow{1 \mapsto \pi} T(K)$ (recall that X is the character group of the split torus T); it corresponds to a semi-stable abelian variety A_π over $\mathscr{O}_K$ with split multiplicative reduction, and $\mathbb{D}(A_\pi)$ sits in a short exact sequence

$$0 \to \mathrm{Hom}(Y, \mathbf{1}) \to \mathbb{D}(A_\pi) \to \mathbb{D}(T) \to 0,$$

of log F-crystals. Let $\mathbb{D}(A_\pi)_J$ be the pull-back of this extension along the natural map $\mathbb{D}(J) \to \mathbb{D}(T)$; then both $\mathbb{D}(A_\pi)_J$ and $\mathbb{D}(A)$ are extensions of $\mathbb{D}(J)$ by $\mathrm{Hom}(Y, \mathbf{1})$. By

construction, their Baer difference is the Dieudonné crystal $\mathbb{D}(\mathscr{G}_\pi)$ associated with Λ, as in (2.2.5.1). Moreover, by *loc. cit.*, $\mathcal{M}(A)$ is identified with $\mathbb{D}(\mathscr{G}_\pi)$ as a φ-module. Since π lifts to u, which satisfies $\varphi(u) = u^p$, we can check from the construction of $\mathbb{D}(A)$ (cf. 1.3) that the underlying φ-module of $\mathbb{D}(A)(S)$ is naturally isomorphic to the one underlying $\mathbb{D}(\mathscr{G}_\pi)(S) = \mathcal{M}(A)$. So $\mathbb{D}(A)(S)$ and $\mathcal{M}(A)$ are naturally identified as φ-modules over S. To see that this identification respects the logarithmic connections on both sides, it suffices by (2.3.2.1) to check that the residues of the connections on either side match up. But this is immediate, since both residues are canonically identified with N_A. □

Proposition 2.3.2.3. *Suppose that we have* $\mathrm{Gal}(\overline{K}/K)$*-invariant tensors* $\{s_\alpha\} \subset \Lambda^\otimes$ *defining a reductive sub-group* $G_{\mathbb{Z}_p} \subset \mathrm{GL}(\Lambda)$*, giving rise to* φ*-invariant tensors* $\{s_{\alpha,\mathrm{st}}\} \subset \left(D_{\mathrm{st}}(A)^\otimes\right)^{N=0}$*. Let* $M_0 = \mathbb{D}(A)(S) \otimes_S W$*. On* $M_0 \otimes k$ *we have the* ***Hodge filtration*** $\mathrm{Fil}^1(M_0 \otimes k)$*, whose defining property is:*

$$\varphi_W^* \left(\mathrm{Fil}^1(M_0 \otimes k)\right) = \ker \varphi_{M_0 \otimes k}. \tag{2.3.2.3.1}$$

1. *We have a natural isomorphism*

$$\mathbb{D}(A)(\mathscr{O}_K)\left[\frac{1}{p}\right] \xrightarrow{\simeq} H^1_{\mathrm{dR}}(A),$$

   respecting weight filtrations.

2. *There is a natural* φ*-equivariant splitting of the weight filtration on* $\mathbb{D}(A)(\mathscr{O}_K) \otimes_{\mathscr{O}_K} K^{\log}$*.*

3. *The tensors* $\{s_\alpha\}$ *give rise to parallel* φ*-invariant tensors* $\{s_{\alpha,S}\} \subset \mathbb{D}(A)(S)^\otimes$ *defining a reductive sub-group* $G_S \subset \mathrm{GL}(\mathbb{D}(A)(S))$ *and reducing to* $\{s_{\alpha,\mathrm{st}}\}$ *under the isomorphism* β_{st} *in (2.2.6.1).*

4. *Let* $G_{\mathscr{O}_K} = G_S \otimes \mathscr{O}_K$*; then the weight and Hodge filtrations on* $\mathbb{D}(A)(\mathscr{O}_K)$ *are* $G_{\mathscr{O}_K}$*-split.*

5. *Let* $P_{\mathrm{wt},k} \subset G_k = G_S \otimes k$ *be the sub-group stabilizing the weight filtration on* $M_0 \otimes k$*. The Hodge filtration* $\mathrm{Fil}^1(M_0 \otimes k)$ *is* $P_{\mathrm{wt},k}$*-split.*

Proof. For (1), we have the isomorphisms:

$$\mathbb{D}(A)(\mathscr{O}_K)\left[\frac{1}{p}\right] \xrightarrow[(2.3.2.2)]{\simeq} \mathcal{M}(A) \otimes_S K \xrightarrow[(2.2.5.1)(4)]{\simeq} D_{\mathrm{dR}}(A) \xrightarrow[(2.3.1.2)]{\simeq} H^1_{\mathrm{dR}}(A). \tag{2.3.2.3.2}$$

Let R_e and $R_e^{\log}$ be as in (2.2.6). Let $\Psi(A) = \Psi(\Lambda)$ in the notation of *loc. cit.*: this is a (φ, N)-module over K_0, and we have natural (φ, ∇)-equivariant isomorphisms

$$\left(\Psi(A) \otimes_{K_0} R_e^{\log}\right) \xrightarrow{\simeq} \mathcal{M}(A) \otimes_S R_e \xrightarrow{\simeq} \mathbb{D}(A)(S) \otimes_S R_e,$$

compatible with weight filtrations. Again, by the crystalline Riemann hypothesis (cf. 2.3.1.1), the weight filtration on $\Psi(A)$ is canonically and φ-equivariantly split. We thus obtain a canonical splitting of the weight filtration on $\mathbb{D}(A)(S) \otimes_S R_e^{\log}$, and thus a canonical splitting of the weight filtration on $\mathbb{D}(A)(\mathscr{O}_K) \otimes_{\mathscr{O}_K} K^{\log}$ via specialization.

Let $\mathfrak{M}(A) = \mathfrak{M}(\Lambda)$ be as in (2.2.4.2). By *loc. cit.*, we obtain φ-invariant tensors $\{s_{\alpha,\mathfrak{S}}\} \subset \mathfrak{M}(A)^{\otimes}$ defining a reductive sub-group $G_{\mathfrak{S}} \subset \mathrm{GL}(\mathfrak{M}(A))$ and reducing to $\{s_{\alpha,\mathrm{st}}\} \in D_{\mathrm{st}}(A)^{\otimes}$ under the isomorphism in (2.2.4.2)(1). Since $\mathcal{M}(A) = \varphi^*\mathfrak{M}(A) \otimes_{\mathfrak{S}} S$ by construction, we simply take $\{s_{\alpha,S}\} = \{\varphi^* s_{\alpha,\mathfrak{S}} \otimes 1\}$, where we also employ the identification of $\mathcal{M}(A)$ with $\mathbb{D}(A)(S)$. This finishes the proof of (3).

We now consider (4): by [Kis10, 1.1.4], since $G_{\mathscr{O}_K}$ is reductive, it suffices to show that the weight filtration on $H^1_{\mathrm{dR}}(A)$ is $G_K = G_{\mathscr{O}_K} \otimes K$-split. By (2.3.1.2), we know that the weight filtration on $H^1_{\mathrm{dR}}(A)$ is the Jacobson-Morosov filtration associated with the operator N on $D_{\mathrm{dR}}(A) = D_{\mathrm{st}}(A) \otimes_{K_0} K$. Further, since $N(s_{\alpha,\mathrm{st}}) = 0$, we have $N \in \mathrm{Lie}\, G_K$. Now it follows from (2.1.2.1)(2) that the weight filtration is indeed G_K-split. A similar argument applies to the Hodge filtration, but this time we need to appeal to (2.2.2.1)(4), which in fact shows that the Hodge filtration on $H^1_{\mathrm{dR}}(A)$ is $P_{\mathrm{wt},K}$-split.

Finally, for (5), to check that the Hodge filtration on $M_0 \otimes k$ is $P_{\mathrm{wt},k}$-split, it is enough, by (2.1.1.6), to check that the induced filtration on $\mathrm{gr}^W_\bullet(M_0 \otimes k)$ is $L_{\mathrm{wt},k}$-split, where $L_{\mathrm{wt},k}$ is the Levi quotient of $P_{\mathrm{wt},k}$. It is of course enough to show that the Hodge filtration on $\mathrm{gr}^W_\bullet \mathbb{D}(A)(\mathscr{O}_K)$ is $L_{\mathrm{wt},\mathscr{O}_K}$-split, where $L_{\mathrm{wt},\mathscr{O}_K}$ is the Levi quotient of $P_{\mathrm{wt},\mathscr{O}_K}$, the parabolic sub-group of $G_{\mathscr{O}_K}$ preserving the weight filtration. Again, by [Kis10, 1.1.4], we can finish by showing that the Hodge filtration on $\mathrm{gr}^W_\bullet H^1_{\mathrm{dR}}(A)$ is $L_{\mathrm{wt},K}$-split. But, in fact, the Hodge filtration on $H^1_{\mathrm{dR}}(A)$ is $P_{\mathrm{wt},K}$-split, as we saw in the proof of (5) above. □

2.4 Families of degenerating abelian varieties

2.4.1

Suppose that we have a local log W-algebra (R, M_R) with residue field k, where R is formally smooth and (R, M_R) is log formally smooth. In more concrete terms, R is isomorphic to $W[|t_1, \ldots, t_r|]$ and M_R is induced by the divisor $t_{n+1}t_{n+2}\cdots t_r = 0$, for some n between 1 and $r-1$. Let $P = \mathrm{M}_R/R^\times$, and let $x_0 : R \to k$ be the natural surjection. This induces a log structure $\mathrm{M}_k = k^\times \oplus P$ on k; let us call the associated log W-algebra k_P (cf. 1.2.1.8). Equip W with the log structure $\mathrm{M}_W = W^\times \oplus P$ and call the resulting log W-algebra W_P; this is now a formal divided power thickening of k_P. We will also equip W_P with a Frobenius lift φ_{W_P} (cf. 1.4.3), so that any log F-crystal over k_P, when evaluated at W_P, will give rise to a φ-module over W.

Let $U \subset \mathrm{Spec}\, R$ be the locus where the log structure is trivial: that is, it is the complement of the divisor defining the log structure. Let A be a semi-abelian scheme over R that restricts to a polarizable abelian scheme over U; then, by (1.2.4.2), we can find a log

1-motif $[Y \xrightarrow{\iota} J^{\log}]$ over R corresponding to A. Here Y is a free abelian group (after finite étale base change, which we will assume), and J is a semi-abelian extension

$$0 \to T \to J \to B \to 0,$$

where B is an abelian scheme over R and T is a split torus with character group X (again, we might need a finite étale base change to ensure this).

Proposition 2.4.1.1. *We can naturally associate with A a log F-crystal $\mathbb{D}(A)$ over R equipped with an ascending three-step filtration $W_\bullet\mathbb{D}(A)$ such that:*

1. *There are natural identifications $W_{-1}\mathbb{D}(A) = 0$; $\mathrm{gr}_0^W \mathbb{D}(A) = \mathrm{Hom}(Y, \mathbf{1})$; $\mathrm{gr}_1^W \mathbb{D}(A) = \mathbb{D}(B)$; $W_1\mathbb{D}(A) = \mathbb{D}(J)$; and $\mathrm{gr}_2^W \mathbb{D}(A) = \mathbb{D}(T)$.*

2. *For any continuous map $x : R \to \mathscr{O}_L$ of log W-algebras, where $L \subset \overline{K}_0$ is a finite extension of K_0, let A_x be the corresponding semi-stable abelian variety over L. Then there is a natural isomorphism of log F-crystals over $\mathscr{O}_K$:*

$$\mathbb{D}(A_x) \xrightarrow{\simeq} x^*\mathbb{D}(A),$$

   preserving weight filtrations.

3. *Set $M_0 = (x_0^*\mathbb{D}(A))(W_P)$; then we have natural isomorphisms*

$$\mathbb{D}(A_x)(S_L) \otimes_{S_L} W(l) \xrightarrow{\simeq} M_0 \otimes W(l); \text{ and}$$

$$D_{\mathrm{st}}(A_x) \xrightarrow{\simeq} M_0 \otimes_W L_0,$$

   where l is the residue field of L and $L_0 = W(l)\left[\frac{1}{p}\right]$ is the maximal absolutely unramified sub-extension of L. Here, S_L is the W-algebra associated with some choice of uniformizer in L, as in (3.3.4).

4. *For x as in (2), we have a natural isomorphism*

$$\mathbb{D}(A)(R) \otimes_{R,x} L \xrightarrow{\simeq} H^1_{\mathrm{dR}}(A_x).$$

5. *For any pair of continuous maps $x, x' : R \to \mathscr{O}_L$ of log W-algebras, we have a natural 'parallel transport' isomorphism*

$$\eta_{x,x'} : H^1_{\mathrm{dR}}(A_x) \otimes_L L^{\log} \xrightarrow{\simeq} H^1_{\mathrm{dR}}(A_{x'}) \otimes_L L^{\log}$$

such that the following diagram commutes:

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(A_x)\otimes_L L^{\log} & \xrightarrow[\simeq]{\eta_{x,x'}} & H^1_{\mathrm{dR}}(A_{x'})\otimes_L L^{\log} \\ \simeq\downarrow \eta_{A_x} & & \eta_{A_{x'}}\downarrow\simeq \\ D_{\mathrm{st}}(A_x)\otimes_{L_0} L^{\log} & \underset{\simeq}{\rightarrow} M_0\otimes_W L^{\log} \underset{\simeq}{\leftarrow} & D_{\mathrm{st}}(A_{x'})\otimes_{L_0} L^{\log}. \end{array}$$

Here $L^{\log}$ is as described in (2.2.1.1) and the vertical isomorphisms are the Coleman-Iovita isomorphisms (cf. 2.3.1.2).

Proof. We have the log 1-motif $[Y \xrightarrow{\iota} J^{\log}]$ over R; $\mathbb{D}(A)$ will just be the log F-crystal associated with this log 1-motif by the theory of (1.3). The compatibility in (2) under pull-back follows from (1.3.3.3).

(3) follows immediately from the pull-back compatibility in (2) and the crystalline nature of $\mathbb{D}(A)$: Note that, with the log structure induced from S_L, $W(l)$ isomorphic to the log W-algebra $W(l)_{\mathbb{N}}$ (cf. 1.2.1.8). This is a formal divided power thickening of $l_{\mathbb{N}}$, viewed as the residue field of S_L endowed with its induced log structure. Let $x'_0 : \mathscr{O}_L \to l_{\mathbb{N}}$ be the natural surjection of log W-algebras; then $\mathbb{D}(A_x)(S_L)\otimes_{S_L} W$ is canonically isomorphic to $((x'_0)^*\mathbb{D}(A_x))(W(l)_{\mathbb{N}})$. This in turn is identified with $((x'_0\circ x)^*\mathbb{D}(A))(W(l)_{\mathbb{N}})$. Now the map $x'_0\circ x : R \to l_{\mathbb{N}}$ factors through $x_0 : R \to k_P$, so we can naturally identify this last $W(l)$-module with $((x_0\circ y_0)^*\mathbb{D}(A))(W(l)_{\mathbb{N}})$, for the map $y_0 : k_P \to l_{\mathbb{N}}$ such that $x'_0\circ x = x_0\circ y_0$, and such that the induced map of fields $k \to l$ is simply the natural inclusion. We can lift y_0 to a map $y : W_P \to W(l)_{\mathbb{N}}$ inducing the canonical map $W \hookrightarrow W(l)$ lifting $k \hookrightarrow l$. Then $((x_0\circ y_0)^*\mathbb{D}(A))(W(l)_{\mathbb{N}})$ is canonically isomorphic to $y^*M_0 = M_0\otimes_W W(l)$. The second isomorphism in (3) follows via the isomorphism β_{st} in (2.2.6.1).

(4) follows from (2) and (2.3.2.3)(1).

For (5), let R^{an} be the ring of functions of the rigid analytic open polydisk $(\mathrm{Spf}\ R)^{\mathrm{an}}$: this has a natural log structure $\mathrm{M}_{R^{\mathrm{an}}}$. Set

$$R^{\mathrm{an,log}} = \frac{R^{\mathrm{an}}\left[l_\alpha : \alpha\in \mathrm{M}^{\mathrm{gp}}_{R^{\mathrm{an}}}\right]}{\left(l_{\alpha\beta}-l_\alpha-l_\beta, \text{ for } \alpha,\beta\in \mathrm{M}^{\mathrm{gp}}_{R^{\mathrm{an}}}; l_\alpha = \log(\alpha), \text{ for } \alpha\in (R^{\mathrm{an}})^\times \text{ and } |\alpha-1|<1.\right)}.$$

By $|\alpha-1|<1$ we mean that, for every point y in the rigid analytic open polydisk $(\mathrm{Spf}\ R)^{\mathrm{an}}$, $|\alpha(y)-1|<1$. We can equip $R^{\mathrm{an,log}}$ with a logarithmic connection $\nabla : l_\alpha \mapsto -1\otimes \mathrm{dlog}(\alpha)$. Set

$$\Psi(A) = \left(\mathbb{D}(A)(R)\otimes_R R^{\mathrm{an,log}}\right)^{\nabla=0}.$$

By [Vol03, Lemma 8], $\Psi(A)$ has a canonical structure of a finite-dimensional φ-module over K_0, and by [Vol03, Theorem 9], the inclusion $\Psi(A)\hookrightarrow \mathbb{D}(A)(R)\otimes_R R^{\mathrm{an,log}}$ induces a

∇-equivariant isomorphism:

$$\Psi(A) \otimes_{K_0} R^{\mathrm{an,log}} \xrightarrow{\simeq} \mathbb{D}(A)(R) \otimes_R R^{\mathrm{an,log}}.$$

Any continuous map $x : R \to \mathscr{O}_L$ of log W-algebras can naturally be lifted to a map $x : R^{\mathrm{an,log}} \to L^{\log}$, and we get a natural isomorphism

$$\epsilon_x : \Psi(A) \otimes_{K_0} L^{\log} \xrightarrow{\simeq} \mathbb{D}(A)(R) \otimes_{R,x} L^{\log} \xrightarrow{\simeq} H^1_{\mathrm{dR}}(A_x) \otimes_L L^{\log}.$$

If we now have $x, x' : R \to \mathscr{O}_L$, then we define $\eta_{x,x'}$ so that the following diagram commutes:

$$\begin{array}{ccc} \Psi(A) \otimes_{K_0} L^{\log} & \xrightarrow{\epsilon_x} & H^1_{\mathrm{dR}}(A_x) \otimes_L L^{\log} \\ & \searrow^{\epsilon_{x'}} & \downarrow{\scriptstyle \eta_{x,x'}} \\ & & H^1_{\mathrm{dR}}(A_{x'}) \otimes_L L^{\log}. \end{array}$$

Let $\Lambda_x = H^1(A_{x,\overline{K}_0}, \mathbb{Z}_p)$ and let $\Psi(A_x) = \Psi(\Lambda_x)$ be the L_0-module associated with Λ_x in (2.2.6). Then $\Psi(A_x)$ is φ-equivariantly identified with $\Psi(A) \otimes_{K_0} L_0$, and the isomorphism ϵ_x is simply the composition

$$\Psi(A_x) \otimes_{L_0} L^{\log} \xrightarrow[\beta_{\mathrm{dR}}]{\simeq} D_{\mathrm{dR}}(A_x) \otimes_L L^{\log} \xrightarrow[\eta_{Ax}^{-1}]{\simeq} H^1_{\mathrm{dR}}(A_x) \otimes_L L^{\log},$$

where η_{A_x} is the Coleman-Iovita isomorphism from (2.3.1.2). In order to show that the diagram in (5) commutes, it suffices to show that the corresponding diagrams for the associated graded pieces of the weight filtrations commute. For gr^W_0, the diagram looks like

$$\begin{array}{ccc} \mathrm{Hom}(Y, L^{\log}) & = & \mathrm{Hom}(Y, L^{\log}) \\ \| & & \| \\ \mathrm{Hom}(Y, L^{\log}) & = & \mathrm{Hom}(Y, L^{\log}). \end{array}$$

For gr^W_2, it looks like

$$\begin{array}{ccc} X \otimes L^{\log} & = & X \otimes L^{\log} \\ \| & & \| \\ X \otimes L^{\log} & = & X \otimes L^{\log}. \end{array}$$

For gr_1^W, before tensoring with $L^{\log}$, and using [CI99, II.7.12], it looks like

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(B_x) & \xrightarrow{\simeq} & H^1_{\mathrm{dR}}(B_{x'}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1_{\mathrm{cris}}(B_{x_0}) \otimes_W L & = & H^1_{\mathrm{cris}}(B_{x_0}) \otimes_W L, \end{array}$$

where the vertical isomorphisms are the Berthelot-Ogus isomorphisms (cf. [BO83]). The first two diagrams obviously commute, and the third commutes by [BO83, Remark 2.9]; see also the proof of [Kis10, 2.3.5]. □

CHAPTER 3
LOCAL MODELS AT THE BOUNDARY

3.1 Deformations of log 1-motives

The aim of this section is to construct deformation rings for certain log 1-motives over perfect fields. The construction closely follows that of the local models in [FC90, Ch. IV], and in fact allows us to view the complete local rings at closed points of these local models as deformation rings of log 1-motives.

3.1.1

We will now study the deformation theory of log 1-motives. Let k be a perfect field of characteristic $p > 0$ and let $W = W(k)$ be its ring of Witt vectors. Let P be a sharp, fs monoid (cf. 1.2.1.2), and let k_P be the associated log ring as in (1.2.1.8). Suppose that we have a polarized log 1-motif (Q_0, λ_0) over k_P. Let $(B_0, Y, X, c_0, c_0^\vee, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau_0)$ be the tuple corresponding to (Q_0, λ_0) via (1.2.2.8). Then $Q_0 = [Y \xrightarrow{\iota_0} J_0^{\log}]$, where J_0 is a semi-abelian variety over k that is the extension

$$0 \to T_0 \to J_0 \to B_0 \to 0$$

of B_0 by the torus T_0 with character group X classified by $c_0^\vee$. We will assume that Y and X are constant, and we will also suppose that λ_0 is a prime-to-p polarization (cf. 1.2.2.7) of degree r.

Definition 3.1.1.1. Let C be a complete local log W-algebra with maximal ideal $\mathfrak{m}_C$ and residue field $k(C)$. A **deformation over** C of $(B_0, \lambda_0^{\mathrm{ab}})$ is a tuple $\left((B_C, \lambda_C^{\mathrm{ab}}), i_C\right)$ where:

1. (B_C, λ_C) is a polarized abelian scheme over C.
2. $i_C : (B_C, \lambda_C) \otimes_C k(C) \xrightarrow{\simeq} B_0 \otimes_k k(C)$ is an isomorphism of polarized abelian varieties over $k(C)$.

The category of deformations over C of $(B_0, \lambda_0^{\mathrm{ab}})$ will be denoted $\mathrm{Def}_{(B_0, \lambda_0^{\mathrm{ab}})}(C)$.

Definition 3.1.1.2. A **deformation over** C of $(B_0, \lambda_0^{\mathrm{ab}}, c_0, c_0^\vee)$ is a tuple $(B_C, \lambda_C^{\mathrm{ab}}, c, c^\vee, i_C)$ where:

1. (B_C, i_C) is a deformation over C of (B_0, λ_0).

2. $c : Y \to B_C$ and $c^\vee : X \to B_C^\vee$ are homomorphisms such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{c} & B_C \\ {\scriptstyle \lambda^{\text{ét}}}\downarrow & & \downarrow{\scriptstyle \lambda_C^{\text{ab}}} \\ X & \xrightarrow[c^\vee]{} & B_C^\vee \end{array}$$

commutes.

3. The diagrams

$$\begin{array}{ccc} Y & \xrightarrow{c} & B_C \otimes_C k(C) \\ & {\scriptstyle c_0}\searrow & \downarrow{\scriptstyle i_C} \\ & & B_0 \otimes_k k(C) \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{c^\vee} & B_C^\vee \otimes_C k(C) \\ & {\scriptstyle c_0^\vee}\searrow & \uparrow{\scriptstyle i_C^\vee} \\ & & B_0^\vee \otimes_k k(C) \end{array}$$

commute.

The category of deformations over C of $(B_0, \lambda_0^{\text{ab}}, c_0, c_0^\vee)$ will be denoted $\text{Def}_{(B_0, \lambda_0^{\text{ab}}, c_0, c_0^\vee)}(C)$.

Definition 3.1.1.3. Let C be a complete local log W-algebra with maximal ideal $\mathfrak{m}_C$ and residue field $k(C)$. A **deformation over** C of J_0 is a tuple (J_C, i_C) where:

1. J_C is a semi-abelian scheme over C.
2. $i_C : J_C \otimes_C k(C) \xrightarrow{\simeq} J_0 \otimes_k k(C)$ is an isomorphism of semi-abelian varieties over $k(C)$.

A **deformation over** C of $(J_0, \lambda_0^{\text{ab}})$ is a tuple $\left((J_C, \lambda_C^{\text{ab}}), i_C\right)$ where:

1. (J_C, i_C) is a deformation over C of J_0.

2. Let B_C be the maximal abelian scheme quotient of J_C; then $(B_C, \lambda_C^{\mathrm{ab}})$ is a deformation over C of $(B_0, \lambda_0^{\mathrm{ab}})$.

3. The diagram

$$\begin{array}{ccc} B_C \otimes_C k(C) & \underset{\simeq}{\xrightarrow{i_C}} & B_0 \otimes_k k(C) \\ \Big\downarrow{\scriptstyle \lambda_C^{\mathrm{ab}} \otimes 1} & & \Big\downarrow{\scriptstyle \lambda_0^{\mathrm{ab}} \otimes 1} \\ B_C^\vee \otimes_C k(C) & \underset{i_C^\vee}{\xleftarrow{\simeq}} & B_0^\vee \otimes_k k(C) \end{array}$$

commutes.

The category of deformations over C of $(J_0, \lambda_0^{\mathrm{ab}})$ will be denoted $\mathrm{Def}_{(J_0, \lambda_0^{\mathrm{ab}})}(C)$.

Remark 3.1.1.4. In the definition above, we are implicitly using the following fact already used in (1.1.3): every deformation of J_0 over C is an extension

$$0 \to T_C \to J_C \to B_C \to 0,$$

where B_C is an abelian scheme over C reducing to $B_0 \otimes_k k(C)$ along the isomorphism i_C, and T_C is the split torus over C with character group X.

Lemma 3.1.1.5. *There is a canonical equivalence of categories*

$$\mathrm{Def}_{(B_0, \lambda_0^{\mathrm{ab}}, c_0, c_0^\vee)}(C) \xrightarrow{\simeq} \mathrm{Def}_{(J_0, \lambda_0^{\mathrm{ab}})}(C).$$

Proof. If $(B_C, \lambda_C^{\mathrm{ab}}, c, c^\vee, i_C)$ is an object on the left hand side, then the classifying map $c^\vee : X \to B_C^\vee$ gives us a semi-abelian scheme J_C over C and thereby an object $(J_C, \lambda_C^{\mathrm{ab}}, i_C)$ on the right. Suppose we have an object $(J_C, \lambda_C^{\mathrm{ab}}, i_C)$ on the right. Then we can consider its classifying map $c^\vee : X \to B_C^\vee$, and the composition $c^\vee \circ \lambda^{\text{ét}} : Y \to B_C^\vee$. The question now is if this composition factors through $\lambda_C^{\mathrm{ab}} : B_C \to B_C^\vee$.

Since $\ker \lambda_C^{\mathrm{ab}}$ is a prime-to-p torsion group, and in particular, étale, the map $\ker \lambda_C^{\mathrm{ab}}(C) \to \ker \lambda_0^{\mathrm{ab}}(k(C))$ is a bijection. Similarly, the map

$$H^1(C, \ker \lambda_C^{\mathrm{ab}}) \to H^1(k(C), \ker \lambda_0^{\mathrm{ab}})$$

is also a bijection.

Considering the following diagram:

$$\begin{array}{ccccccc}
\ker \lambda_C^{\mathrm{ab}}(C) & \hookrightarrow & B_C(C) & \xrightarrow{\lambda_C^{\mathrm{ab}}} & B_C^\vee(C) & \twoheadrightarrow & H^1(C, \ker \lambda_C^{\mathrm{ab}}) \\
\| & & \downarrow & & \downarrow & & \| \\
\ker \lambda_0^{\mathrm{ab}}(k(C)) & \hookrightarrow & B_0(k(C)) & \xrightarrow[\lambda_0^{\mathrm{ab}}]{} & B_0^\vee(k(C)) & \twoheadrightarrow & H^1(k(C), \ker \lambda_0^{\mathrm{ab}}),
\end{array}$$

we see that, for any $y \in B_0(k(C))$, the fibers over y and $\lambda_0^{\mathrm{ab}}(y)$ of the vertical arrows are isomorphic. This tells us that $c^\vee \circ \lambda^{\text{ét}}$ must factor through λ_C^{ab}. □

Definition 3.1.1.6. Let C be a complete local log W-algebra with an fs log structure and with maximal ideal $\mathfrak{m}_C$. We will equip the residue field $k(C) = C/\mathfrak{m}_C$ with its induced log structure. A **deformation over C of** (Q_0, λ_0) is a tuple $((Q_C, \lambda_C), j_C, i_C)$ where:

1. (Q_C, λ_C) is a polarized log 1-motif over C, with $Q_C = [Y \xrightarrow{\iota_C} J_C^{\log}]$.
2. $j_C : k_P \to k(C)$ is a map of log W-algebras.
3. $i_C : (Q_C, \lambda_C) \otimes_C k(C) \xrightarrow{\simeq} j_C^*(Q_0, \lambda_0)$ is an isomorphism of polarized log 1-motifs.

We will denote by $\mathrm{Def}_{(Q_0,\lambda_0)}(C)$ the category of deformations over C of (Q_0, λ_0).

We have natural functors

$$\mathrm{Def}_{(Q_0,\lambda_0)}(C) \to \mathrm{Def}_{(J_0,\lambda_0^{\mathrm{ab}})}(C) \to \mathrm{Def}_{(B_0,\lambda_0^{\mathrm{ab}})}(C)$$

Remark 3.1.1.7. All these deformation problems are **rigid**. More precisely, all the categories above are groupoids and the automorphism group of any of their objects is trivial. Indeed, for an object in $\mathrm{Def}_{(B_0,\lambda_0^{\mathrm{ab}})}(C)$, this is a consequence of [Kat81, 1.1.3]. The statement immediately follows for objects in $\mathrm{Def}_{(J_0,\lambda_0^{\mathrm{ab}})}(C)$ and $\mathrm{Def}_{(Q_0,\lambda_0)}(C)$: For the former, by (3.1.1.5), it is equivalent to the statement that deformations over C of the tuple $(B_0, \lambda_0^{\mathrm{ab}}, c_0, c_0^\vee)$ have no non-trivial automorphisms, and this is clear. For the latter, we are saying that deformations of $([Y \xrightarrow{\iota_0} J_0^{\log}], \lambda_0)$ over C do not admit automorphisms: this is again clear, since neither Y nor any deformation (J_C, i_C) of J_0 over C admits any non-trivial automorphisms reducing to the identity over k.

Let $\mathrm{Art}_W^{\log}$ be the category of Artin local log W-algebras; we obtain a tower of rigid deformation functors from $\mathrm{Art}_W^{\log}$ into sets:

$$\begin{array}{c} \mathrm{Def}_{(Q_0,\lambda_0)} \\ \downarrow \\ \mathrm{Def}_{(J_0,\lambda_0^{\mathrm{ab}})} \\ \downarrow \\ \mathrm{Def}_{(B_0,\lambda_0^{\mathrm{ab}})} \end{array}$$

It is the rigidity of the deformation problems that makes it permissible to view them as functors into the category of sets. We can now expect (pro-)representability results for the tower. This expectation will be realized in a precise manner under additional hypotheses in the next sub-section.

3.1.2

Consider $\mathrm{Def}_{(B_0,\lambda_0^{\mathrm{ab}})}$: by classical results, this is pro-represented by a formally smooth ring R^{ab} equipped with a universal deformation $(B_{\mathrm{univ}}, \lambda_{\mathrm{univ}}^{\mathrm{ab}}, i_{R^{\mathrm{ab}}})$ of (B_0, λ_0).

Next, consider $\mathrm{Def}_{(J_0,\lambda_0^{\mathrm{ab}})}$: this is represented over R^{ab} by the Hom-scheme

$$\underline{\mathrm{Hom}}^{c_0^\vee}(X, B_{\mathrm{univ}}^\vee) = \{c^\vee : X \to B_{\mathrm{univ}}^\vee : \text{ the reduction of } c^\vee \text{ is } c_0^\vee : X \to B_0^\vee\}.$$

This is evidently a torsor over the formal group of the abelian scheme $\underline{\mathrm{Hom}}(X, B_{\mathrm{univ}}^\vee)$ over R^{ab}. It is therefore relatively pro-representable over R^{ab} by a formally smooth R^{ab}-algebra R^{sab}.

Finally, we would like to consider $\mathrm{Def}_{(Q_0,\lambda_0)}$ as a deformation problem over R^{sab}. For this we need some preparation. Over R^{sab}, we have the universal pair $(J_{\mathrm{univ}}, \lambda_{\mathrm{univ}}^{\mathrm{ab}})$. By (3.1.1.5), we have maps $c_{\mathrm{univ}} : Y \to B_{\mathrm{univ}}$ and $c_{\mathrm{univ}}^\vee : X \to B_{\mathrm{univ}}^\vee$ over R^{sab}, the latter of which classifies J_{univ}. Over $B_{\mathrm{univ}} \times_{R^{\mathrm{sab}}} B_{\mathrm{univ}}^\vee$ we have the Poincaré bundle $\mathcal{P}_{B_{\mathrm{univ}}}$. The map

$$c_{\mathrm{univ}} \times c_{\mathrm{univ}}^\vee : Y \times X \to B_{\mathrm{univ}} \times_{R^{\mathrm{sab}}} B_{\mathrm{univ}}^\vee$$

allows us to pull $\mathcal{P}_{B_{\mathrm{univ}}}^{-1}$ back to the line bundle $I = (c_{\mathrm{univ}} \times c_{\mathrm{univ}}^\vee)^* \mathcal{P}_{B_{\mathrm{univ}}}^{-1}$ over $Y \times X$. I has the structure of a $\mathbb{G}_m$-bi-extension over $Y \times X$ (cf. discussion above (1.2.2.3)), and $(1 \times \lambda^{\text{ét}})^* I$ has the structure of a symmetric $\mathbb{G}_m$-bi-extension over $Y \times Y$ (cf. discussion above (1.2.2.8)).

Set

$$\mathbf{B}_{\lambda^{\text{ét}}} = \{\text{Pairings } \langle\ ,\ \rangle : Y \times X \to \mathbb{Z} : \text{ such that } \langle y, \lambda^{\text{ét}}(y')\rangle = \langle y, \lambda^{\text{ét}}(y')\rangle, \text{ for all } y, y' \in Y\}.$$

Let $\mathbf{S}_{\lambda^{\text{ét}}} = \mathbf{B}_{\lambda^{\text{ét}}}^{\vee}$ be its dual abelian group: this is a certain quotient of the tensor product $Y \otimes X$. Let $\mathbf{E}_{\lambda^{\text{ét}}}$ be the split torus over R^{sab} with character group $\mathbf{S}_{\lambda^{\text{ét}}}$.

Consider the functor $\mathbf{\Xi}_{\lambda^{\text{ét}}}$ that assigns to each R^{sab}-algebra C, the set

$$\mathbf{\Xi}_{\lambda^{\text{ét}}}(C) = \begin{pmatrix} \text{Trivializations } \tau : \mathbf{1}_{Y\times X} \xrightarrow{\simeq} I \text{ over } C \\ \text{of } \mathbb{G}_m\text{-bi-extensions of } Y \times X \\ \text{inducing a symmetric trivialization of} \\ \text{the symmetric } \mathbb{G}_m\text{-bi-extension } (1 \times \lambda^{\text{ét}})^* I \text{ of } Y \times Y. \end{pmatrix}$$

One checks that this is a torsor over $\mathbf{E}_{\lambda^{\text{ét}}}$, and is therefore representable by an R^{sab}-scheme, which we will again denote by $\mathbf{\Xi}_{\lambda^{\text{ét}}}$. In fact, it is the $\mathbf{E}_{\lambda^{\text{ét}}}$-torsor that assigns to every element

$$\sum_i [y_i \otimes x_i] \in \mathbf{S}_{\lambda^{\text{ét}}}$$

the $\mathbb{G}_m$-torsor

$$\otimes_i (c(y_i) \times c^{\vee}(x_i))^* \mathcal{P}_{B_{\text{univ}}}^{-1}.$$

By the very definition of $\mathbf{\Xi}_{\lambda^{\text{ét}}}$, there is a canonical trivialization over $\mathbf{\Xi}_{\lambda^{\text{ét}}}$ of $\mathbb{G}_m$-bi-extensions

$$\tau_{\text{univ}} : \mathbf{1}_{Y\times X} \xrightarrow{\simeq} I$$

of $Y \times X$.

Let $N_0 = N_{Q_0} : Y \to \text{Hom}(X, P^{\text{gp}})$ be the monodromy map associated with Q_0 (cf. 1.2.2.2). Viewing N_0 as a pairing $Y \times X \to P^{\text{gp}}$, we see that it must satisfy the identity

$$N_0(y, \lambda^{\text{ét}}(y')) = N_0(y', \lambda^{\text{ét}}(y))$$

for all $y, y' \in Y$. This follows, for example, from the description of N_0 in terms of the trivializations $\tau(y, x)$ in (1.2.2.3)(4) and from the symmetry condition in (1.2.2.8) satisfied by these trivializations. In particular, N_0 determines an element of $\mathbf{B}_{\lambda^{\text{ét}}} \otimes P^{\text{gp}}$, and thus a map $\mathbf{S}_{\lambda^{\text{ét}}} \to P^{\text{gp}}$.

To proceed further, we need to make an additional

Assumption 3.1.2.1. The polarized log 1-motif (Q_0, λ_0) is *positive*; that is

$$N_0(y, \lambda_0^{\text{ét}}(y)) \in P \setminus \{1\},$$

for all $y \in Y$ (cf. 1.2.4.1).

For any map $f : P \to \mathbb{N}$ of monoids, let $\langle\ ,\ \rangle_{N_0,f}$ be the pairing on $Y \times X$ given by

$$\langle y, x\rangle_{N_0,f} = f(N_0(y, x)).$$

Define a sub-monoid

$$\sigma = \left\{ \begin{array}{c} \langle\ ,\ \rangle \in \mathbf{B}_{\lambda^{\text{ét}}} \otimes \mathbb{R} :\ \text{such that for } y \in Y,\ x \in X \\ \langle y, x\rangle > 0 \text{ whenever } \langle y, x\rangle_{N_0,f} > 0 \text{ for all continuous maps } f : P \to \mathbb{N}. \end{array} \right\}.$$

Set

$$\mathbf{C}_{\lambda^{\text{ét}}} = \left\{ \begin{array}{l} \langle\ ,\ \rangle \in \mathbf{B}_{\lambda^{\text{ét}}} \otimes \mathbb{R} :\ \text{such that } (y, y') \mapsto \langle y, y'\rangle \\ \text{induces a positive definite pairing on } Y \otimes \mathbb{R} \end{array} \right\}. \tag{3.1.2.1.1}$$

This is an open convex cone inside $\mathbf{B}_{\lambda^{\text{ét}}} \otimes \mathbb{R}$.

Lemma 3.1.2.2. *We have $\sigma \subset \mathbf{C}_{\lambda^{\text{ét}}}$. Moreover, σ is a non-degenerate, rational polyhedral cone: that is, it is finitely generated as a monoid by elements of $\mathbf{B}_{\lambda^{\text{ét}}} \otimes \mathbb{Q}$, and it does not contain any lines; cf.* [AMRT10, Ch. I].

Proof. For any $f : P \to \mathbb{N}$ and any $y \in Y$, by assumption (3.1.2.1), we have

$$\langle y, \lambda^{\text{ét}}(y)\rangle_{N_0,f} > 0.$$

So, if $\langle\ ,\ \rangle$ lies in σ, then $\langle y, \lambda^{\text{ét}}(y)\rangle > 0$, for all $y \in Y$; this means that σ consists of positive definite pairings on $Y \otimes \mathbb{R}$. It is finitely generated by elements of $\mathbf{B}_{\lambda^{\text{ét}}}$ by Gordan's lemma (cf. [KKMSD73, p. 7]), and it is non-degenerate simply because it is never the case that both a form and its negative are positive definite. □

Let $\mathbf{S}_{\lambda^{\text{ét}},\sigma}$ be the monoid $\sigma^\vee \cap \mathbf{S}_{\lambda^{\text{ét}}}$, where

$$\sigma^\vee = \{n \in \mathbf{S}_{\lambda^{\text{ét}}} \otimes \mathbb{R} : \langle n, s\rangle \geq 0,\ \text{for all } s \in \sigma\}.$$

Explicitly, we have

$$\mathbf{S}_{\lambda^{\text{ét}},\sigma} = \{n \in \mathbf{S}_{\lambda^{\text{ét}}} : f(N_0(n)) \geq 0,\ \text{for all continuous } f : P \to \mathbb{N}\}.$$

Lemma 3.1.2.3. *The map N_0 restricts to a* continuous *map of monoids $N_0 : \mathbf{S}_{\lambda^{\text{ét}},\sigma} \to P$, and in particular induces an embedding*

$$\mathbf{S}_{\lambda^{\text{ét}},\sigma}/\mathbf{S}^\times_{\lambda^{\text{ét}},\sigma} \hookrightarrow P.$$

Proof. This is clear. □

Let $\mathbf{E}_{\lambda^{\text{ét}},\sigma} = \operatorname{Spec} R^{\text{sab}}[\mathbf{S}_{\lambda^{\text{ét}},\sigma}]$, so that we have a toric embedding $\mathbf{E}_{\lambda^{\text{ét}}} \hookrightarrow \mathbf{E}_{\lambda^{\text{ét}},\sigma}$ over R^{sab}; and let $\mathbf{\Xi}_{\lambda^{\text{ét}},\sigma}$ be the contraction product $\mathbf{\Xi}_{\lambda^{\text{ét}}} \times^{\mathbf{E}_{\lambda^{\text{ét}}}} \mathbf{E}_{\lambda^{\text{ét}},\sigma}$: it is a log scheme over R^{sab} in an evident way with the log structure induced by the divisor that is the complement of $\mathbf{\Xi}_{\lambda^{\text{ét}}}$.

The $\mathbb{G}_m$-bi-extension I of $Y \times X$ over $\mathbf{\Xi}_{\lambda^{\text{ét}}}$ extends to a $\mathbb{G}_m^{\log}$-bi-extension $I^{\log}$ of $Y \times X$ over $\mathbf{\Xi}_{\lambda^{\text{ét}},\sigma}$, and the trivialization τ_{univ} extends to a trivialization of $I^{\log}$ over $\mathbf{\Xi}_{\lambda^{\text{ét}},\sigma}$. Thus, by (1.2.2.4), τ_{univ} gives rise to a log 1-motif $Q_\sigma = [Y \xrightarrow{\iota_\sigma} J^{\log}_{\text{univ}}]$ over $\mathbf{\Xi}_{\lambda^{\text{ét}},\sigma}$. Q_σ is naturally equipped with a prime-to-p polarization λ_σ by (1.2.2.8), since τ_{univ} satisfies the symmetry condition in (1.2.2.8). $\mathbf{\Xi}_{\lambda^{\text{ét}},\sigma}$ has a natural stratification arising from the stratification of $\mathbf{E}_{\lambda^{\text{ét}},\sigma}$ by $\mathbf{E}_{\lambda^{\text{ét}}}$-orbits. There is a unique closed stratum corresponding to the unique closed orbit in $\mathbf{E}_{\lambda^{\text{ét}},\sigma}$.

Lemma 3.1.2.4. *Let (H, M_H) be an fs log scheme over R^{sab}, and let $(J_H, \lambda_H^{\text{ab}})$ be the pair over H obtained by pull-back from the universal pair $(J_{\text{univ}}, \lambda^{\text{ab}}_{\text{univ}})$ over R^{sab}. Then giving a map $H \to \Xi_{\lambda^{\text{ét}},\sigma}$ of log schemes over R^{sab} is equivalent to giving a polarized log 1-motif (Q_H, λ_H), with $Q_H = [Y \xrightarrow{\iota_H} J_H^{\log}]$ such that:*

1. *$\lambda_H^{\text{ét}} = \lambda^{\text{ét}}$.*
2. *Let $N_H : Y \times X \to \mathbb{G}^{\log}_{m,H}/\mathbb{G}_{m,H}$ be the monodromy pairing, considered as a linear map*
$$N_H : \mathbf{S}_{\lambda^{\text{ét}}} \to \mathbb{G}^{\log}_{m,H}/\mathbb{G}_{m,H};$$
then, for all geometric points $\bar{x}$ of H, the image of the induced map
$$N_{H,\bar{x}} : \mathbf{S}_{\lambda^{\text{ét}}} \to \mathrm{M}^{\text{gp}}_{H,\bar{x}}/\mathscr{O}^\times_{H,\bar{x}}$$
restricts to a continuous map of monoids
$$N_{H,\bar{x}} : \mathbf{S}_{\lambda^{\text{ét}},\sigma} \to \mathrm{M}_{H,\bar{x}}/\mathscr{O}^\times_{H,\bar{x}}$$

Proof. Indeed, giving a map $H \to \Xi_{\lambda^{\text{ét}},\sigma}$ of log R^{sab}-schemes is equivalent to giving a trivialization τ_H of the $\mathbb{G}_m^{\log}$-bi-extension $I_H^{\log}$ of $Y \times X$ over H, satisfying the symmetry condition with respect to $\lambda^{\text{ét}}$ and λ^{ab} as in (1.2.2.8), and also satisfying the positivity condition expressed in condition (2) of the statement of the lemma. By (1.2.2.8), this is equivalent to giving a polarized log 1-motif (Q_H, λ_H) over H satisfying the conditions of the lemma. □

In particular, the log 1-motif (Q_0, λ_0) over k_P corresponds to a map $x_{0,\sigma} : \operatorname{Spec} k_P \to \Xi_{\lambda^{\text{ét}},\sigma}$. Let R_1 be the complete local ring of $\Xi_{\lambda^{\text{ét}},\sigma}$ at the point corresponding to $x_{0,\sigma}$, and equip it with the induced log structure. Over R_1, we have the polarized log 1-motif obtained via pull-back of $(Q_\sigma, \lambda_\sigma)$ from $\Xi_{\lambda^{\text{ét}},\sigma}$; we will denote this pull-back also by $(Q_\sigma, \lambda_\sigma)$. Let $k(R_1)$ be the residue field of R_1 equipped with the induced log structure, and let $(Q_{\sigma,0}, \lambda_{\sigma,0})$ be the reduction of $(Q_\sigma, \lambda_\sigma)$ to $k(R_1)$. We have maps of log R^{sab}-algebras

$$R_1 \to k(R_1) \xrightarrow{j_{R_1}} k_P,$$

and an identification

$$j_{R_1}^*\left((Q_{\sigma,0},\lambda_{\sigma,0})\right) = (Q_0,\lambda_0).$$

Just as in (3.1.1.6), we can consider the deformation problem $\mathrm{Def}_{(Q_{0,\sigma},\lambda_{0,\sigma})}$ for the log 1-motif $(Q_{0,\sigma},\lambda_{0,\sigma})$ over $k(R_1)$: For any log W-algebra C, $\mathrm{Def}_{(Q_{0,\sigma},\lambda_{0,\sigma})}(C)$ is the category of tuples $((Q_C,\lambda_C),j_C,i_C)$ where:

1. (Q_C,λ_C) is a polarized log 1-motif over C, with $Q_C = [Y \xrightarrow{\iota_C} J_C^{\log}]$.
2. $j_C : k(R_1) \to k(C)$ is a map of log W-algebras.
3. $i_C : (Q_C,\lambda_C)\otimes_C k(C) \xrightarrow{\simeq} j_C^*(Q_{0,\sigma},\lambda_{0,\sigma})$ is an isomorphism of polarized log 1-motifs.

We have the diagram:

$$\begin{array}{ccc} \mathrm{Def}_{(Q_0,\lambda_0)} & \longrightarrow & \mathrm{Def}_{(Q_{0,\sigma},\lambda_{0,\sigma})} \\ & \searrow & \downarrow \\ & & \mathrm{Def}_{(J_0,\lambda_0^{\mathrm{ab}})} = \mathrm{Spf}\, R^{\mathrm{sab}}. \end{array}$$

Corollary 3.1.2.5. *Let the notation be as above. Then the triple $(R_1,Q_{0,\sigma},\lambda_{0,\sigma})$ pro-represents the deformation problem $\mathrm{Def}_{(Q_{0,\sigma},\lambda_{0,\sigma})}$.*

Proof. This is immediate from (3.1.2.4). □

3.1.3

We have now reduced to showing relative representability of the map

$$\mathrm{Def}_{(Q_0,\lambda_0)} \to \mathrm{Def}_{(Q_{0,\sigma},\lambda_{0,\sigma})}.$$

We will do this under certain restrictive hypotheses. First, for $N \in \mathbb{Z}_{>0}$, let $\mathbf{S}_{\lambda^{\text{ét}},N} = \frac{1}{N}\mathbf{S}_{\lambda^{\text{ét}}}$, and let $\mathbf{S}_{\lambda^{\text{ét}},N,\sigma} = \sigma^\vee \cap \mathbf{S}_{\lambda^{\text{ét}},N}$.

Assumption 3.1.3.1. There exist

- $N \in \mathbb{Z}_{>0}$ with $(N,pr) = 1$ (recall that r is the degree of λ_0);
- A free $\mathbb{Z}/N\mathbb{Z}$-module $\Lambda_{N,g}$ of rank $2g$ equipped with a symplectic pairing into $\mathbb{Z}/N\mathbb{Z}$;
- A free, isotropic $\mathbb{Z}/N\mathbb{Z}$-module $\Sigma_{N,g} \subset \Lambda_{N,g}$ such that the quotient $\Lambda_{N,g}/\Sigma_{N,g}$ is again free over $\mathbb{Z}/N\mathbb{Z}$;

such that (Q_0, λ_0) is equipped with a level N structure (cf. 1.2.3.3)

$$\alpha_N = (\varphi_{0,N}^{\mathrm{ab}}, \varphi_{0,N}^{\text{ét}}, \varphi_{0,N}^{\mathrm{mult}}, c_{0,N}, c_{0,N}^{\vee}, \tau_{0,N}, \delta).$$

of type $(\Lambda_{N,g}, \Sigma_{N,g})$.

Moreover, the map $N_0 : \mathbf{S}_{\lambda^{\text{ét}}} \to P^{\mathrm{gp}}$ induces an identification

$$\mathbf{S}_{\lambda^{\text{ét}},N,\sigma}/\mathbf{S}^{\times}_{\lambda^{\text{ét}},N,\sigma} = P.$$

With the notation of this assumption, set

$$\mathbf{S}_Q = \mathbf{S}_{\lambda^{\text{ét}},N}; \qquad \mathbf{S}_{Q,\sigma} = \mathbf{S}_{\lambda^{\text{ét}},N,\sigma}.$$

The map N_0 then induces an identification

$$\mathbf{S}_{Q,\sigma}/\mathbf{S}^{\times}_{Q,\sigma} = P.$$

Definition 3.1.3.2. For any complete log W-algebra C with residue field $k(C)$, $\mathrm{Def}_{(Q_0,\lambda_0,c_{0,N},\tau_{0,N})}(C)$ will be the set of isomorphism classes of tuples $((Q_C, \lambda_C, \alpha_{C,N}, j_C, i_C)$, where $((Q_C, \lambda_C), j_C, i_C)$ is a deformation over C of (Q_0, λ_0), and $\alpha_{C,N}$ is a level N structure on (Q_C, λ_C) of type $(\Lambda_{N,g}, \Sigma_{N,g})$ reducing to the level structure $j_C^*\alpha_{0,N}$ on the reduction of (Q_C, λ_C) to $k(C)$.

Lemma 3.1.3.3. *The natural 'forgetting level N structure' map*

$$\mathrm{Def}_{(Q_0,\lambda_0,\alpha_{0,N})} \to \mathrm{Def}_{(Q_0,\lambda_0)}$$

is an isomorphism of deformation problems.

Proof. We have to show that there is a unique lift of the level N structure $\alpha_{0,N}$ to any deformation of (Q_0, λ_0). As always, the key is that N is prime to p. So let us suppose that $((Q_C, \lambda_C), j_C, i_C)$ is a deformation over C of (Q_0, λ_0) corresponding to a tuple $(B_C, Y, X, c_C, c_C^{\vee}, \lambda_C^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau_C)$.

The tuple $(\varphi_{0,N}^{\mathrm{ab}}, \varphi_{0,N}^{\text{ét}}, \varphi_{0,N}^{\mathrm{mult}})$ consists of isomorphisms between finite flat groups schemes that are extensions of multiplicative groups by étale ones. In particular, it lifts uniquely over C.

Let us now lift $c_{0,N}$ to a map $c_{C,N} : \frac{1}{N}Y \to B_C$ restricting to c_C on Y. This follows from the following

Claim. If $b' \in B_C(C)$ and $b_0 \in B_0(k(C))$ are such that $[N]b_0$ is equal to the image of b' in $B_0(k(C))$, then there exists a unique $b \in B_C(C)$ such that $[N]b = b'$.

Indeed, the b such that $[N]b = b'$ form a torsor under the étale group scheme $B_C[N]$ of N-torsion points of B_C. This torsor is trivial if and only if the associated $B_0[N] \otimes k(C)$-torsor is trivial, and any trivialization of this latter torsor lifts uniquely to a trivialization of the $B_C[N]$-torsor.

Similarly, one can also lift $c_{0,N}^{\vee}$ over C.

It remains to lift $\tau_{0,N}$: Note that it makes sense, for $y \in \frac{1}{N}Y$ and $x \in X$, to ask for a lift of the trivialization $\tau_{0,N}(y,x)$ to a trivialization $\tau_{C,N}(y,x)$ of the $\mathbb{G}_m^{\log}$-bundle $I_{y,x}^{\log}$ over C. The argument for finding this lift is the same as that for the previous claim. The space of trivializations $\tau_{C,N}(y,x)$ such that $\tau_{C,N}(y,x)^{\otimes N} = \tau_C(Ny,x)$ is a torsor under

$$\ker(\mathbb{G}_m^{\log} \xrightarrow{\uparrow N} \mathbb{G}_m^{\log}) = \mu_N,$$

the étale group scheme of N^{th}-roots of unity. So a similar argument shows that there is a unique such trivialization lifting $\tau_{0,N}(x,y)$. □

Let $\mathbf{E}_Q$ be the torus with character group $\mathbf{S}_Q$, and let $\mathbf{E}_{Q,\sigma}$ be the toric embedding of $\mathbf{E}_Q$ associated with σ. From the proof of (3.1.3.3), it follows that over R^{sab} we have maps

$$c_{\text{univ},N} : \frac{1}{N}Y \to B_{\text{univ}}; \qquad c_{\text{univ},N}^{\vee} : \frac{1}{N}X \to B_{\text{univ}}^{\vee}$$

lifting $c_{0,N}$ and $c_{0,N}^{\vee}$.

Let I_N be the $\mathbb{G}_m$-bi-extension over $\frac{1}{N}Y \times X$ given by

$$I_N = (c_{\text{univ},N} \times c_{\text{univ}}^{\vee})^* \mathcal{P}_{B_{\text{univ}}}^{-1},$$

and let $\mathbf{\Xi}_Q$ be the $\mathbf{\Xi}_{\lambda^{\text{ét}}}$-scheme of trivializations $\tau_N : \mathbf{1}_{\frac{1}{N}Y \times X} \xrightarrow{\simeq} I_N$ lifting the tautological trivialization $\tau_{\text{univ}} : \mathbf{1}_{Y \times X} \xrightarrow{\simeq} I$ over $\mathbf{\Xi}_{\lambda^{\text{ét}}}$. It is a torsor under $\mathbf{E}_Q$. Set

$$\mathbf{\Xi}_{Q,\sigma} = \mathbf{\Xi}_Q \times^{\mathbf{E}_Q} \mathbf{E}_{Q,\sigma}.$$

Then there exists a polarized log 1-motif (Q,λ) over $\mathbf{\Xi}_{Q,\sigma}$ equipped with universal level N-structure of type $(\Lambda_{N,g}, \Sigma_{N,g,r})$ $\alpha_{\text{univ},N}$. There is a map $x_0 : \operatorname{Spec} k_P \to \mathbf{\Xi}_{Q,\sigma}$ of log W-algebras and an identification

$$i_R : x_0^* \left((Q, \lambda, \alpha_{\text{univ},N}) \right) = (Q_0, \lambda_0, \alpha_{0,N}).$$

Let R be the complete local ring of $\Xi_{Q,\sigma}$ at x_0 equipped with the induced log structure and the polarized log 1-motif (Q,λ) obtained via pull-back from $\Xi_{Q,\sigma}$.

Proposition 3.1.3.4. *The log R^{sab}-algebra R equipped with the tuple $((Q,\lambda), j_R, i_R)$ is the universal deformation ring pro-representing* $\mathrm{Def}_{(Q_0,\lambda_0)}$.

Proof. The proof is immediate from the construction, assumption (3.1.3.1), and (3.1.3.3) above. □

3.2 Explicit deformation rings for log 1-motifs

We begin as in §3.1 with a positive prime-to-p polarized log 1-motif (Q_0, λ_0) over a perfect field k of characteristic $p > 0$. In *loc. cit.*, we constructed, under certain assumptions, a deformation ring R for (Q_0, λ_0). We will continue to maintain these assumptions and also the notation used above, and we will make the additional assumption that k is *finite*.

Our goal in this section is to give an explicit description of R. We suggest that the reader first look at the constructions of explicit deformation rings for p-divisible groups due to Faltings as found in [Fal99, §7] or [Moo98, §4]. The main utility of an explicit construction, which is essentially group-theoretic, is to give us a hands-on construction of the Dieudonné log F-crystal associated with the universal deformation of (Q_0, λ_0). This in turn will enable us to easily work with 'Tate cycles' (more precisely, φ-invariant, parallel tensors) over the Dieudonné log F-crystals associated with deformations of our log 1-motif.

The idea is to first give an explicit model R^+ of R^{sab}: this essentially follows the aforementioned construction of Faltings, and is done in (3.2.3.4). Then we exhibit R as an explicit completed toric embedding over R^+ in (3.2.5) and give a concrete description of the log F-crystal associated with the universal deformation over R in (3.2.6.1).

The construction depends on certain choices of co-characters. In the interest of streamlining our presentation, we have opted to work with tensors right from the beginning, and to make our choices compatible with these tensors (cf. 3.2.3).

3.2.1

Let $\mathbb{D}(Q_0)$ be the log F-crystal over k_P associated with Q_0 by the theory of § 1.3. The polarization λ_0 gives rise to a symplectic Frobenius-equivariant pairing

$$\psi_0 : \mathbb{D}(Q_0) \times \mathbb{D}(Q_0) \to \mathbf{1}(1)$$

of log F-crystals over k_P (cf. [KT03, §4.7]).

Let $W = W(k)$ be the ring of Witt vectors over k, and let W_P be the associated log ring as in (1.2.1.8). Choose any map of log rings $W_P \twoheadrightarrow k_P$ inducing the identity on P and with underlying map of rings the canonical surjection $W \to k$. This will be a formal log PD thickening, and we can evaluate any log F-crystal over k on W_P along this surjection. Set $M_0 = \mathbb{D}(Q_0)(W_P)$; to endow this with the structure of a φ-module, we have to choose a Frobenius lift φ on W_P. Recall from (1.4.3) that this amounts to fixing a splitting

$$\beta : \mathrm{M}^{\mathrm{gp}}_{W_P} \xrightarrow{\simeq} \mathrm{M}^{\mathrm{gp}}_{k_P} \oplus (1 + pW).$$

Each choice of such lift gives us a map $\varphi_{M_0,\beta} : \varphi_W^* M_0 \to M_0$.

Set $M_0^{\text{ét}} = W_0 M_0 = \mathrm{Hom}(Y, W)$; $M_0^{\mathrm{sab}} = M_0/W_0 M_0 = \mathbb{D}(J_0)(W)$; $M_0^{\mathrm{ab}} = \mathrm{gr}_1^W M_0 = \mathbb{D}(B_0)(W)$; and $M_0^{\mathrm{mult}} = \mathrm{gr}_2^W M_0 = X \otimes W(1)$. All these modules have canonical φ-module structures, and we have φ-equivariant (for any choice of β) short exact sequences

$$0 \to M_0^{\text{ét}} \to M_0 \to M_0^{\mathrm{sab}} \to 0;$$

$$0 \to M_0^{\mathrm{ab}} \to M_0^{\mathrm{sab}} \to M_0^{\mathrm{mult}} \to 0.$$

Note that the φ-module structures on $M_0^{\text{ét}}$, M_0^{sab}, M_0^{ab} and M_0^{mult} do *not* depend on the splitting β.

The polarization λ on Q_0, via the pairing ψ on log F-crystals above, induces a perfect φ-equivariant (again, for any choice of β) pairing

$$\psi_0 : M_0 \otimes_W M_0 \to W(1).$$

By functoriality, the weight filtration $W_\bullet M_0$ is $\mathrm{GSp}(M_0, \psi_0)$-split. This just means that $M_0^{\text{ét}}$ is ψ_0-isotropic and its annihilator in M_0 is $W_1 M_0$. In particular, ψ_0 induces φ-equivariant perfect pairings

$$\psi_0^{\text{ét}} : M_0^{\text{ét}} \otimes M_0^{\mathrm{mult}} \to W(1);$$

$$\psi_0^{\mathrm{sab}} : M_0^{\mathrm{sab}} \otimes W_1 M_0 \to W(1);$$

$$\psi_0^{\mathrm{ab}} : M_0^{\mathrm{ab}} \otimes M_0^{\mathrm{ab}} \to W(1).$$

Let $P_{\mathrm{wt}} \subset \mathrm{GSp}(M_0, \psi_0)$ be the parabolic sub-group stabilizing $W_\bullet M_0$; $U_{\mathrm{wt}} \subset P_{\mathrm{wt}}$ its unipotent radical; and $U_{\mathrm{wt}}^{-2} \subset U_{\mathrm{wt}}$ the sub-group of elements that act trivially on M_0^{sab}. Since $W_1 M_0$ and M_0^{sab} are each identified with the dual of the other under ψ_0, we find that $U_{\mathrm{wt}}^{-2} \subset P_{\mathrm{wt}}$ is also the sub-group acting trivially on $M_0^{\mathrm{sab}} \oplus W_1 M_0$.

Let $\mathbf{B}_Q$ be the $\mathbb{Z}$-dual of $\mathbf{S}_Q$: it is naturally contained in $\mathbf{B}_{\lambda^{\text{ét}}}$, and we have a canonical identification

$$\mathbf{B}_Q \otimes \mathbb{Z}_{(p)} = \mathbf{B}_{\lambda^{\text{ét}}} \otimes \mathbb{Z}_{(p)}.$$

Lemma 3.2.1.1. *There is a canonical identification*

$$\mathbf{B}_Q \otimes_{\mathbb{Z}} W = \mathrm{Lie}\, U_{\mathrm{wt}}^{-2}$$

Proof. By functoriality, the pairing $\psi_0^{\text{ét}}$ is given by the formula

$$\psi_0^{\text{ét}} : \mathrm{Hom}(Y, W) \otimes (X \otimes W(1)) \to W(1) \tag{3.2.1.1.1}$$

$$\psi_0^{\text{ét}} \left(\varphi, (\lambda^{\text{ét}} \otimes 1)(y \otimes 1)\right) = \varphi(y). \tag{3.2.1.1.2}$$

Since $\lambda^{\text{ét}} \otimes 1 : Y \otimes W \to X \otimes W(1)$ is an isomorphism, this is a well-defined pairing. We now claim that we have a natural identification

$$\mathrm{Lie}\, U_{\mathrm{wt}}^{-2} = \left\{ \begin{array}{c} \text{Pairings } N : Y \times X \to W \text{ such that} \\ N(y, \lambda^{\text{ét}}(y')) = N(y', \lambda^{\text{ét}}(y)), \text{ for all } y, y' \in Y \end{array} \right\}.$$

In particular, we have an identification

$$\mathbf{B}_Q \otimes W = \mathbf{B}_{\lambda^{\text{ét}}} \otimes W = \mathrm{Lie}\, U_{\mathrm{wt}}^{-2}. \tag{3.2.1.1.3}$$

Let us prove our claim above about $\operatorname{Lie} U_{\mathrm{wt}}^{-2}$. By definition, it is the sub-space

$$\{N \in \operatorname{Lie}(\mathrm{GSp}(M_0, \psi_0)) : W_1 M_0 \subset \ker N; \operatorname{im} N \subset M_0^{\text{ét}}\}.$$

More explicitly, $\operatorname{Lie} U_{\mathrm{wt}}^{-2}$ consists of those maps $N : M_0^{\mathrm{mult}} \to M_0^{\text{ét}}$ such that the diagram

$$\begin{array}{ccc} M_0^{\mathrm{mult}} & \xrightarrow{\quad N \quad} & M_0^{\text{ét}} \\ \psi_0^{\text{ét}} \downarrow \simeq & & \simeq \downarrow \psi_0^{\text{ét}} \\ (M_0^{\text{ét}})^\vee(1) & \xrightarrow[N^\vee(1)]{} & (M_0^{\mathrm{mult}})^\vee(1) \end{array}$$

commutes. Here the vertical isomorphisms are the ones induced by the perfect pairing $\psi_0^{\text{ét}}$.

Using the identifications of $M_0^{\text{ét}}$ with $\operatorname{Hom}(Y, W)$, and M_0^{mult} with $X \otimes W(1)$, and the explicit formula (3.2.1.1.1) for $\psi_0^{\text{ét}}$, we see that $\operatorname{Lie} U_{\mathrm{wt}}^{-2}$ is the space of maps $N : X \otimes W \to \operatorname{Hom}(Y, W)$ such that the diagram

$$\begin{array}{ccc} X \otimes W & \xrightarrow{N} & \operatorname{Hom}(Y, W) \\ \lambda^{\text{ét}} \uparrow & & \uparrow (\lambda^{\text{ét}})^\vee \\ Y \otimes W & \xrightarrow[N^\vee]{} & \operatorname{Hom}(X, W) \end{array} \tag{3.2.1.1.4}$$

commutes. If we now think of an element N of $\operatorname{Lie} U_{\mathrm{wt}}^{-2}$ as a pairing $X \times Y \to W$ via the formula

$$\begin{aligned} N : Y \times X &\to W \\ (y, x) &\mapsto N(x \otimes 1)(y \otimes 1), \end{aligned}$$

then the commuting of the diagram (3.2.1.1.4) is equivalent to requiring that

$$N(y, \lambda^{\text{ét}}(y')) = N(y', \lambda^{\text{ét}}(y)),$$

for all $y, y' \in Y$. □

3.2.2

Now we introduce 'Tate cycles' into the picture. Since the polarized log 1-motif (Q, λ) over R is positive by construction, it corresponds to a polarized abelian scheme (A, λ) over the locus $U \subset \operatorname{Spec} R$ where the log structure is trivial by (1.2.4.2).

Suppose that we have a continuous map $x : R \to \mathscr{O}_K$ of log W-algebras, for $K \subset \overline{K}_0$ a finite extension of K_0 with residue field k. Let (A_x, λ_x) be the corresponding principally polarized semi-stable abelian variety over K. Let $\Lambda_x = H^1\left(A_{x,\overline{K}_0}, \mathbb{Z}_p\right)$; then the polarization λ_x induces a perfect Galois-equivariant Weil pairing

$$\psi : \Lambda_x \otimes_{\mathbb{Z}_p} \Lambda_x \to \mathbb{Z}_p(-1).$$

We will suppose also that we have Galois-invariant tensors $\{s_{\alpha,x,\text{ét}}\} \subset \Lambda_x^{\otimes}$ whose pointwise stabilizer is a reductive sub-group $G_{\mathbb{Z}_p} \subset \mathrm{GSp}(\Lambda_x, \psi_x)$. By (2.3.2.3) and (2.4.1.1)(3), the corresponding φ-invariant tensors $\{s_{\alpha,x,\text{st}}\} \subset \left(D_{\text{st}}(A_x)^{\otimes}\right)^{N=0}$ give rise to φ-invariant tensors $\{s_{\alpha,0}\} \subset M_0^{\otimes}$ defining a reductive group $G \subset \mathrm{GSp}(M_0, \psi_0)$.

Recall that we have identified $\mathbf{B}_Q \otimes W$ with $\mathrm{Lie}\, U_{\text{wt}}^{-2}$ in (3.2.1.1) above. Let $\mathbf{B}_{Q,G} = \mathbf{B}_Q \cap \mathrm{Lie}\, G$, and let $\mathbf{S}_{Q,G}$ be the quotient of $\mathbf{S}_Q$ that is dual to $\mathbf{B}_{Q,G}$. Let σ_G be the polyhedral cone $\sigma \cap (\mathbf{B}_{Q,G} \otimes \mathbb{R})$, and let $\mathbf{S}_{\sigma_G} \subset \mathbf{S}_{Q,G}$ be the corresponding sub-monoid.

Remark 3.2.2.1. Note that the monodromy at x, $N_x = N_{A_x}$, lies in σ_G by (2.3.1.2), which says that N_x agrees (up to a $\mathbb{Q}$-multiple) with Fontaine's monodromy operator on $D_{\text{st}}(A_x)$, and so kills the tensors $\{s_{\alpha,0}\}$.

Lemma 3.2.2.2. *The following is true in our situation:*

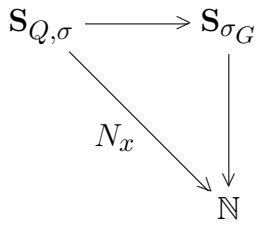
1. *The weight filtration $W_\bullet M_0$ is G-split.*
2. *Let $P_{\text{wt},G}, U_{\text{wt},G}, U_{\text{wt},G}^{-2}$ denote the intersections with G of the groups $P_{\text{wt}}, U_{\text{wt}}$ and U_{wt}^{-2}, respectively. Then $P_{\text{wt},G} \subset G$ is a parabolic sub-group and $U_{\text{wt},G}$ is its unipotent radical. The Hodge filtration $\mathrm{Fil}^1(M_0 \otimes k) \subset M_0 \otimes k$ is $P_{\text{wt},G} \otimes k$-split.*
3. *$\mathbf{S}_{\sigma_G}$ is the saturation of the image of $\mathbf{S}_{Q,\sigma}$ in S_G, and, moreover, the map $\mathbf{S}_{Q,\sigma} \to \mathbf{S}_{\sigma_G}$ is continuous.*
4. *We can choose a lift $\iota_{\sigma,0}^{\sharp} : \mathbf{S}_{Q,\sigma} \to \mathrm{M}_{k_P}$ of the identification*

$$\mathbf{S}_{Q,\sigma}/\mathbf{S}_{Q,\sigma}^{\times} = \mathrm{M}_R/R^{\times} = P$$

such that the composition $\mathbf{S}_{Q,\sigma} \to \mathrm{M}_{k_P} \xrightarrow{x^{\sharp}} \mathrm{M}_{\mathscr{O}_K/\mathfrak{m}_K}$ factors through $\mathbf{S}_{\sigma_G}$.

Proof. (1) follows from (2.2.4.3), and (2) from (2.3.2.3)(6). For (3), the first point follows from [Har89, 3.1]. Recall that a map $f : P \to Q$ of monoids is **continuous** if $f(p)$ is invertible in Q only when p is already invertible in P. To check continuity of our given

map, simply observe that we have a diagram



where we are considering $N_x \in \sigma \cap \mathbf{B}_{Q,G}$ as a map $N_x : \mathbf{S}_{Q,\sigma} \to \mathbb{N}$. It is enough to show that N_x is continuous, but this follows because the map of algebras $x : R \to \mathscr{O}_K$ is continuous (as a map of local rings).

As for (4), note that lifts $\mathbf{S}_{Q,\sigma} \to \mathrm{M}_{k_P}$ of the identification

$$\mathbf{S}_{Q,\sigma}/\mathbf{S}_{Q,\sigma}^{\times} = \mathrm{M}_R\,/R^{\times} = P$$

form a torsor Φ under the group $\mathrm{Hom}(\mathbf{S}_Q, k^{\times})$. Lifts $\mathbf{S}_{Q,\sigma} \to \mathrm{M}_{\mathscr{O}_K/\mathfrak{m}_K}$ of the map N_x also form a torsor Φ' under the same group. In particular, Φ and Φ' are in bijection with one another; the bijection is given by post-composition with the map $\mathrm{M}_{k_P} \to \mathrm{M}_{\mathscr{O}_K/\mathfrak{m}_K}$ induced by the map of log algebras $R \to \mathscr{O}_K/\mathfrak{m}_K$. As we observed in the remark directly above the statement of the lemma, the map N_x factors through $\mathbf{S}_{\sigma_G}$. This means precisely that the $\mathrm{Hom}(\mathbf{S}_Q, k^{\times})$-torsor Φ' has a 'reduction of structure group' to $\mathrm{Hom}(\mathbf{S}_G, k^{\times})$ given by the sub-space of lifts of N_x that factor through $\mathbf{S}_{\sigma_G}$. Pick any lift in this sub-space, and let $\iota_{\sigma,0}^{\sharp} \in \Phi$ be the corresponding lift under the bijection between Φ' and Φ. □

3.2.3

Here we will construct the explicit model R^+ for R^{sab}. Choose any co-character $\mu_0 : \mathbb{G}_m \otimes k \to P_{\mathrm{wt},G} \otimes k$ that splits the Hodge filtration $\mathrm{Fil}^1(M_0 \otimes k) \subset M_0 \otimes k$. This is possible by (3.2.2.2)(2). Let $A_0 \subset P_{\mathrm{wt},G} \otimes k$ be any maximal k-split torus that contains the image of μ_0, and choose a Levi sub-group $L \subset P_{\mathrm{wt},G}$ such that $A_0 \subset L \otimes k$. Then we can lift μ_0 to a co-character $\mu : \mathbb{G}_m \to L$ inducing a splitting $M_0 = \mathrm{Fil}^1 M_0 \oplus {M_0}'$. Let $P_F \subset \mathrm{GSp}(M_0, \psi_0)$ and $P_{F,G} \subset G$ be the parabolic sub-groups associated with the filtration $\mathrm{Fil}^1 M_0 \subset M_0$, and let $U_F^{\mathrm{op}} \subset \mathrm{GSp}(M_0, \psi_0)$ and $U_{F,G}^{\mathrm{op}} \subset G$ be the opposite unipotent sub-groups associated with μ. Then, every $N \in \mathrm{Lie}\, U_F^{\mathrm{op}}$ satisfies $N^2 = 0$ in $\mathrm{End}(M_0)$, and the exponential $N \mapsto 1 + N$ induces an isomorphism of groups $\mathrm{Lie}\, U_F^{\mathrm{op}} \xrightarrow{\simeq} U_F^{\mathrm{op}}$.

The choice of Levi L gives us a co-character $w : \mathbb{G}_m \to G$ splitting $W_\bullet M_0$, and, by construction, μ commutes with w. This ensures the following:

3.2.3.1. $M_0^{\text{ét}} \subset {M_0}' \subset W_1 M_0$.

3.2.3.2. $U_{\mathrm{wt}}^{-2} \subset U_F^{\mathrm{op}} \subset P_{\mathrm{wt}}$.

3.2.3.3. U_F^{op} is stable under conjugation by $w(\mathbb{G}_m)$.

Indeed, we first note that, by the choice of μ, the filtration $W_\bullet M_0$ is $\mu(\mathbb{G}_m)$-stable, and so $W_iM = (W_iM \cap M_0{}') \oplus (W_iM \cap \mathrm{Fil}^1 M_0)$, for $i = 0, -1$. Since we have $\mathrm{Fil}^1 M_0 + W_1 M_0 = M_0$ and $M_0^{\text{ét}} \cap \mathrm{Fil}^1 M_0 = 0$, (3.2.3.1) follows. Now (3.2.3.2) is an immediate consequence. Finally, (3.2.3.3) follows simply because w and μ commute.

Let $U_F^+ = U_F^{\mathrm{op}}/U_{\mathrm{wt}}^{-2}$; note that this group acts naturally on M_0^{sab}. Our choice of cocharacter w splitting the weight filtration preserves U_F^{op} and so gives us a section $U_F^+ \to U_F^{\mathrm{op}}$. Let $\hat{U}^+$ be the completion of U_F^+ along the identity section. This is a formal affine scheme. Call the associated formally smooth W-algebras R^+. Let $I_{R^+} \subset R^+$ be the augmentation ideal corresponding to the identity section, and equip R^+ with an I_{R^+}-contracting Frobenius lift φ_{R^+} (that is, we have $\varphi_{R^+}\left(I_{R^+}\right) \subset I_{R^+}^2$). We can then define an object $\underline{M}^{+,\mathrm{sab}}$ in $\mathcal{MF}_{[0,1]}(R^+)$ (cf. 1.4.1.5) in the following way:

As an R^+-module $M^{+,\mathrm{sab}} = M_0^{\mathrm{sab}} \otimes_W R^+$. We set $\mathrm{Fil}^1 M^+ = \mathrm{Fil}^1 M_0^{\mathrm{sab}} \otimes_W R$. We equip M^+ with the φ_{R^+}-semi-linear map $\varphi_{M^+} = g^+(\varphi_{M_0^{\mathrm{sab}}} \otimes \varphi_{R^+})$, where $g^+ \in \hat{U}^+(R^+)$ is the universal element of $\hat{U}^+$. By (1.4.2.4), it follows that there exists a unique φ-compatible, topologically quasi-nilpotent connection $\nabla_{M^{+,\mathrm{sab}}} : M^{+,\mathrm{sab}} \to M^{+,\mathrm{sab}} \otimes_{R^+} \hat{\Omega}^1_{R^+/W}$. By (3.2.3.2), the φ-stable filtration $W_\bullet M_0^{\mathrm{sab}}$ of M_0^{sab} also extends naturally to an φ-stable filtration $W_\bullet M^{+,\mathrm{sab}}$ on M^+, giving us a short exact sequence

$$0 \to \underline{M}^{\mathrm{ab},+} \to \underline{M}^{\mathrm{sab},+} \to \underline{M}^{\mathrm{mult},+} \to 0,$$

in $\mathcal{MF}_{[0,1]}(R^+)$. This gives a deformation

$$0 \to T^+[p^\infty] \to \mathscr{G}^{\mathrm{sab},+} \to {}'B^+[p^\infty]' \to 0$$

over R^+ of the extension of p-divisible groups

$$0 \to T_0[p^\infty] \to J_0[p^\infty] \to B_0[p^\infty] \to 0$$

over k. Here, T^+ is the split torus over R^+ with character group X. Using Serre-Tate theory, we find a unique deformation B^+ of the abelian variety B_0 to R^+ so that '$B^+[p^\infty]$' is in fact the p-divisible group $B^+[p^\infty]$ associated with B^+. Moreover, B^+ is equipped with a lift $\lambda^{\mathrm{ab},+}$ of the polarization λ_0^{ab} on B_0, since $M^{\mathrm{ab},+} = W_1 M^{\mathrm{sab},+} = M_0^{\mathrm{ab}} \otimes_W R^+$ carries the polarization $\psi_0^{\mathrm{ab}} \otimes 1$.

Lemma 3.2.3.4. *There exists a semi-abelian scheme J^+ over R^+ sitting in a short exact sequence:*

$$0 \to T^+ \to J^+ \to B^+ \to 0,$$

equipped with an isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^+[p^\infty] & \longrightarrow & J^+[p^\infty] & \longrightarrow & B^+[p^\infty] & \longrightarrow & 0 \\ & & \| & & \simeq\downarrow & & \| & & \\ 0 & \longrightarrow & T^+[p^\infty] & \longrightarrow & \mathscr{G}^{\mathrm{sab},+} & \longrightarrow & B^+[p^\infty] & \longrightarrow & 0 \end{array}$$

Moreover, let Art_W *be the category of artinian local* W*-algebras* $(R, \mathfrak{m}_R)$ *equipped with an identification* $R/\mathfrak{m}_R = k$*. The triple* $(R^+, J^+, \lambda^{\mathrm{ab},+})$ *pro-represents the deformation functor (cf. 3.1)*

$$\mathrm{Def}_{(J_0,\lambda_0^{\mathrm{ab}})} : \mathrm{Art}_W \to \mathrm{Set}$$

$$\mathrm{Def}_{(J_0,\lambda_0^{\mathrm{ab}})}(A) = \left(\begin{array}{c} \textit{Pairs } (J', \lambda^{\mathrm{ab},\prime}) \textit{ where } J' \textit{ is a deformation of } J_0 \textit{ over } A \\ \textit{and } \lambda^{\mathrm{ab},\prime} \textit{ is a lift to } B' \textit{ of the polarization } \lambda_0^{\mathrm{ab}} \textit{ on } B_0. \end{array} \right).$$

In particular, the classifying map $R^{\mathrm{sab}} \to R^+$ *corresponding to the deformation* (J^+, λ^+) *over* R^+ *is an isomorphism of* W*-algebras.*

Proof. To begin, we remark that, over any $R \in \mathrm{Art}_W$ equipped with a deformation J' of J_0, the inclusion $T_0 \hookrightarrow J_0$ deforms uniquely to an inclusion $T' \hookrightarrow J'$ (T' is the split torus over A with character group X), and so the quotient B' of J' by T' is unambiguously determined; cf. proof of (1.1.3.1). In particular, our deformation problem makes sense.

By (1.1.3.2), our deformation problem is equivalent to the one for the pair $\mathrm{Def}_{(J_0[p^\infty],\lambda_0^{\mathrm{ab}})}$. The corresponding deformation functor is pro-represented by a formally smooth W-algebra $T = W[|t_1, \ldots, t_r|]$ and a universal deformation $(\mathscr{G}^{\mathrm{sab}}, \lambda^{\mathrm{ab}})$ over T. The universal deformation gives rise, via the Dieudonné functor, to a Dieudonné F-crystal $\underline{M}^{\mathrm{sab}}$ over T. If we fix the Frobenius lift φ_T on T taking t_i to t_i^p, we can reinterpret $\underline{M}^{\mathrm{sab}}$ as a tuple $(M^{\mathrm{sab}}, \varphi_{M^{\mathrm{sab}}}, \mathrm{Fil}^1 M^{\mathrm{sab}}, \nabla_{M^{\mathrm{sab}}})$ in $\mathcal{MF}_{[0,1]}(T)$, and we can choose the co-ordinates t_i so that reducing $\underline{M}^{\mathrm{sab}}$ modulo $(t_1, \ldots, t_r)$ gives the Dieudonné module $\underline{M}_0^{\mathrm{sab}}$ over W.

The versality of T gives us a map $f : T \to R^+$ and an identification

$$f^* \underline{M}^{\mathrm{sab}} = \underline{M}^{\mathrm{sab},+}.$$

Let

$$\mathrm{KS}_{M^{+,\mathrm{sab}}} : \mathrm{Lie}\, U_F^+ \otimes_W R^+ \to \mathrm{Hom}\left(\mathrm{Fil}^1 M^{+,\mathrm{sab}}, \frac{M^{+,\mathrm{sab}}}{\mathrm{Fil}^1 M^{+,\mathrm{sab}}} \right)$$

be the Kodaira-Spencer map arising from the connection on $M^{+,\mathrm{sab}}$. Then we see from

(1.4.2.2)(3) that $\mathrm{KS}_{M^+,\mathrm{sab}} \otimes_{R^+} W$ is simply the inclusion

$$\mathrm{Lie}\, U_F^+ \hookrightarrow \mathrm{Hom}\left(\mathrm{Fil}^1 M_0^{\mathrm{sab}}, \frac{M_0^{\mathrm{sab}}}{\mathrm{Fil}^1 M_0^{\mathrm{sab}}}\right).$$

Now we note:

- $\frac{M_0^{\mathrm{ab}}}{\mathrm{Fil}^1 M_0^{\mathrm{ab}}} = \frac{M_0^{\mathrm{sab}}}{\mathrm{Fil}^1 M_0^{\mathrm{sab}}}$ is identified with the dual of $\mathrm{Fil}^1 M_0^{\mathrm{ab}}$ under the polarization ψ_0^{ab}.
- $\mathrm{Lie}\, U_F^+$ consists of those elements of $\mathrm{Hom}\left(\mathrm{Fil}^1 M_0^{\mathrm{sab}}, \frac{M_0^{\mathrm{sab}}}{\mathrm{Fil}^1 M_0^{\mathrm{sab}}}\right)$ that restrict to symmetric maps from $\mathrm{Fil}^1 M_0^{\mathrm{ab}}$ to $(\mathrm{Fil}^1 M_0^{\mathrm{ab}})^\vee$ under the identification via ψ_0^{ab} of the latter space with $\frac{M_0^{\mathrm{sab}}}{\mathrm{Fil}^1 M_0^{\mathrm{sab}}}$.

It is easy to see that the latter is also a description of the image of the tangent space of T under the Kodaira-Spencer map $\mathrm{KS}_{M^{\mathrm{sab}}} \otimes_T W$. This implies that the map $f : T \to R^+$ has to be an isomorphism. □

From now on we will identify the triple $(R^{\mathrm{sab}}, J_{\mathrm{univ}}, \lambda_{\mathrm{univ}}^{\mathrm{ab}})$ with $(R^+, J^+, \lambda^{\mathrm{ab},+})$.

3.2.4

Let $\iota_{\sigma,0}^\sharp : \mathbf{S}_{Q,\sigma} \to \mathrm{M}_{k_P}$ be the lift chosen in (3.2.2.2)(4). It gives us a map $\mathbf{S}_Q \to \mathrm{M}_{k_P}^{\mathrm{gp}}$; restricting this to $\mathbf{S}_{\lambda^{\text{ét}}}$ gives us an element of $\mathbf{B}_{\lambda^{\text{ét}}} \otimes \mathrm{M}_{k_P}^{\mathrm{gp}}$, which amounts to a pairing

$$\langle\ ,\ \rangle_{\iota_{\sigma,0}} : X \times Y \to \mathrm{M}_{k_P}^{\mathrm{gp}}$$

such that $\langle \lambda^{\text{ét}}(y), y' \rangle_{\iota_{\sigma,0}} = \langle \lambda^{\text{ét}}(y'), y \rangle_{\iota_{\sigma,0}}$, for all $y, y' \in Y$. This in turn provides us with a log 1-motif $[Y \xrightarrow{\iota_{\sigma,0}} T_0^{\log} = \mathrm{Hom}(X, \mathbb{G}_m^{\log})]$ over k_P, where

$$\iota_{\sigma,0}(y)(x) = \langle x, y \rangle_{\iota_{\sigma,0}}.$$

The difference $u_0 = \iota_0 - \iota_{\sigma,0}$ gives us a classical 1-motif $Q_0^{\mathrm{cl}} = [Y \xrightarrow{u_0} J_0]$ over k that is naturally equipped with a polarization λ_0^{cl}. Let

$$\mathscr{G}_{u_0} = \varinjlim_n \left([Y \xrightarrow{u_0} J_0] \otimes^L \mathbb{Z}/p^n\mathbb{Z}[-1]\right)$$

be its associated p-divisible group over k. It is polarized, and we know from the construction in § 1.3 that $\mathbb{D}(\mathscr{G}_{u_0})(W)$ is naturally isomorphic to $(M_0, \varphi_{M_0}, \psi_0)$ as a polarized Dieudonné

module over W. Here φ_{M_0} is associated with some choice of section $\beta : \mathrm{M}_{k_P} \to \mathrm{M}_{W_P}$ (cf. 3.2.1). We will fix this φ-module structure on M_0, from now on.

We now consider the space $\hat{U}^{\mathrm{cl}}$, the completion of U_F^{op} along the identity section. Let R^{cl} be its ring of global sections, and let $I_{R^{\mathrm{cl}}} \subset R^{\mathrm{cl}}$ be its augmentation ideal at the identity. Equip R^{cl} with an $I_{R^{\mathrm{cl}}}$-contracting Frobenius lift $\varphi_{R^{\mathrm{cl}}}$ that lifts φ_{R^+}. Set $M^{\mathrm{cl}} = M_0 \otimes_W R^{\mathrm{cl}}$, $\mathrm{Fil}^1 M^{\mathrm{cl}} = \mathrm{Fil}^1 M_0 \otimes_W R^{\mathrm{cl}}$, $\varphi_{M^{\mathrm{cl}}} = g(\varphi_{M_0} \otimes \varphi_{R^{\mathrm{cl}}})$. Here $g \in \hat{U}^{\mathrm{cl}}(R^{\mathrm{cl}})$ is the universal element. Just as it was the case for M^+ above, we can use (1.4.2.3) to extend this to an object $(\underline{M}^{\mathrm{cl}}, \psi^{\mathrm{cl}})$ in $\mathcal{MF}^{\mathrm{pol}}_{[0,1]}(R^{\mathrm{cl}})$. This corresponds to a deformation of $\mathscr{G}_{u_0}$ along with its polarization, and thus of the polarized 1-motif $(Q_0^{\mathrm{cl}}, \lambda_0^{\mathrm{cl}})$, over R^{cl} by (1.1.3.2).

Lemma 3.2.4.1. *Let us denote by $(Q^{\mathrm{cl}}, \lambda^{\mathrm{cl}})$ the deformation over R^+ of $(Q_0^{\mathrm{cl}}, \lambda_0^{\mathrm{cl}})$ found above.*

1. *$(Q^{\mathrm{cl}}, \lambda^{\mathrm{cl}})$ is a universal deformation of the polarized 1-motif $(Q_0^{\mathrm{cl}}, \lambda_0^{\mathrm{cl}})$.*
2. *The choice of weight co-character $w : \mathbb{G}_m \to P_{\mathrm{wt},G}$ made in (3.2.3) gives us a deformation (Q^+, λ^+) over R^+ of the 1-motif $(Q_0^{\mathrm{cl}}, \lambda_0^{\mathrm{cl}})$.*

Proof. The argument for (1) is similar to the one in (3.2.3.4) and will be omitted: it uses the fact from (1.1.3.1) that deforming a 1-motif is equivalent to deforming its associated p-divisible group.

For (2), we note that w gives us a section $\mathrm{Lie}\, U_F^+ \hookrightarrow \mathrm{Lie}\, U_F^{\mathrm{op}}$, and hence a section $\hat{U}^+ \hookrightarrow \hat{U}^{\mathrm{cl}}$. This is because both U^+ and U_F^{op} are vector groups isomorphic to their Lie algebras (cf. 3.2.3). By assertion (1), the section $\hat{U}^+ \hookrightarrow \hat{U}^{\mathrm{cl}}$ will determine a deformation (L^+, λ^+) over R^+ of the polarized 1-motif $(Q_0^{\mathrm{cl}}, \lambda_0^{\mathrm{cl}})$. □

3.2.5

Let R_σ be the complete local W-algebra obtained by completing the toric scheme $\mathrm{Spec}\, W[\mathbf{S}_{Q,\sigma}]$ along the point corresponding to the map of monoids $\iota^\sharp_{\sigma,0} : \mathbf{S}_{Q,\sigma} \to \mathrm{M}_{k_P}$ chosen in (3.2.2.2)(4). R_σ has a natural log structure and we get a map $\iota_\sigma : R_\sigma \to k_P$ of log W-algebras. Just as above, over R_σ we have the log 1-motif $[Y \xrightarrow{u_\sigma} T^{\log}]$ induced by the natural maps $\mathbf{S}_{\lambda^{\text{ét}}} \hookrightarrow \mathbf{S}_Q \to \mathrm{M}^{\mathrm{gp}}_{R_\sigma}$.

Let us summarize what we have so far:

- There is a formally smooth W-algebra R^+ equipped with a pair $(J^+, \lambda^{\mathrm{ab},+})$ deforming (J_0, λ_0). It is identified with the deformation ring R^{sab} for the pair (J_0, λ_0).
- There is a formally smooth and log formally smooth log W-algebra R_σ equipped with a log 1-motif $[Y \xrightarrow{u_\sigma} T^{\log}]$ reducing to the log 1-motif $[Y \xrightarrow{\iota_{\sigma,0}} T_0^{\log}]$ over k_P. u_σ is induced by the natural map of monoids $\mathbf{S}_{\lambda^{\text{ét}},\sigma} \to R_\sigma$ and $\iota_{\sigma,0}$ is induced by the map $\iota^\sharp_{\sigma,0}$. The difference between this last log 1-motif and the polarized log 1-motif (Q_0, λ_0) over k_P is a classical polarized 1-motif $(Q_0^{\mathrm{cl}}, \lambda_0^{\mathrm{cl}})$ over k.

- The choice of weight co-character $w : \mathbb{G}_m \to P_{\mathrm{wt},G}$ gives us a polarized 1-motif (Q^+, λ^+) over R^+ deforming the 1-motif $(Q_0^{\mathrm{cl}}, \lambda_0^{\mathrm{cl}})$.

Let $R' = R^+ \hat{\otimes} R_\sigma$: this is a local log W-algebra with the log structure induced from the one on R_σ. Let $x_0' : R' \to k_P$ be the natural surjection. Over R' we have the polarized 1-motif (Q^+, λ^+), where $Q^+ = [Y \xrightarrow{u^+} J_{R'}]$ arising from the one over R^+, and the log 1-motif $[Y \xrightarrow{u_\sigma} T^{\log}_{R'}]$, arising from the one over R_σ. The map $\iota' = u^+ + u^\sigma$ gives us a polarized log 1-motif (Q', λ') over R' (with $Q' = [Y \xrightarrow{\iota'} J^{\log}_{R'}]$) reducing to the polarized log 1-motif (Q_0, λ_0) along x_0'.

R' with the deformation (Q', λ') of (Q_0, λ_0) will be our explicit model for the deformation ring R. We codify this in the next:

Proposition 3.2.5.1. *Giving a continuous map $f : R' \to C$ of local log W-algebras is equivalent to giving a deformation over C of (Q_0, λ_0) (cf. 3.1.1.6). In particular, the triples $(R', (Q', \lambda'))$ and $(R, (Q, \lambda))$ are naturally isomorphic.*

Proof. If we have such a map f, then clearly the pull-back $f^*(Q', \lambda')$ gives rise to a deformation over C of (L_0, λ_0). Conversely, suppose we have a deformation $((Q_C, \lambda_C), j_C, i_C)$ over C of (L_0, λ_0). In particular, we have a deformation $(J_C, \lambda_C^{\mathrm{ab}})$ over C of the pair $(J_0, \lambda_0^{\mathrm{ab}})$; by (3.2.3.4), this corresponds to a map $f^+ : R^+ \to C$ such that $f^+(J^+, \lambda^{\mathrm{ab},+}) = (J_C, \lambda_C^{\mathrm{ab}})$. Moreover, from the construction in (3.2.4), we have the polarized 1-motif (L^+, λ^+) over R^+ and this gives us a classical 1-motif $(Q_C^{\mathrm{cl}}, \lambda_C^{\mathrm{cl}})$ over C when pulled back along f^+.

Note that Q_C is of the form $[Y \xrightarrow{\iota_C} J_C^{\log}]$ and Q_C^{cl} is of the form $[Y \xrightarrow{\iota_C^{\mathrm{cl}}} J_C]$. Moreover, the difference $\iota_C - \iota_C^{\mathrm{cl}}$ factors through $T_C^{\log}$ and gives us a map from Y to $T^{\log}(C)$. This last map can also be viewed as a pairing $Y \times X \to \mathrm{M}_C^{\mathrm{gp}}$, and the presence of the polarization λ_T ensures that this pairing factors through a *linear* map $\mathbf{S}_{\lambda^{\text{ét}}} \to \mathrm{M}_C^{\mathrm{gp}}$. We claim that we can extend this to a map $\mathbf{S}_Q \to \mathrm{M}_C^{\mathrm{gp}}$. To do this, consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}\left(\dfrac{\mathbf{S}_Q}{\mathbf{S}_{\lambda^{\text{ét}}}}, \mathrm{M}_C^{\mathrm{gp}}\right) & \hookrightarrow & \mathrm{Hom}(\mathbf{S}_Q, \mathrm{M}_C^{\mathrm{gp}}) & \longrightarrow & \mathrm{Hom}(\mathbf{S}_{\lambda^{\text{ét}}}, \mathrm{M}_C^{\mathrm{gp}}) & \twoheadrightarrow & \mathrm{Ext}^1\left(\dfrac{\mathbf{S}_Q}{\mathbf{S}_{\lambda^{\text{ét}}}}, \mathrm{M}_C^{\mathrm{gp}}\right) \\
\downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \simeq \\
\mathrm{Hom}\left(\dfrac{\mathbf{S}_Q}{\mathbf{S}_{\lambda^{\text{ét}}}}, \mathrm{M}_{l_C}^{\mathrm{gp}}\right) & \hookrightarrow & \mathrm{Hom}(\mathbf{S}_Q, \mathrm{M}_{l_C}^{\mathrm{gp}}) & \longrightarrow & \mathrm{Hom}(\mathbf{S}_{\lambda^{\text{ét}}}, \mathrm{M}_{l_C}^{\mathrm{gp}}) & \twoheadrightarrow & \mathrm{Ext}^1\left(\dfrac{\mathbf{S}_Q}{\mathbf{S}_{\lambda^{\text{ét}}}}, \mathrm{M}_{l_C}^{\mathrm{gp}}\right)
\end{array}$$

All the rows of the diagram are exact, and the middle square is Cartesian, since the vertical arrows at the ends are isomorphisms. To see this last fact, first observe that the quotient group $K = \dfrac{\mathbf{S}_Q}{\mathbf{S}_{\lambda^{\text{ét}}}}$ is finite of prime-to-p order. So any map from K to $\mathrm{M}_C^{\mathrm{gp}}$ or $\mathrm{M}_{l_C}^{\mathrm{gp}}$ must land in $k^\times$; this shows that the vertical arrow on the extreme left is an isomorphism. For the other isomorphism, the argument is similar: Since $1 + \mathfrak{m}_C$ ($\mathfrak{m}_C$ is the maximal ideal of

C) is a pro-p group, the groups $\mathrm{Ext}^i(K, 1+\mathfrak{m}_C)$ vanish for all $i \geq 0$. This implies that the arrow on the extreme right is also an isomorphism, since we have the short exact sequence

$$1 \to 1+\mathfrak{m}_C \to \mathrm{M}_C^{\mathrm{gp}} \to \mathrm{M}_{l_C}^{\mathrm{gp}} \to 1.$$

Now we return to our map $\iota_C^{\log} : \mathbf{S}_{\lambda^{\text{ét}}} \to \mathrm{M}_C^{\mathrm{gp}}$: The induced map $\mathbf{S}_{\lambda^{\text{ét}}} \to \mathrm{M}_{l_C}^{\mathrm{gp}}$ agrees with the restriction to $\mathbf{S}_{\lambda^{\text{ét}}}$ of the map $j_T^\sharp : \mathbf{S}_Q \to \mathrm{M}_{l_C}^{\mathrm{gp}}$ induced from $j_T : k_P \to k_L$ and the surjection $x_0' : R_\sigma \to k_P$. Since the middle square above is Cartesian, we obtain a unique map $f^\sharp : \mathbf{S}_Q \to \mathrm{M}_C^{\mathrm{gp}}$ inducing $\iota_C^{\log}$ and $j_T^\sharp$.

We claim that the restriction of $f^\sharp$ to $\mathbf{S}_{Q,\sigma}$ lands in M_C; indeed, we only have to check that the induced map $j_T^\sharp$ lands in M_{k_C}, and this is true by hypothesis. By definition, $f^\sharp$ then amounts to giving a continuous map $f_\sigma : R_\sigma \to C$ of log W-algebras.

Now, the map $f = f^+ \hat{\otimes} f_\sigma : R' \to C$ is the one inducing the deformation $((Q_C, \lambda_C), j_C, i_C)$. □

From now on we will identify R with R' along with the polarized log 1-motifs (and hence the degenerating family of polarized abelian varieties) over them.

3.2.6

There is a natural element $\Theta_\sigma \in \mathbf{B}_Q \otimes \hat{\Omega}^{1,\log}_{R_\sigma/W} = \mathrm{Lie}\, U_{\mathrm{wt}}^{-2} \otimes_W \hat{\Omega}^{1,\log}_{R_\sigma/W}$ induced by the map

$$\begin{aligned} \mathbf{S}_Q &\to \hat{\Omega}^{1,\log}_{R_\sigma/W} \\ s &\mapsto \mathrm{dlog}(s). \end{aligned}$$

Set $M = M_0 \otimes_W R$ and equip it with the constant filtrations $\mathrm{Fil}^1 M = \mathrm{Fil}^1 M_0 \otimes_W R$, $W_\bullet M = W_\bullet M_0 \otimes_W R$, and the constant polarization $\psi = \psi_0 \otimes 1$.

Let $\Theta^+ \in \mathrm{Lie}\, U_F^+ \otimes \hat{\Omega}^1_{R^+/W}$ be the connection matrix associated with the connection on $M^{\mathrm{sab},+}$. The choice of co-character $w : \mathbb{G}_m \to P_{\mathrm{wt},G}$ splitting $W_\bullet M_0$ made in (3.2.3) gives us a section $\mathrm{Lie}\, U_F^+ \hookrightarrow \mathrm{Lie}\, U_F^{\mathrm{op}}$, and we can use this to view Θ^+ as an element of $\mathrm{Lie}\, U_F^{\mathrm{op}} \otimes \hat{\Omega}^1_{R/W}$. This also gives us a section $U_F^+ \to U_F^{\mathrm{op}}$, which we can use to view elements of U_F^+ as automorphisms of M_0. Let $\Theta_\sigma \in \mathrm{Lie}\, U_{\mathrm{wt}}^{-2} \otimes_W \hat{\Omega}^{1,\log}_{R/W}$ be the element arising from the natural inclusion $R_\sigma \hookrightarrow R$. Let ∇_M be the connection on M with connection matrix $\Theta = \Theta^+ + \Theta_\sigma$. Fix the Frobenius lift φ_σ on R_σ that restricts to the p-power map on $\mathbf{S}_{Q,\sigma} \subset R_\sigma$, let $\varphi_R = \varphi_{R^+} \hat{\otimes} \varphi_\sigma$, and let φ_M be the map

$$\varphi_R^* M = \varphi_W^* M_0 \otimes_W R \xrightarrow{\varphi_{M_0} \otimes 1} M_0 \otimes_W R \xrightarrow{g^+} M_0 \otimes_W R = M,$$

where $g^+ \in U_F^+(R)$ is the image of the universal element in $U_F^+(R^+)$. By construction, φ_M is parallel for the connection ∇_M.

Lemma 3.2.6.1. *The tuple* $\underline{M} = (M, \varphi_M, \mathrm{Fil}^1 M, \nabla_M, \psi)$ *corresponds to the polarized Dieudonné log* F*-crystal* $\mathbb{D}(A)$ *attached to the degenerating polarized abelian scheme* (A, λ) *over* R*.*

Proof. This is immediate from the construction. □

3.3 G-admissibility

Let R be the explicit log deformation ring constructed in § 3.2. In this section, we construct an explicit map $R \mapsto R_G$ (it will be the normalization of a surjection) that will be the local model for maps between complete local rings at the boundary of integral models of appropriate Shimura varieties. Given the theory of the previous section, the construction is quite simple; however, showing that it has the correct properties, and in particular that R_G has the right dimension, is more involved and requires certain additional assumptions that will be verified in applications.

There are two main results here. One is (3.3.3.6), which is a formal analogue of the rationality result for Hodge cycles found in Lemma 1 of the introduction; the second is (3.3.4.10), which gives a criterion for the normalization of a quotient $\mathscr{O}$ of R to be identified with R_G. Both these results involve the notion of (strong) G-admissibility of points in $(\mathrm{Spf}\, R)^{\mathrm{an}}$ away from the boundary divisor. One should think of this condition as follows: the points which are (strongly) G-admissible correspond to semi-stable abelian varieties appearing near the (p-adic) boundary of an appropriate Shimura variety associated with G.

3.3.1

Let $U^+_{F,G} = U^{\mathrm{op}}_{F,G}/U^{-2}_{\mathrm{wt},G}$, and let $\hat{U}^+_G$ be the completion of $U^+_{F,G}$ along the identity section. Call the associated formally smooth W-algebra R^+_G: this is a quotient of R^+. Let R_{σ_G} be the complete local log W-algebra obtained from the toric scheme associated to the monoid $\mathbf{S}_{\sigma_G}$ completed along the map $x_0^\sharp : \mathbf{S}_{\sigma_G} \to k$.

Lemma 3.3.1.1. *The natural map* $R_\sigma \to R_{\sigma_G}$ *of log* W*-algebras is continuous, and is the normalization of a surjection. Moreover, we have* $\dim R_\sigma = \dim U^{-2}_{\mathrm{wt},G} = \mathrm{rk}_{\mathbb{Z}}\, \mathbf{B}_Q + 1$*, and* $\dim R_{\sigma,G} = \mathrm{rk}_{\mathbb{Z}}\, \mathbf{B}_{Q,G} + 1$*.*

Proof. The continuity follows from that of the map of monoids $\mathbf{S}_{Q,\sigma} \to \mathbf{S}_{\sigma_G}$ (3.2.2.2)(3). That the map is then the normalization of a surjection follows from [Har89, 3.1]. The last assertion about dimensions is immediate from the definitions. □

Let $R_G = R^+_G \hat{\otimes} R_{\sigma_G}$: by the lemma above, this is the normalization of a continuous quotient of R. R_G with its inherited (from R) family of degenerating abelian varieties will be our 'local model with Tate cycles'. The first difficulty is to show that it has the right dimension. To deal with this issue, we will need a little detour.

The notation used in this sub-section will be strictly local to it. We fix a sharp, fs monoid P (cf. 1.2.1.2). Let L/K_0 be a finite extension, and let A be the logarithmic $\mathscr{O}_L$-algebra $\mathscr{O}_L[P]$ with the log structure given by the monoid P, and let $\hat{A}$ be its completion along the ideal generated by $P \setminus \{1\}$. Let $M(A)$ be the set of continuous maps of log W-algebras $\hat{A} \to \mathscr{O}_{\overline{K}_0}$. For any quotient ring T of $\hat{A}$, let $M(T) = M(A) \cap T(\mathscr{O}_{\overline{L}})$. Consider the map

$$\nu_T : M(T) \xrightarrow{x \mapsto \nu_x} \mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{Q}).$$

Definition 3.3.2.1. The **ν-dimension** of a quotient T of $\hat{A}$ is the dimension of the vector sub-space of $\mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{Q})$ generated by the image of ν_T.

Lemma 3.3.2.2. *Suppose $T = \hat{A}/\mathfrak{q}$, for some prime ideal $\mathfrak{q} \subset \hat{A}$, and that $m \notin \mathfrak{q}$, for all $m \in P$. If T is moreover W-flat (that is, if $p \notin \mathfrak{q}$), then we have $\nu\text{-}\dim(T) \geq \dim(T) - 1$.*

Before we give a proof, here are two examples with $P = \mathbb{N}^2$ and $L = K_0$, so that $\hat{A} = W[|t_1, t_2|]$. Let e_1, e_2 be the standard basis elements of $\mathbb{Z}^2$, and, for any $x \in M(A)$, set $\nu_1(x) = \nu_x(e_1)$ and $\nu_2(x) = \nu_x(e_2)$.

Example 3.3.2.3. First, take $\mathfrak{q} = (t_1 - t_2)$; then, for any $x \in M(T)$, we must have $\nu_1(x) = \nu_2(x)$. This is the only constraint, and the image of ν_T generates the sub-space of maps $f : \mathbb{Z}^2 \to \mathbb{Q}$ such that $f(e_1) = f(e_2)$. So $\nu\text{-}\dim(T) = 1 = \dim(T) - 1$ in this case.

For the second example, take $\mathfrak{q} = (t_1 - pt_2)$; then, for any $x \in M(T)$, we have $\nu_1(x) = \nu_2(x) + 1$. If $(a_1, a_2) \in \mathbb{Z}^2$, then we have

$$\nu_x(a_1, a_2) = (a_1 + a_2)\nu_2(x) + a_1.$$

Since there is no constraint on $\nu_2(x)$ (other than that it be positive) for varying $x \in M(T)$, this means that the common kernel of all the ν_x for $x \in M(T)$ is 0, and so the sub-space of $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Q})$ generated by $\operatorname{im} \nu_T$ must be everything. In particular, $\nu\text{-}\dim(T) = 2 > 1 = \dim(T) - 1$.

Remark 3.3.2.4. The examples show that the following is a reasonable interpretation of ν-$\dim(T)$: it is the dimension of the smallest 'toric' sub-scheme of $\operatorname{Spf} \hat{A}$ that contains $\operatorname{Spf} T$. In particular, they show that the ν-dimension of T conveys non-trivial information about the special fiber of T over k. Indeed, the generic fibers of the sub-schemes of $\operatorname{Spf} \hat{A}$ corresponding to the two quotients above are conjugate in $(\operatorname{Spf} \hat{A})^{\mathrm{an}}$ under the action of the torus with character group P^{gp}.

Remark 3.3.2.5. The lemma should be a consequence of the following theorem in non-archimedean tropical geometry: The closure of the image of ν_T in $\mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{R})$ is a locally finite union of $\mathbb{Q}$-rational d-dimensional polytopes, where $d = \dim T - 1$. See [EKL06] or [Gub07]. Nonetheless, the specific result we need admits an elementary proof, which we present below.

Proof. We will prove this by induction on $n = \operatorname{rk} P^{\mathrm{gp}}$. For the purposes of the induction, we will allow P to be merely fine, and do not demand that it be saturated as well.

If $n = 1$, then the normalization of $\hat{A}$ is $\mathscr{O}_L[|t|]$, and there exists some prime $\mathfrak{q}' \subset \mathscr{O}_L[|t|]$ such that $\mathfrak{q}' \cap \hat{A} = \mathfrak{q}$ and such that $T' = \mathscr{O}_L[|t|]/\mathfrak{q}'$ satisfies the same conditions as T. Since $\dim T' = \dim T$ and $\operatorname{im} \nu_{T'}$ is contained in $\operatorname{im} \nu_T$, it is enough to prove the lemma with T replaced by T' and $\hat{A}$ replaced by $\mathscr{O}_L[|t|]$. In this case, either $\mathfrak{q} = (0)$ or $\mathfrak{q}$ is principal, generated by $q(t)$ for some irreducible polynomial $q(t)$ not equal to t or p. In both cases, the lemma is easily checked by hand.

So suppose $n > 1$, and define $\Sigma = \{N \subset P^{\mathrm{gp}} \text{ a rank 1 summand} : N \cap P = \{1\}\}$. For every $N \in \Sigma$, the image P_N of P in $P_N^{\mathrm{gp}} = P^{\mathrm{gp}}/N$ is again a fine monoid without non-trivial invertible elements, and, if $\hat{A}_N = \mathscr{O}_L[|P_N|]$, we get a surjection $\hat{A} \to \hat{A}_N$ of local log W-algebras. Let $\mathfrak{p}_N \subset \hat{A}$ be the kernel of this surjection: this is a height 1 prime in $\hat{A}$.

Fix $x \in M(T)$ (this exists, by our hypotheses on T). Let $\Sigma_x = \{N \in \Sigma : N \not\subset \ker \nu_x\}$; then we have the following:

Claim.

$$\bigcap_{\substack{\text{minimal primes } \mathfrak{p}' \supset \mathfrak{q} + \mathfrak{p}_N \\ N \in \Sigma_x}} \mathfrak{p}' = \mathfrak{q}.$$

To prove this claim, it is enough to show that the collection Δ of ideals $\{\mathfrak{q} + \mathfrak{p}_N : N \in \Sigma_x\}$ is infinite. That will imply that the collection of primes minimal over ideals in Δ is infinite. If $\mathfrak{a}$ is the intersection on the left hand side and is *not* equal to $\mathfrak{q}$, then all the primes minimal over ideals in Δ will be minimal over $\mathfrak{a}$, because $\mathfrak{p}_N$ has height 1 for every N. But this is impossible, since there can only be finitely many primes minimal over any ideal of the Noetherian ring $\hat{A}$.

To show the infinitude of Δ, choose $m_1, m_2 \in P$ that generate two distinct lines in P^{gp} and are such that $\nu_x(m_1) = \nu_x(m_2)$. Then $\nu_x(m_1 m_2^{-l}) \neq 0$, for all $l \geq 2$. Let $N_l \subset P^{\mathrm{gp}}$ be the line generated by $m_1 m_2^{-l}$; then $N_l \in \Sigma_x$, and $m_1 - m_2^l \in \mathfrak{p}_{N_l} = \mathfrak{p}_l$. If Δ were finite, we can find $l > k \geq 2$ such that $\mathfrak{q} + \mathfrak{p}_k = \mathfrak{q} + \mathfrak{p}_l$. In particular, we will have

$$m_2^k(1 - m_2^{l-k}) = m_2^k - m_2^l = (m_1 - m_2^l) - (m_1 - m_2^k) \in \mathfrak{q}.$$

Since $1 - m_2^{l-k}$ is a unit in $\hat{A}$, this implies that $m_2^k \in \mathfrak{q}$, which contradicts our hypothesis that $\mathfrak{q}$ contains no element of P.

Now, choose finitely many generators $m_1, \ldots, m_l$ for P, and let $a = p \prod_i m_i$. By hypothesis, $a \notin \mathfrak{q}$, so by the claim above it follows that there exists $N \in \Sigma_x$ and a minimal prime $\mathfrak{q}' \supset \mathfrak{q} + \mathfrak{p}_N$ such that $a \notin \mathfrak{q}'$. Let $T' = \hat{A}/\mathfrak{q}'$; then T' is a quotient of $\hat{A}_N$, and, since $a \notin \mathfrak{q}'$, it satisfies all the conditions that T did. So by induction (since $\operatorname{rk} P_N^{\mathrm{gp}} = n - 1$) it follows that $\nu\text{-}\dim(T') \geq \dim(T') - 1$, and we have

$$\nu\text{-}\dim(T) \geq \nu\text{-}\dim(T') + 1 \geq \dim(T') - 1 + 1 \geq \dim(T) - 1.$$

Here, the first inequality follows since N is contained in the common kernel of the image of $\nu_{T'}$, but not in that of the image of ν_T (it is not killed by ν_x). The last inequality follows because $\hat{A}$ is catenary and $\mathfrak{p}_N$ has height 1.

And we have the conclusion of the lemma. □

3.3.3

We return now to the notation and setting of §3.2.

Definition 3.3.3.1. A continuous map $y : R_\sigma \to \mathscr{O}_{\overline{K}_0}$ of log W-algebras is **G-admissible** if the induced map $N_y : \mathbf{S}_Q \to \overline{K}_0 \xrightarrow{\nu} \mathbb{Q}$, viewed as an element of $\mathbf{B}_Q \otimes \mathbb{Q}$, lies in $\mathbf{B}_{Q,G} \otimes \mathbb{Q}$. Here ν is the p-adic valuation on $\overline{K}_0$ taking p to 1.

Remark 3.3.3.2. $\exp(N_y)$ can be viewed as the monodromy of $\underline{M}$ around y. Having y be G-admissible is therefore equivalent to asking that the the tensors $\{s_\alpha\}$ be invariant under monodromy around y.

Definition 3.3.3.3. A continuous map $y : R \to \mathscr{O}_{\overline{K}_0}$ of W-algebras is **G-admissible** if

- The induced map $y_\sigma = y|_{R_\sigma}$ is G-admissible.
- The induced map $y^+ = y|_{R^+}$ factors through R^+_G.

As above, let $M(R)$ be the set of continuous maps of log W-algebras $R \to \mathscr{O}_{\overline{K}_0}$. For any quotient ring $\mathscr{O}$ of R, we set $M(\mathscr{O}) = M(R) \cap \mathscr{O}(\mathscr{O}_{\overline{K}_0})$.

Definition 3.3.3.4. We say that a quotient $\mathscr{O} = R/\mathfrak{q}$ of R is **adapted to** G if the following conditions hold:

1. $\mathfrak{q}$ is prime (so that $\mathscr{O}$ is a domain).
2. $\mathscr{O}$ is flat over W.
3. $\dim \mathscr{O} = \mathrm{rk}_W \operatorname{Lie} U^{\mathrm{op}}_{F,G} + 1$.
4. Every element of $M(\mathscr{O})$ is G-admissible.
5. The original lift $x : R \to \mathscr{O}_K$ chosen in (3.2.2) factors through $\mathscr{O}$.

We will consider the following restriction on our setup.

Assumption 3.3.3.5. (Rationality) $\mathbf{B}_{Q,G}$ generates $\operatorname{Lie} U^{-2}_{\mathrm{wt},G}$ as a W-module; in other words, $\mathrm{rk}_{\mathbb{Z}} \mathbf{B}_{Q,G} = \mathrm{rk}_W \operatorname{Lie} U^{-2}_{\mathrm{wt},G}$.

Lemma 3.3.3.6. *Suppose that there exists a quotient $\mathscr{O}$ of R adapted to G, and suppose $\sigma \subset \mathbf{B}_Q \otimes \mathbb{R}$ has maximal dimension. Let $x^+ : R^+ \to \mathscr{O}_K$ be the restriction of our original lift x to R^+. Suppose also that we can find a prime $\mathfrak{p} \subset \mathscr{O}$ minimal over $\ker(x^+)\mathscr{O}$ such that $\mathscr{O}' = \mathscr{O}/\mathfrak{p}$ satisfies the following:*

1. *$\mathscr{O}'$ is flat over W.*
2. *For all $s \in \mathbf{S}_{Q,\sigma}$, the image of s in $\mathscr{O}$ does not lie in $\mathfrak{p}$.*
3. *$\dim \mathscr{O}' \geq \mathrm{rk}_W U^{-2}_{\mathrm{wt},G} + 1$.*

Then assumption (3.3.3.5) holds.

Proof. In order to show that $\mathbf{B}_{Q,G}$ generates $\mathrm{Lie}\, U^{-2}_{\mathrm{wt},G}$, it suffices to show

$$\dim_{\mathbb{Q}}(\mathbf{B}_{Q,G} \otimes \mathbb{Q}) \geq s \coloneqq \mathrm{rk}_W \mathrm{Lie}\, U^{-2}_{\mathrm{wt},G}.$$

The condition that σ has maximal dimension ensures that R_σ is simply the completed monoid ring $W[|\mathbf{S}_{Q,\sigma}|]$. Let $x^+ : R^+ \to \mathscr{O}_K$ and $\mathfrak{p} \subset \ker(x^+)\mathscr{O}$ be as in our hypotheses. Then $R \otimes_{R^+,x^+} \mathscr{O}_K$ is isomorphic to $\mathscr{O}_K[|\mathbf{S}_{Q,\sigma}|]$, and $\mathscr{O}' = \mathscr{O}/\mathfrak{p}$ is a quotient domain of this ring.

The hypotheses of (3.3.2.2) are now valid with $P = \mathbf{S}_{Q,\sigma}$, $L = K$, and $T = \mathscr{O}'$, and so is therefore its conclusion. By condition (4) of (3.3.3.4), every element of $M(\mathscr{O}')$ is G-admissible: this means that, for every $y \in M(\mathscr{O}')$, the associated element $N_y \in \mathbf{B}_Q \otimes \mathbb{Q}$ lies in $\mathbf{B}_{Q,G} \otimes \mathbb{Q}$. On the other hand, by (3.3.2.2) and hypothesis (3) above, the sub-space of $\mathbf{B}_Q \otimes \mathbb{Q}$ generated by N_y for $y \in M(\mathscr{O}')$ has dimension at least s. So we have the desired inference. □

In fact, under some mild conditions, we can get the numbered hypotheses of the lemma above for free.

Lemma 3.3.3.7. *Suppose $\mathscr{O}$ is a quotient of R adapted to G. Suppose also that we have*

- *A flat map $f : X \to Y$ of flat, finite type, integral W-schemes;*
- *A closed sub-scheme $Z \subset X$; and*
- *a point $z \in Z(k)$;*

such that the completion of the diagram

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ & & \downarrow{\scriptstyle f} \\ & & Y \end{array}$$

at z is isomorphic over W to the diagram

$$\begin{array}{ccc} \operatorname{Spf} \mathscr{O} & \hookrightarrow & \operatorname{Spf} R \\ & & \downarrow{\scriptstyle \operatorname{Spf} \iota^+} \\ & & \operatorname{Spf} R^+. \end{array}$$

Then the numbered hypotheses of (3.3.3.6) are valid.

Proof. First, we claim that $M(\mathscr{O})$ is Zariski dense in the rigid analytic space $(\operatorname{Spf} \mathscr{O})^{\mathrm{an}}$ associated with $\mathscr{O}$. Since x factors through $\mathscr{O}$ and is a map of log W-algebras, no $s \in \mathbf{S}_{Q,\sigma}$ maps to 0 in $\mathscr{O}$. Indeed, if some s did map to 0 in $\mathscr{O}$, then it would have to map to 0 under x as well, which is impossible, since x is a map of log W-algebras.

Now, we find, using conditions (1),(2) and (4) of (3.3.3.4), that the restriction $\operatorname{Spf} \iota^+|_{\operatorname{Spf} \mathscr{O}}$ factors through $\operatorname{Spf} R^+_G$. This is because the restriction of every element of $M(\mathscr{O})$ to $\operatorname{Spf} R^+$ factors through $\operatorname{Spf} R^+_G$ (implied by G-admissibility), and because $M(\mathscr{O})$ is dense in $(\operatorname{Spf} \mathscr{O})^{\mathrm{an}}$.

Let $Y' \subset Y$ be the closure of the image of Z in Y; then, it follows that, for any small enough affine neighborhood V of $s = f(z)$ in Y', we have $\dim V \leq \dim R^+_G$. Replacing Y' by V, and Z by any affine neighborhood of z in the pre-image of V, we can assume that $Z \to Y'$ is induced by a map $B \leftarrow A$ of finitely generated domains over W, with $\dim B = \dim \mathscr{O}$ and $\dim A \leq \dim R^+_G$.

Let $\mathfrak{m}_z \subset B$ be the maximal ideal corresponding to z, and let $\mathfrak{m}_s \subset A$ be the one corresponding to s. The map $x^+ : R^+_G \to \mathscr{O}_K$ induces a map $j : A \to K$, such that $I_A \coloneqq \ker j \subset \mathfrak{m}_s$, and the fiber $B/I_A B\left[\frac{1}{p}\right]$ of $B\left[\frac{1}{p}\right]$ over j has dimension at least

$$\dim \mathscr{O} - \dim R^+_G = \operatorname{rk}_W U^{\mathrm{op}}_{F,G} - \operatorname{rk}_W U^+_G = \operatorname{rk}_W U^{-2}_{\mathrm{wt},G},$$

by the upper semi-continuity of dimensions of fibers. Let $\mathfrak{p}_B \supset I_A B$ be any minimal prime such that $(B/\mathfrak{p}_B)[1/p]$ has dimension at least $\operatorname{rk}_W U^{-2}_{\mathrm{wt},G}$, and such that $\mathfrak{p}_B \subset \mathfrak{m}_z$.

Then the closure of $\mathfrak{p}_B$ in $\mathscr{O}$ under the identification $\hat{B}_{\mathfrak{m}_z} = \mathscr{O}$ will give us the prime $\mathfrak{p} \subset \mathscr{O}$ needed in the hypotheses of (3.3.3.6). Indeed, the closure of I_A in $\mathscr{O}$ is $\ker(x^+)\mathscr{O}$, and so $\mathfrak{p}$ will be minimal over $\ker(x^+)\mathscr{O}$. By construction, $\mathscr{O}' = \mathscr{O}/\mathfrak{p}$ is flat over W and its dimension satisfies the lower bound in hypothesis (3) of (3.3.3.6). Moreover, we have the map $x : R \to \mathscr{O}_K$, which factors through $\mathscr{O}$ by condition (5) of (3.3.3.4), and so in fact factors through $\mathscr{O}/(\ker x^+)\mathscr{O}$. In particular, for any $s \in \mathbf{S}_{Q,\sigma}$, the image of s in $\mathscr{O}/(\ker x^+)\mathscr{O}$ is non-zero, since its image in $\mathscr{O}_K$ under x is non-zero as already observed at the beginning of this proof. This implies hypothesis (2) of (3.3.3.6), since, if some $s \in \mathbf{S}_{Q,\sigma}$ belonged to $\mathfrak{p}$, then some high enough power s^N would lie in $\ker x^+$ and thus would map to 0 in $\mathscr{O}/(\ker x^+)\mathscr{O}$. □

3.3.4

Let L_0/K_0 be an unramified extension. We endow the power series ring $L_0[|u|]$ with a φ_{L_0}-semi-linear endomorphism φ taking u to u^p. We also endow it with the log structure

$$\mathrm{M}_{L_0[|u|]} = L_0[|u|] \setminus \{0\} \hookrightarrow L_0[|u|],$$

thus making $L_0[|u|]$ a log L_0-algebra. The endomorphism φ can be viewed as a φ_{L_0}-semi-linear endomorphism of $L_0[|u|]$ as a log L_0-algebra.

Suppose that D_0 is a φ-module over L_0: that is, D_0 is a finite dimensional L_0-vector space equipped with an isomorphism

$$\varphi_{D_0} : \varphi_{L_0}^* D_0 \xrightarrow{\simeq} D_0.$$

Let $D = D_0 \otimes_{L_0} L_0[|u|]$, and suppose that D is equipped with a logarithmic connection $\nabla_D : D \to D \otimes L_0[|u|]\,\mathrm{dlog}(u)$ and an isomorphism

$$\varphi_D : \varphi^* D \xrightarrow{\simeq} D$$

such that:

- φ_D reduces modulo u to the endomorphism φ_{D_0}.
- φ_D is ∇_D-parallel.

Then the residue $N : D_0 \to D_0$ at $u = 0$ of ∇_D is an operator satisfying $N\varphi_{D_0} = p\varphi_{D_0}N$, and so D_0 has the structure of a (φ, N)-module over L_0.

Let $L_0[|u|]^{\log} = L_0[|u|][l_u]$ be the polynomial ring in the variable l_u over $L_0[|u|]$. $L_0[|u|]^{\log}$ can be equipped with a natural logarithmic connection $\nabla : l_u \mapsto -1 \otimes \mathrm{dlog}(u)$, and a φ-semi-linear map φ given by $\varphi(l_u) = pl_u$. The logarithmic connection corresponds to a $L_0[|u|]$-derivation $N : l_u \mapsto -l_u$ of $L_0[|u|]^{\log}$, and we have

$$L_0[|u|] = (L_0[|u|]^{\log})^{N=0}.$$

Set

$$\Psi(D) = \left(D \otimes_{L_0[|u|]} L_0[|u|]^{\log}\right)^{\nabla=0},$$

where $D \otimes_{L_0[|u|]} L_0[|u|]^{\log}$ is equipped with the tensor product logarithmic connection. This is naturally a (φ, N)-module over L_0: the endomorphism N of $\Psi(D)$ is given by $(\nabla_D \otimes 1)|_{\Psi(D)}$. Moreover, by [Vol03, Theorem 9], the inclusion $\Psi(D) \hookrightarrow D \otimes_{L_0[|u|]} L_0[|u|]^{\log}$ induces a (φ, ∇)-equivariant isomorphism

$$\Psi(D) \otimes_{L_0} L_0[|u|]^{\log} \xrightarrow{\simeq} D \otimes_{L_0[|u|]} L_0[|u|]^{\log}.$$

The map $L_0[|u|]^{\log} \to L_0$ sending u and l_u to 0 gives rise to an isomorphism

$$\Psi(D) \xrightarrow{\simeq} D_0$$

of (φ, N)-modules over L_0, and so we obtain a (φ, ∇)-equivariant isomorphism

$$\xi : D_0 \otimes_{L_0} L_0[|u|]^{\log} \xrightarrow{\simeq} D \otimes_{L_0[|u|]} L_0[|u|]^{\log}.$$

In particular, if $d \in D_0$ is such that $N(d) = 0$, then $\xi(d)$ will be a parallel section of $D = \left(D \otimes_{L_0[|u|]} L_0[|u|]^{\log}\right)^{N=0}$.

Lemma 3.3.4.1. *Let $G \subset \mathrm{GL}(D_0)$ be a reductive sub-group, which is the pointwise stabilizer of a collection of φ-invariant tensors $\{s_\alpha, 0\} \subset (D_0^{\otimes})^{N=0}$. For each α, let $\tilde{s}_\alpha = \xi(s_{\alpha,0}) \in D^{\otimes}$ be the unique ∇_D-parallel, φ-invariant element lifting $s_{\alpha,0}$. Let $\theta \in \mathrm{GL}(D)$ be the composition*

$$D = D_0 \otimes L_0[|u|] \xrightarrow{\varphi_{D_0}^{-1} \otimes 1} \varphi_{L_0}^* D_0 \otimes L_0[|u|] = \varphi^* D \xrightarrow{\varphi_D} D;$$

Suppose that D_0 is equipped with a filtration $\mathrm{Fil}^\bullet D_0$ such that:

1. *The tuple $(D_0, \varphi_{D_0}, \mathrm{Fil}^\bullet D_0)$ gives a weakly admissible φ-module over L_0;*
2. *$\mathrm{Fil}^\bullet D_0$ is split by a co-character $\mu : \mathbb{G}_m \to G$;*
3. *θ lies in $U^{\mathrm{op}}(L_0[|u|])$, where U^{op} is the opposite unipotent associated with μ;*
4. *The tensors $\tilde{s}_\alpha$ lie in $\mathrm{Fil}^0(D^{\otimes})$, where we equip D with the constant filtration $\mathrm{Fil}^\bullet D = \mathrm{Fil}^\bullet D_0 \otimes L_0[|u|]$.*

Then θ lies in $(U^{\mathrm{op}} \cap G)(L_0[|u|])$, and, for all α, $\tilde{s}_\alpha = s_{\alpha,0} \otimes 1$ (we are using the trivial identification of D with $D_0 \otimes L_0[|u|]$).

Proof. This is just a slight generalization of [Kis10, 1.5.6], and the proof of that result goes through for us verbatim. □

Let L/K_0 be a finite extension with residue field l, and let $L_0 = W(l)\left[\frac{1}{p}\right]$ be its maximal absolutely unramified sub-extension. Fix some uniformizer $\pi_L \in L$ and let S_L be the associated log W-algebra equipped with the Frobenius lift $\varphi : u \mapsto u^p$ (cf. 2.2.5). Let $J = \ker(S_L \to \mathscr{O}_K)$: this is a PD-ideal, and we have the PD-filtration $J^{[i]}$ given by the divided powers of J. Let $\hat{S}_L = \lim_{\leftarrow} S_L/J^{[i]}$ be the PD-completion of S_L along J: this inherits a log structure from S_L and also admits an embedding in $L_0[|u|]$.

Proposition 3.3.4.2. *Let $y : R \to \mathscr{O}_L$ be a continuous map of log W-algebras. Let A_y be the semi-stable abelian variety induced over L. Set $\mathcal{M}_y = \mathbb{D}(A_y)(S_L)$, and suppose that there exist φ-invariant, ∇-parallel tensors $\{\tilde{s}_\alpha\} \subset \mathcal{M}_y^{\otimes}$ lifting $\{s_\alpha \otimes 1\} \subset M_0 \otimes_W W(l)$ such that:*

1. *The pointwise stabilizer of $\{\tilde{s}_\alpha\}$ is a reductive sub-group $G_S \subset \mathrm{GL}(\mathcal{M}_y)$;*
2. *The tensors $\{\tilde{s}_\alpha\}$ reduce to tensors $\{s_{\alpha,\mathscr{O}_L}\} \subset \mathbb{D}(A_y)(\mathscr{O}_L)^{\otimes}$ defining a reductive sub-group $G_{\mathscr{O}_L} \subset \mathrm{GL}\left(\mathbb{D}(A_y)(\mathscr{O}_L)\right)$, and the Hodge filtration on $\mathbb{D}(A_y)(\mathscr{O}_L)$ is $G_{\mathscr{O}_L}$-split.*

Then:

1. *The induced map $y : R \to \mathscr{O}_L$ is G-admissible.*
2. *Suppose, in addition, that (3.3.3.5) holds; then the induced map of monoids $\mathbf{S}_{Q,\sigma} \to \left(\mathrm{M}_{\mathscr{O}_L}/l^\times\right)$ factors through $\mathbf{S}_{\sigma_G}$.*
3. *In particular, if (3.3.3.5) holds, and if the induced map $y_{\sigma,0} : R_\sigma \to \mathscr{O}_L/\mathfrak{m}_L$ of log W-algebras factors through R_{σ_G}, then y factors through R_G.*

Proof. Set $M_{y,0} = \mathcal{M}_y \otimes_{S_L} W(l)$. Note that $M_{y,0}$ is identified with $M_0 \otimes_W W(l)$ by (2.4.1.1)(3); so it makes sense to ask for lifts to $\mathcal{M}_y^\otimes$ of tensors in $(M_0 \otimes_W W(l))^\otimes$. In hypothesis (2), we are using notation and results from (2.3.2.3).

Let $\hat{\mathcal{M}}_y = \mathcal{M}_y \otimes_{S_L} \hat{S}_L$; by the argument in [Kis10, 1.5.8], we can find a G_S-split filtration $\mathrm{Fil}^1 \hat{\mathcal{M}}_y$ on $\mathcal{M}_y$ lifting both the $G_{W(l)}$-split filtration $\mathrm{Fil}^1 M_0 \otimes_W W(l)$ and the $G_{\mathscr{O}_L}$-split Hodge filtration $\mathrm{Fil}^1 \mathbb{D}(A_y)(\mathscr{O}_L)$. Applying (1.3.4.1), this gives us a deformation $[Y \xrightarrow{\tilde{\iota}_y} J^{\log}_{\tilde{y}}]$ over $\hat{S}_L$ of the log 1-motif $[Y \xrightarrow{\iota_y} J^{\log}_y]$ corresponding to A_y. So we have a map $\tilde{y} : R \to \hat{S}_L$ inducing this deformation (cf. 3.2.5.1).

Let $M_0 \otimes_W \hat{S}_L = \mathbb{D}(A) \otimes_R \hat{S}_L = \hat{\mathcal{M}}_y$ be the identification induced by $\tilde{y}$.

Claim 3.3.4.3. Under this identification, the tensors $\{s_{\alpha,0} \otimes 1\} \subset M_0^\otimes \otimes_W \hat{S}_L$ are identified with $\{\tilde{s}_\alpha\} \subset \mathcal{M}_y^\otimes$. In particular, they are φ-invariant and ∇-parallel.

Let $D_0 = M_0 \otimes_W L_0$, and let $D = \mathcal{M}_y \otimes_{S_L} L_0[|u|]$. D_0 is equipped with the filtration $\mathrm{Fil}^1 D_0 = \mathrm{Fil}^1 M_0 \otimes L_0$, and the lift $\tilde{y}$ gives us an identification $D = D_0 \otimes_{L_0} L_0[|u|]$. We note:

- The filtered φ-module D_0 is weakly admissible.
- The filtration $\mathrm{Fil}^1 D_0$ is G_{L_0}-split.
- For every α, $\tilde{s}_\alpha$ lies in $\mathrm{Fil}^0 D^\otimes$.
- The composition
$$D_0 \xrightarrow{\varphi_{D_0}^{-1}} \varphi^* D \xrightarrow{\varphi_D} D = D_0 \otimes_{L_0} L_0[|u|]$$
is an element of $U_F^{\mathrm{op}}(L_0[|u|])$.

Only the last assertion is not immediate. To see it, first observe that $\varphi_{D_0}^{-1} = \varphi_{M_0}^{-1} g_y$, for some $g_y \in U^{-2}_{\mathrm{wt},G}(L_0)$; this was observed above. Also above, we saw that $\varphi^* D$ had the factorization

$$\varphi_W^* M_0 \otimes_W L_0[|u|] \xrightarrow{\epsilon} \varphi_W^* M_0 \otimes_W L_0[|u|] \xrightarrow{\varphi_{M_0} \otimes 1} M_0 \otimes_W L_0[|u|] \xrightarrow{\tilde{y}^* g_+} M_0 \otimes_W L_0[|u|].$$

So it suffices to show that $(\varphi_{M_0} \otimes 1) \circ \epsilon \circ \varphi_{M_0}^{-1}$ lies in $U_F^{\mathrm{op}}(L_0[|u|])$. This follows from the argument in [Kis10, 1.5.3]: one just has to replace the derivations ∂t_i with their logarithmic analogues. Now, the hypotheses of (3.3.4.1) are valid and we see that $\tilde{s}_\alpha$ must equal $s_\alpha \otimes 1$: our claim is proven.

To prove assertion (1), it will be enough to show two things:

- The induced map $\tilde{y}^+ : R^+ \to \hat{S}_L$ factors through R_G^+.
- The monodromy N_y for A_y lies in $\mathbf{B}_{Q,G} \otimes \mathbb{Q}$.

Let us tackle the first statement. The map $\varphi_{\hat{\mathcal{M}}_y}$ is given by the composition

$$\varphi_{\hat{\mathcal{M}}_y} : \varphi_W^* M_0 \otimes_W \hat{S}_L \xrightarrow{\epsilon}_{\simeq} \varphi_W^* M_0 \otimes_W \hat{S}_L \xrightarrow{\tilde{y}^* \varphi_M} M_0 \otimes_W \hat{S}_L.$$

Here, the isomorphism ϵ is given by parallel transport (cf. 1.4.2). Since the tensors $\{\varphi_W^* s_{\alpha,0} \otimes 1\}$ are parallel for the connection on $\varphi^* \mathcal{M}_y$, they must be preserved by ϵ, and, since they are taken to $\{s_{\alpha,0} \otimes 1\}$ by $\varphi_{\hat{\mathcal{M}}_y}$, they must be taken to $\{s_{\alpha,0} \otimes 1\}$ by $\tilde{y}^* \varphi_M$ as well. But, by construction, $\tilde{y}^* \varphi_M$ is given as the following composition:

$$\varphi_W^* M_0 \otimes \hat{S}_L \xrightarrow{\varphi_{M_0} \otimes 1} M_0 \otimes \hat{S}_L \xrightarrow{\tilde{y}^* g^+} M_0 \otimes \hat{S}_L,$$

where $g^+ \in U_F^+(R)$ is the image of the universal element in $U_F^+(R^+)$. In particular, we see that $\tilde{y}^* g^+$ lies in $U_{F,G}^+$, which means precisely that $\tilde{y}|_{R^+}$ factors through R_G^+.

For the second statement, note that the residue at $u = 0$

$$\mathrm{res}_{\hat{S}_L} \nabla_{\hat{\mathcal{M}}_y} : M_{y,0} \to M_{y,0} \otimes \left(\mathrm{M}_{\hat{S}_L} / \hat{S}_L^\times \right) = M_{y,0}$$

of the logarithmic connection on $\hat{\mathcal{M}}_y$, is identified with a rational multiple of the monodromy N_y. Since the tensors $\{\tilde{s}_\alpha\}$ are parallel for $\nabla_{\hat{\mathcal{M}}_y}$, it follows that $N_y(s_{\alpha,0}) = 0$, for all α, and so $N_y \in \mathbf{B}_{Q,G} \otimes \mathbb{Q}$. Together with what we proved in the previous paragraph, this shows (1).

Suppose that (3.3.3.5) is valid. To show, as in assertion (2), that the map $\mathbf{S}_{Q,\sigma} \to \mathrm{M}_{\mathscr{O}_L} / l^\times$ of monoids factors through $\mathbf{S}_{\sigma_G}$, it is enough to show that the map $\mathbf{S}_Q \to \mathrm{M}_{\hat{S}_L}^{\mathrm{gp}} / l^\times$ of their group envelopes factors through $\mathbf{S}_{Q,G}$. We observe that the following square is cartesian:

$$\begin{array}{ccc} \mathrm{M}_{\hat{S}_L} / l^\times & \longrightarrow & \mathrm{M}_{\hat{S}_L} / W(l)^\times \\ \downarrow & & \downarrow \\ \mathrm{M}_{W(l)_{\mathbb{N}}} / l^\times & \longrightarrow & \mathbb{N} \end{array}$$

With this observation in hand, we can now finish the proof of (2) with the following two claims:

Claim 3.3.4.4. The induced map $\mathbf{S}_Q \to \mathrm{M}^{\mathrm{gp}}_{\hat{S}_L}/W(l)^\times$ factors through $\mathbf{S}_{Q,G}$.

Let $\Theta_y \in \mathrm{Lie}\, U_F^{\mathrm{op}} \otimes \hat{S}_L\, \mathrm{dlog}(u)$ be the connection matrix for $\mathcal{M}_y$; then, since the tensors $\{s_{\alpha,0} \otimes 1\}$ are parallel, Θ_y actually has values in $\mathrm{Lie}\, U^{\mathrm{op}}_{F,G}$. Recall that our choice of weight co-character w gave us a splitting

$$\mathrm{Lie}\, U^{\mathrm{op}}_{F,G} = \mathrm{Lie}\, U^{+}_{F,G} \oplus \mathrm{Lie}\, U^{-2}_{\mathrm{wt},G}.$$

By construction, the projection of Θ_y onto $\mathrm{Lie}\, U^{-2}_{\mathrm{wt},G} \otimes \hat{S}_L\, \mathrm{dlog}(u)$ is the matrix corresponding to the map

$$\mathbf{S}_Q \xrightarrow{\tilde{y}^\sharp_\sigma} \mathrm{M}^{\mathrm{gp}}_{\hat{S}_L} \longrightarrow \mathrm{M}^{\mathrm{gp}}_{\hat{S}_L}/W(l)^\times \xrightarrow[\mathrm{dlog}]{\subset} \hat{S}_L\, \mathrm{dlog}(u).$$

It follows that the composition of the first two maps factors through $\mathbf{S}_{Q,G}$. Note that we are using the rationality assumption (3.3.3.5) here. This finishes the proof of claim (3.3.4.4).

Claim 3.3.4.5. The induced map $\mathbf{S}_Q \to \mathrm{M}^{\mathrm{gp}}_{W(l)_{\mathbb{N}}}/l^\times$ factors through $\mathbf{S}_{Q,G}$.

Let us call this map $\bar{y}$. Let $\varphi_{\mathbb{N}}$ be the Frobenius lift on $W_{\mathbb{N}}$ induced from that on $\hat{S}_L$; by (1.4.3.1), this Frobenius lift gives us a splitting

$$\mathrm{M}^{\mathrm{gp}}_{W(l)_{\mathbb{N}}} = \mathrm{M}^{\mathrm{gp}}_{l_{\mathbb{N}}} \oplus (1 + pW(l))$$

compatible with the Teichmüller splitting $W(l)^\times = l^\times \oplus (1 + pW(l))$. This splitting is defined as follows: We define a map

$$\begin{aligned} \Phi : \mathrm{M}^{\mathrm{gp}}_{W(l)_{\mathbb{N}}} &\to 1 + pW(l) \\ m &\mapsto \varphi_{\mathbb{N}}(m)m^{-p}. \end{aligned}$$

The section $\mathrm{M}^{\mathrm{gp}}_{W(l)_{\mathbb{N}}} \to 1 + pW(l)$ associated with $\varphi_{\mathbb{N}}$ is now $(\Phi|_{1+pW(l)})^{-1} \circ \Phi$. Dividing by the sub-group $l^\times$, we obtain a splitting

$$\mathrm{M}^{\mathrm{gp}}_{W(l)_{\mathbb{N}}}/l^\times = (\mathrm{M}_{l_{\mathbb{N}}}/l^\times) \oplus (1 + pW(l)). \tag{3.3.4.5.1}$$

By (1), N_y lies in $\mathbf{B}_{Q,G} \otimes \mathbb{Q}$; moreover, a rational multiple of it corresponds to the map

$$\mathbf{S}_Q \to \mathrm{M}^{\mathrm{gp}}_{l_{\mathbb{N}}}/l^\times = \mathbb{Z}$$

induced from $\bar{y}$. This implies that $\mathbf{S}_Q \to \mathrm{M}^{\mathrm{gp}}_{l_{\mathbb{N}}}/l^\times$ must factor through $\mathbf{S}_{Q,G}$. So, to show that $\bar{y}$ factors through $\mathbf{S}_{Q,G}$, it is enough to show that its projection onto $1 + pW(l)$ via

the splitting (3.3.4.5.1) above also factors through $\mathbf{S}_{Q,G}$. Explicitly, we have to show that the map

$$\begin{aligned} f_y : \quad \mathbf{S}_Q &\longrightarrow 1 + pW(l) \\ m &\longmapsto \varphi_{\mathbb{N}}(\bar{y}(m))\bar{y}(m)^{-p} \end{aligned} \tag{3.3.4.5.2}$$

factors through $S_{Q,G}$. Let us now consider the map $\log(f_y) : \mathbf{S}_Q \to W(l)$: this corresponds to an element $U_y \in \operatorname{Lie} U^{-2}_{\mathrm{wt}} \otimes W(l)$ via the identification (cf. 3.2.1.1)

$$\operatorname{Hom}(\mathbf{S}_Q, W) = \mathbf{B}_Q \otimes W = \operatorname{Lie} U^{-2}_{\mathrm{wt}}.$$

Since our rationality assumption (3.3.3.5) is in force, to show that f_y factors through $\mathbf{S}_{Q,G}$, it is enough to show that U_y lies in $\operatorname{Lie} U^{-2}_{\mathrm{wt},G} \otimes W(l)$.

Now we observe that we have two *different* φ-semi-linear maps on $M_{y,0} = \mathcal{M}_y \otimes_{S_L} W(l)$. There is the map $\varphi_{M_0} \otimes 1$ obtained from the identification $M_0 \otimes_W W(l) = M_{y,0}$ induced by $\tilde{y}$, and there is also the map $\varphi_{M_{y,0}}$ induced from the reduction of $\varphi_{\mathcal{M}_y}$. Let us see how these two maps are related.

Let $\varphi_{\mathbb{N}}$ be as above, and let φ_P be the Frobenius lift on W_P induced from that on R_σ. Note that, by the choice of Frobenius lift on R_σ (cf. 3.2.6), $\varphi_P(m) = m^p$, for all m in the image of the map $\mathbf{S}_Q \to \mathrm{M}^{\mathrm{gp}}_{W_P}$. The map $\tilde{y} : R \to \hat{S}_L$ induces a map of log W-algebras $\tilde{y}_0 : W_P \to W(l)_{\mathbb{N}}$. Let $\tilde{y}_0^\sharp : \mathrm{M}_{W_P} \to \mathrm{M}_{W(l)_{\mathbb{N}}}$ be the induced map of monoids, and set

$$\begin{aligned} \Phi_y : \mathrm{M}^{\mathrm{gp}}_{W_P} &\to 1 + pW(l) \\ m &\mapsto \varphi_{\mathbb{N}}(\tilde{y}_0^\sharp(m))\tilde{y}_0^\sharp(\varphi_P(m))^{-1}. \end{aligned}$$

Note that the induced composition

$$\mathbf{S}_Q \to \mathrm{M}^{\mathrm{gp}}_{W_P} \xrightarrow{\Phi_y} 1 + pW(l)$$

is simply the map f_y considered in (3.3.4.5.2). This follows because $\varphi_P(m) = m^p$, for all m in the image of $\mathbf{S}_Q$. Therefore, by (1.4.3.5), we have:

$$\varphi_{M_{y,0}} = (\varphi_{M_0} \otimes 1) \circ (1 + U_y).$$

Note that the tensors $\{s_{\alpha,0} \otimes 1\}$, being the reductions of $\{\tilde{s}_\alpha\}$, are φ-invariant in $M^{\otimes}_{y,0}$, and so U_y must lie in $\operatorname{Lie} U^{-2}_{\mathrm{wt},G} \otimes_W W(l)$. This finishes the proof of claim (3.3.4.5) and hence also the proof of assertion (2).

Finally, for assertion (3), to show that y factors through R_G, we need to show that $y^+ = y|_{R^+}$ factors through R^+_G and that $y_\sigma = y|_{R_\sigma}$ factors through R_{σ_G}. The first of these conditions holds because of assertion (1), and, for the second, we only have to check that

the map $y_\sigma^\sharp : \mathbf{S}_{Q,\sigma} \to \mathrm{M}_{\mathscr{O}_L}$ of monoids factors through $\mathbf{S}_{\sigma_G}$. By (2), we already know that the induced map $\mathbf{S}_{Q,\sigma} \to \mathrm{M}_{\mathscr{O}_L}/l^\times$ factors through $\mathbf{S}_{\sigma_G}$. Observe that the following square is cartesian:

$$\begin{array}{ccc} \mathrm{M}_{\mathscr{O}_L} & \longrightarrow & \mathrm{M}_{\mathscr{O}_L/\mathfrak{m}_L} \\ \downarrow & & \downarrow \\ \mathrm{M}_{\mathscr{O}_L}/l^\times & \longrightarrow & \mathrm{M}_{\mathscr{O}_L}/\mathscr{O}_L^\times. \end{array}$$

So to show that $y_\sigma^\sharp$ factors through $\mathbf{S}_{\sigma_G}$ it is now enough to show that the induced map $\mathbf{S}_{Q,\sigma} \to \mathrm{M}_{\mathscr{O}_L/\mathfrak{m}_L}$ factors through $\mathbf{S}_{\sigma_G}$. But this is precisely the hypothesis of (3), and so we are done. □

Definition 3.3.4.6. A continuous map $y : R \to \mathscr{O}_L$ of log W-algebras is **strongly G-admissible** if there exists a diagram of log W-algebras

$$\begin{array}{ccc} R & \xrightarrow{\tilde{y}} & \hat{S}_L \\ & \searrow^{y} & \downarrow \\ & & \mathscr{O}_L \end{array}$$

such that $\tilde{y}$ satisfies the hypotheses of (3.3.4.2). In particular, a strongly G-admissible map is G-admissible.

A continuous map of log W-algebras $y : R \to \mathscr{O}_{\overline{K}_0}$ is **strongly G-admissible** if there is a finite extension L/K_0 inside $\overline{K}_0$, and a strongly G-admissible map $y' : R \to \mathscr{O}_L$ such that y factors as $R \xrightarrow{y'} \mathscr{O}_L \hookrightarrow \mathscr{O}_{\overline{K}_0}$.

Proposition 3.3.4.7 (Criterion for strong G-admissibility)**.** *Let $L \subset \overline{K}$ be a finite extension of K; let $y : R \to \mathscr{O}_L$ be a continuous map of log W-algebras, let A_y be the associated polarized semi-stable abelian variety over L, and let $\Lambda_y = H^1\left(A_{y,\overline{K}}, \mathbb{Z}_p\right)$. Suppose that we have Galois-invariant tensors $\{s_{\alpha,\text{ét},y}\} \subset \Lambda_y^\otimes$, and let $\{s_{\alpha,\mathrm{dR},y}\} \subset H^1_{\mathrm{dR}}(A_y)^\otimes$ be the corresponding tensors obtained via the p-adic de Rham comparison isomorphism. Suppose in addition that the parallel transport isomorphism*

$$H^1_{\mathrm{dR}}(A_y) \otimes_L L^{\log} \xrightarrow{\eta_{x,y}} H^1_{\mathrm{dR}}(A_x \otimes_K L) \otimes_L L^{\log} = H^1_{\mathrm{dR}}(A_x) \otimes_K L^{\log}$$

carries $\{s_{\alpha,\mathrm{dR},y} \otimes 1\}$ to $\{s_{\alpha,\mathrm{dR},x} \otimes 1\}$. Then y is strongly G-admissible, and is in particular G-admissible.

Proof. Let $\{s_{\alpha,\mathrm{st},y}\} \subset (D_{\mathrm{st}}(A_y)^{\otimes})^{N=0}$ be the tensors obtained from $\{s_{\alpha,\text{ét},y}\}$ via the functor D_{st}. By (2.3.2.3)(4), we obtain φ-invariant ∇-parallel tensors $\{s_{\alpha,S,y}\} \subset \mathbb{D}(A_y)(S_L)^{\otimes}$ reducing modulo u to $\{s_{\alpha,\mathrm{st},y}\}$ and defining a reductive group $G_S \subset \mathrm{GL}(\mathbb{D}(A_y)(S_L))$. Write $\mathcal{M}$ for $\mathbb{D}(A_y)(S_L)$: it is equipped with a polarization $\psi_{\mathcal{M}}$.

With the given hypotheses, and the commutativity of the diagram in (2.4.1.1)(6), the tensors $\{s_{\alpha,\mathrm{dR},y} \otimes 1\}$ map to the tensors $\{s_{\alpha,0} \otimes 1\}$ in $(M_0 \otimes_W L_0)^{\otimes}$. Moreover, under the isomorphism in (2.4.1.1)(3), the tensors $\{s_{\alpha,\mathrm{st},y}\}$ map to $\{s_{\alpha,0} \otimes 1\}$. In particular, the tensors $\{s_{\alpha,S,y}\}$ reduce to $\{s_{\alpha,0} \otimes 1\}$ in $(M_0 \otimes_W W(l))^{\otimes}$. The hypotheses of (3.3.4.2) are now satisfied and so y is strongly G-admissible. □

Definition 3.3.4.8. A quotient $\mathscr{O}$ of R is **strongly adapted to** G if it is is adapted to G (3.3.3.4) and satisfies in addition:

(4') Every element in $M(\mathscr{O})$ is *strongly* G-admissible.

(6) There exists a finite extension l/k such that the set

$$\{y \in M(\mathscr{O}) : \ y \text{ factors through } L \subset \overline{K}_0 \text{ with residue field } l\}$$

is Zariski dense in $(\mathrm{Spf}\, \mathscr{O})^{\mathrm{an}}$.

Remark 3.3.4.9. If $\mathscr{O}$ is the completion at a point of a flat, integral, finite-type scheme over W, then Condition (6) above is automatic. This is clear if $\mathscr{O}$ is formally smooth (in this case, even the W valued points in $M(\mathscr{O})$ will be dense in $\mathscr{O}$). In general, [dJ96, 2.13] shows that $\mathscr{O}$ admits a finite, injective map $f : \mathscr{O} \to \mathscr{O}'$, where $\mathscr{O}'$ is formally smooth over a (possibly ramified) extension $\mathscr{O}_L$ of W. Since, for some unramified extension L'/L, the $\mathscr{O}_{L'}$-valued points are dense in $\mathscr{O}'$, the result follows.

Proposition 3.3.4.10. *Suppose that there exists a quotient $\mathscr{O}$ of R strongly adapted to G, and suppose (3.3.3.5) holds. Let $\mathscr{O}^{\mathrm{norm}}$ be the normalization of $\mathscr{O}$. Then the map $R \to \mathscr{O}^{\mathrm{norm}}$ factors through $R_G = R_G^+ \hat{\otimes} R_{\sigma_G}$ and identifies $\mathscr{O}^{\mathrm{norm}}$ with R_G. In particular, $\mathscr{O}^{\mathrm{norm}}$ is (the completion of) a toric embedding over R_G^+ corresponding to the torus with co-character group $\mathbf{B}_{Q,G}$ and the rational polyhedral cone $\sigma_G \subset \mathbf{B}_{Q,G} \otimes \mathbb{R}$.*

Proof. Let l/k be a finite extension for which condition (6) of (3.3.4.8) is valid for $\mathscr{O}$; let $r = \#l^{\times}$. Consider the r-power map $r : \mathbf{S}_{Q,\sigma} \to \mathbf{S}_{Q,\sigma}$; this restricts to the r-power map on $\mathbf{S}_{\sigma_G}$. It induces a map $f_r : R^+[\mathbf{S}_{Q,\sigma}] \to R^+[\mathbf{S}_{Q,\sigma}]$, and localizing the target of this map at the point x_0, and the domain at the point $x_0 \circ f_r$, we obtain a finite flat map $f_r : R_r \to R$ of R^+-algebras. This induces a finite flat map $f_r|_{R_G} : R_{G,r} \to R_G$ of R_G^+-algebras

Choose an element $y : R \to \mathscr{O}_L$ in $M(\mathscr{O})$, for some finite extension L/K with residue field l. By hypothesis, such points are dense in $(\mathrm{Spf}\, \mathscr{O})^{\mathrm{an}}$. Since $\mathscr{O}$ is strongly adapted to G, y admits a lift $\tilde{y} : R \to \hat{S}_L$ satisfying the equivalent conditions of (3.3.4.2). Then y is G-admissible, and so $y|_{R^+}$ factors through R_G^+. By density of such points, it follows that the map $R^+ \to \mathscr{O}^{\mathrm{norm}}$, given by pre-composition of $R \to \mathscr{O}^{\mathrm{norm}}$ with the inclusion $\iota^+ : R^+ \to R$, factors through R_G^+.

By (3.3.4.2)(2), it follows that the induced map $\mathbf{S}_{Q,\sigma} \to \mathrm{M}_{\mathscr{O}_L}/l^{\times}$ factors through $\mathbf{S}_{\sigma_G}$. In particular, the composition $y \circ f_r : R_r \to \mathscr{O}_L$ of log W-algebras factors through $R_{G,r}$. By density of such points, we conclude that the composition $R_r \xrightarrow{f_r} R \to \mathscr{O}^{\mathrm{norm}}$ factors through $R_{G,r}$. In particular, both $\operatorname{Spf} \mathscr{O}^{\mathrm{norm}}$ and $\operatorname{Spf} R_G$, being of the same dimension (cf. 3.3.1.1), are normalizations of irreducible components of $\operatorname{Spec}(R_{G,r} \otimes_{R_r} R)$. This already tells us that $\mathscr{O}^{\mathrm{norm}}$ is the complete local ring of a toric embedding over R_G^+.

By condition (5) of (3.3.3.4), the point x factors through $\mathscr{O}^{\mathrm{norm}}$. We claim that x factors through R_G as well. Indeed, by the choice we made in (3.2.2.2)(4), the map of monoids $x^{\sharp}_{\sigma,0} : \mathbf{S}_{Q,\sigma} \to \mathrm{M}_{\mathscr{O}_K/\mathfrak{m}_K}$ factors through $\mathbf{S}_{\sigma_G}$. The hypotheses of (3.3.4.7) are tautologically true for x, and so x is strongly G-admissible. By (3.3.4.2)(3), it then follows that x factors through R_G.

It now follows that $\operatorname{Spec} R_G$ and $\operatorname{Spec} \mathscr{O}^{\mathrm{norm}}$ must map onto the same irreducible component of $\operatorname{Spec}(R_{G,r} \otimes_{R_r} R)$ and must therefore be identified. $\square$

CHAPTER 4
COMPACTIFICATIONS OF INTEGRAL MODELS OF SHIMURA VARIETIES

4.1 Shimura varieties and absolute Hodge cycles

This is essentially a resumé of the first part of [Kis10, §2].

4.1.1

Definition 4.1.1.1. A **Shimura datum** is a pair (G, X), where G is a connected reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms

$$h : \mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$$

satisfying:

1. The composite
$$\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}(G))$$
defines a Hodge structure of type $(-1,1),(0,0),(1,-1)$ on $\mathrm{Lie}(G)$;
2. $h(i)$ is a Cartan involution of $G_{\mathbb{R}}$;
3. G^{ad} has no $\mathbb{Q}$-simple factors whose $\mathbb{R}$-points form a compact group.

A map $\iota : (G_1, X_1) \to (G_2, X_2)$ of Shimura data consists of a map $\iota : G_1 \to G_2$ of $\mathbb{Q}$-groups inducing a map $X_1 \to X_2$ over $\mathbb{R}$. It is an **embedding** if the underlying map of groups is a closed embedding.

Let $\mathbb{A}_f$ be the ring of finite adéles, let $K \subset G(\mathbb{A}_f)$ be a compact open sub-group of the adélic points of G. We will write $K = K^p K_p$, where $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p \subset \mathbb{A}_f$ denotes the sub-ring of adéles with trivial p-component.

By results of Baily-Borel, Shimura, Deligne, Milne, Borovoi and others (see [Mil90, §4.5]), the double coset space

$$\mathrm{Sh}_K(G,X)_{\mathbb{C}} = G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/K$$

has the natural structure of an algebraic variety over $\mathbb{C}$ with a canonical model $\mathrm{Sh}_K(G,X)$ over a number field $E(G,X)$ (the **reflex field**), which depends only on the Shimura datum (G,X).

Lemma 4.1.1.2. *Let $\iota : (G_1, X_1) \hookrightarrow (G_2, X_2)$ be an embedding of Shimura data, let $K_{2,p} \subset G_1(\mathbb{Q}_p)$ be a compact open sub-group, and let $K_{1,p} = K_{2,p} \cap G_2(\mathbb{Q}_p)$. For any compact open sub-group $K_1^p \subset G_1(\mathbb{A}_f^p)$, we can find a compact open sub-group $K_2^p \subset G_2(\mathbb{A}_f^p)$ containing K_1^p such that ι induces an embedding*

$$\mathrm{Sh}_{K_1^p K_{1,p}}(G, X) \hookrightarrow \mathrm{Sh}_{K_2^p K_{2,p}}(G, X)$$

defined over $E(G_1, X_1)$.

Proof. This is [Kis10, 2.1.2]. □

Definition 4.1.1.3. Let V be a $\mathbb{Q}$-vector-space equipped with a symplectic form ψ. The **Siegel Shimura datum** associated to (V, ψ) is the pair $(\mathrm{GSp}(V, \psi), S^{\pm})$, where $S^{\pm}$ is the $\mathrm{GSp}(V, \psi)(\mathbb{R})$-conjugacy class of maps $h : \mathbb{S} \to \mathrm{GSp}(V, \psi)_{\mathbb{R}}$ such that:

1. h induces a Hodge structure of type $(1, 0), (0, 1)$ on V, so that we have a corresponding decomposition
$$V_{\mathbb{C}} = V_h^{1,0} \oplus V_h^{0,1};$$
2. The symmetric form $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

The reflex field of a Siegel Shimura datum is $\mathbb{Q}$.

4.1.2

Let $(\mathrm{GSp}, S^{\pm})$ be a Siegel Shimura datum associated to (V, ψ), and let $K = K^p K_p \subset \mathrm{GSp}(\mathbb{A}_f)$ be a compact open sub-group. For K^p sufficiently small, $\mathrm{Sh}_K(\mathrm{GSp}, S^{\pm})$ can be interpreted as the fine moduli space of polarized abelian varieties with level structure. To be more precise, we fix some $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ such that ψ restricts to a bilinear form on $V_{\mathbb{Z}}$ and such that $V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$ is stable under K. For any abelian variety A over an algebraically closed field k, let $H^1(A, \hat{\mathbb{Z}}) = \prod_{l \text{ prime}} H^1_{\text{ét}}(A, \mathbb{Z}_l)$. Then, for any algebraically closed extension $k/\mathbb{Q}$, $\mathrm{Sh}_K(\mathrm{GSp}, S^{\pm})(k)$ parametrizes tuples (A, λ, η), where

- A is an abelian variety;
- λ is a polarization of A;
- η is a K-orbit of isomorphisms
$$(V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}, \psi \otimes 1) \xrightarrow{\simeq} (H^1(A, \hat{\mathbb{Z}}), \psi_\lambda)$$
that respect polarizations up to a $\hat{\mathbb{Z}}^\times$-multiple. Here, the right hand side is equipped with the alternating form ψ_λ induced by the Weil pairing and the polarization λ.

For more details, see [Del71, §4] or [RZ96, §6]. We see therefore that, for K^p sufficiently small, there exists a universal abelian scheme $\mathcal{A}$ over $\mathrm{Sh}_K(\mathrm{GSp}, S^{\pm})$.

Definition 4.1.2.1. A Shimura datum (G, X) is of **Hodge type** if it admits an embedding

$$(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$$

into a Siegel Shimura datum.

4.1.3

Let (G, X) be a Shimura datum of Hodge type equipped with an embedding

$$(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm}).$$

Let $K = K^p K_p \subset G(\mathbb{A}_f)$ be a compact open subgroup. By (4.1.1.2), we can find $K' \subset \mathrm{GSp}(\mathbb{A}_f)$ containing K such that the map $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ is an embedding defined over $E = E(G, X)$. Moreover, we can ensure that K^p and K'^p are sufficiently small, and fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ as above, so that $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ admits an interpretation as a fine moduli space of polarized abelian schemes with level structure. Let $h : \mathcal{A} \to \mathrm{Sh}_K(G, X)$ be the pull-back of the universal family of abelian varieties over $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$.

Suppose that we have a finite collection of tensors $\{s_{\alpha,B}\} \subset V^{\otimes}$ whose pointwise stabilizer in GSp is G. Let $\mathcal{V} = H^1_{\mathrm{dR}}(\mathcal{A}/\mathrm{Sh}_K(G, X))$ be the first relative de Rham cohomology of $\mathcal{A}$ over $\mathrm{Sh}_K(G, X)$: this is a vector bundle with flat connection over $\mathrm{Sh}_K(G, X)$. From [Kis10, §2.2], we see that the tensors $\{s_{\alpha,B}\}$, via the de Rham isomorphism, give rise to parallel tensors $\{s_{\alpha,\mathrm{dR}}\} \subset \mathcal{V}^{\otimes}$. Moreover, for any field extension κ of E, any point $x \in \mathrm{Sh}_K(G, X)(\kappa)$, and any choice of algebraic closure $\overline{\kappa}$ of κ, we get a $\mathrm{Gal}(\overline{\kappa}/\kappa)$-invariant tensor $s_{\alpha,\text{ét},x} \in H^1_{\text{ét}}(\mathcal{A}_{x,\bar{\kappa}}, \mathbb{Q}_p)^{\otimes}$. Given any choice of embeddings $\sigma : \bar{\kappa} \hookrightarrow \mathbb{C}$ and $\iota : \mathbb{Q}_p \hookrightarrow \mathbb{C}$, under the isomorphisms

$$H^1_{\mathrm{dR}}(\mathcal{A}_x) \otimes_{\kappa,\sigma} \mathbb{C} \xrightarrow{\simeq} H^1(\mathcal{A}_{x,\sigma}(\mathbb{C}), \mathbb{C}) \xrightarrow{\simeq} H^1_{\text{ét}}(\mathcal{A}_{x,\bar{\kappa},\sigma}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p,\iota} \mathbb{C},$$

$s_{\alpha,\mathrm{dR},x}$ is carried to $s_{\alpha,\text{ét},x}$. This is a consequence of the main result of [DMOS82]: 'Hodge implies absolutely Hodge for abelian varieties over $\mathbb{C}$'.

We also have one additional piece of compatibility between $s_{\alpha,\mathrm{dR},x}$ and $s_{\alpha,\text{ét},x}$. For this, consider the case where κ is a finite extension of E_v, the completion at v for some place $v|p$ of E. Then we also have the p-adic comparison isomorphism

$$H^1_{\mathrm{dR}}(\mathcal{A}_x) \otimes_{\kappa} B_{\mathrm{dR}} \xrightarrow{\simeq} H^1_{\text{ét}}(\mathcal{A}_{x,\bar{\kappa}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

Proposition 4.1.3.1. *Under the p-adic comparison isomorphism above, $s_{\alpha,\mathrm{dR},x}$ is carried to $s_{\alpha,\text{ét},x}$.*

Proof. This is essentially the main result of [Bla94], which applies directly when $\mathcal{A}_x$ is in fact defined over a number field. For the generality we need, as pointed out in [Moo98, 5.6.3], we can either appeal to a trick of Lieberman as in [Vas99, 5.2.16], or we can directly use the fact that $\mathcal{A}_x$ arises from the family $\mathcal{A}$ defined over the number field E. □

4.2 Toroidal compactifications of integral canonical models

4.2.1

Let $(\mathrm{GSp}, S^{\pm})$ be a Siegel Shimura datum associated with a symplectic space (V, ψ). Suppose that $K_p \subset \mathrm{GSp}(\mathbb{Q}_p)$ is a hyperspecial sub-group. For us, this means that $K_p = \mathrm{GSp}(V_{\mathbb{Z}_p}, \psi)$, where $V_{\mathbb{Z}_p} = V_{\mathbb{Z}_{(p)}} \otimes \mathbb{Z}_p$, for a $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V$ such that ψ induces a $\mathbb{Z}_{(p)}$-valued symplectic form on $V_{\mathbb{Z}_{(p)}}$. Let $K^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ be compact open sub-group such that $K = K^p K_p \subset \mathrm{GSp}(\mathbb{A}_f)$ is **neat**[1]. For example, suppose that we fix an $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Z}_{(p)}}$ stable under the pairing ψ and such that the pairing is perfect on $V_{\mathbb{Z}}\left[\frac{1}{r}\right]$, for some $r \in \mathbb{Z}_{>0}$. Choose N such that $(N, pr) = 1$; then it makes sense to consider the compact open sub-group $\prod_{l \nmid r} \mathrm{GSp}(V_{\mathbb{Z}_l})$ of $\mathrm{GSp}(\mathbb{A}_f)$, and within it the congruence sub-group $K(N)$ given by

$$K(N) = \ker\left(\prod_{l \nmid r} \mathrm{GSp}(V_{\mathbb{Z}_l}) \to \mathrm{GSp}(V \otimes \mathbb{Z}/N\mathbb{Z})\right).$$

By Serre's lemma, it follows that $K(N)$ is a neat sub-group for $N \geq 3$, and we clearly have $K(N)_p = K_p$. In particular, any compact open sub-group $K = K^p K_p$ contained in $K(N)$ for some N will be neat.

Fix a choice of $V_{\mathbb{Z}} \subset V_{\mathbb{Z}_{(p)}}$ and a neat sub-group $K = K(N)$ as above. Then, the moduli problem represented by $\mathrm{Sh}_K(\mathrm{GSp}, S^{\pm})$ over $\mathbb{Q}$ (cf. 4.1.2) extends naturally to a moduli problem over $\mathbb{Z}_{(p)}$ that is representable by a smooth $\mathbb{Z}_{(p)}$-scheme $\mathscr{S} = \mathscr{S}_K(\mathrm{GSp}, S^{\pm})$ (cf. [Kis10, 2.3.3]).

The moduli scheme $\mathscr{S}$ is not compact, and there arises the problem of finding a good compactification for it. This problem was solved by Faltings and Chai in [FC90], but [Lan08] will be a better source of precise statements for us. To explain these results, we will need a fresh panoply of definitions. It might be helpful at this point to skim over their relatives in (1.2.3).

Set $V_{\mathbb{Z}/N\mathbb{Z}} = V_{\mathbb{Z}} \otimes (\mathbb{Z}/N\mathbb{Z})$.

Definition 4.2.1.1. A **cusp label** Φ **for** $(V_{\mathbb{Z}}, \psi)$ **at level** K (cf. [Lan08, 5.4.1]) is a tuple $(Y, X, \lambda^{\text{ét}}, \Psi_{N,r}, \varphi_N^{\text{ét}}, \varphi_N^{\text{mult}}, \delta)$, where:

1. cf [Lan08, 1.4.1.8].

1. Y and X are free $\mathbb{Z}$-modules of rank r and $\lambda^{\text{ét}} : Y \to X$ is an injective map of groups.

2. $\Psi_{N,r} \subset V_{\mathbb{Z}/N\mathbb{Z}}$ is a free isotropic $\mathbb{Z}/N\mathbb{Z}$-sub-module such that the quotient $V_{\mathbb{Z}/N\mathbb{Z}}/\Psi_{N,r}$ is again free over $\mathbb{Z}/N\mathbb{Z}$.

3. $$\varphi_N^{\text{ét}} : V_{\mathbb{Z}/N\mathbb{Z}}/\Psi_{N,r}^{\perp} \xrightarrow{\simeq} Y/NY;$$
$$\varphi_N^{\text{mult}} : \Psi_{N,r} \xrightarrow{\simeq} \mathrm{Hom}(X, \mathbb{Z}/N\mathbb{Z})$$
are isomorphisms of groups such that the pairing
$$\left(V_{\mathbb{Z}/N\mathbb{Z}}/\Psi_{N,r}^{\perp}\right) \times \Psi_{N,r} \xrightarrow{\varphi_N^{\text{ét}} \times \varphi_N^{\text{mult}}} Y/NY \times \mathrm{Hom}(X, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\langle \lambda^{\text{ét}}(\cdot), \cdot \rangle} \mathbb{Z}/N\mathbb{Z}$$
is equal to the perfect pairing induced from ψ.

4. δ is a symplectic splitting of the filtration
$$0 \subset \Psi_{N,r} \subset \Psi_{N,r}^{\perp} \subset V_{\mathbb{Z}/N\mathbb{Z}}.$$

There is a large collection of objects associated with a cusp label Φ:

- Choose any isotropic free and co-free $\mathbb{Z}$-sub-module $\Psi_r \subset V_{\mathbb{Z}}$ whose reduction modulo N is $\Psi_{N,r}$. Then the induced alternating form on $V_{\mathbb{Z}}^{\Phi} = \Psi_r^{\perp}/\Psi_r$ is non-degenerate after extending scalars to $\mathbb{Q}$. Let us take $K^{\Phi} \subset \mathrm{GSp}(V_{\mathbb{A}}^{\Phi})$ to be
$$K^{\Phi} = \ker\left(\prod_{l \nmid r} \mathrm{GSp}(V_{\mathbb{Z}_l}^{\Phi}) \to \mathrm{GSp}(V_{\mathbb{Z}}^{\Phi} \otimes \mathbb{Z}/N\mathbb{Z})\right).$$
Then we can consider the Shimura variety $\mathrm{Sh}_{K^{\Phi}}(\mathrm{GSp}(V^{\Phi}), S^{\pm})$ and its model $\mathscr{S}_{\Phi} = \mathscr{S}_{K^{\Phi}}(\mathrm{GSp}(V^{\Phi}), S^{\pm})$ over $\mathbb{Z}_{(p)}$: the latter is a fine moduli space over $\mathbb{Z}_{(p)}$ for polarized abelian schemes (B, λ^{ab}) with level N structure of type $\Psi_{N,r}^{\perp}/\Psi_{N,r}$ (cf. 1.2.3.1). It is shown in [Lan08, 5.2.7.5] that the space $\mathscr{S}_{\Phi}$ and the moduli problem it represents are independent of the choice of Ψ_r.

- Let $(\mathcal{B}, \lambda^{\text{ab}})$ be the universal polarized abelian scheme over $\mathscr{S}_{\Phi}$. Consider the $\mathscr{S}_{\Phi}$-scheme:
$$\ddot{\mathscr{P}}_{\Phi} = \underline{\mathrm{Hom}}\left(\frac{1}{N}Y, \mathcal{B}\right) \times_{\underline{\mathrm{Hom}}(Y, \mathcal{B}^{\vee})} \underline{\mathrm{Hom}}\left(\frac{1}{N}X, \mathcal{B}^{\vee}\right).$$

This is the fiber product of the diagram:

$$\begin{array}{ccc} & & \underline{\mathrm{Hom}}\left(\frac{1}{N}Y, \mathcal{B}\right) \\ & & \downarrow \\ \underline{\mathrm{Hom}}\left(\frac{1}{N}X, \mathcal{B}^\vee\right) & \longrightarrow & \underline{\mathrm{Hom}}(Y, \mathcal{B}^\vee), \end{array}$$

where the vertical arrow is restriction followed by post-composition with λ^{ab}, and the horizontal arrow is pull-back along the map $Y \xrightarrow{\lambda^{\text{ét}}} X \hookrightarrow \frac{1}{N}X$. $\ddot{\mathscr{P}}_\Phi$ is a smooth, proper group scheme over $\mathscr{S}_\Phi$. It is shown in [Lan08, 6.2.3.4] that there is a natural map

$$\partial : \ddot{\mathscr{P}}_\Phi \to \underline{\mathrm{Hom}}\left(\frac{1}{N}Y/Y, \mathcal{B}[N]\right)$$

of group schemes whose fibers are abelian schemes over $\mathscr{S}_\Phi$. The cusp label Φ gives us a distinguished element b_Φ in the image of ∂ (cf. [Lan08, 6.2.3.1]). Let $\mathscr{P}_\Phi$ be the fiber of ∂ over b_Φ: this is an abelian scheme over $\mathscr{S}_\Phi$.

- Over $\mathscr{P}_\Phi$, we have the tautological maps

$$c_{N,\Phi} : \frac{1}{N}Y \to \mathcal{B};$$

$$c^\vee_{N,\Phi} : \frac{1}{N}X \to \mathcal{B}^\vee.$$

Let $c_\Phi = c_{N,\Phi}|_Y$ and let $c^\vee_\Phi = c^\vee_{N,\Phi}|_X$. Set

$$I_{N,\Phi} = (c_{N,\Phi} \times c^\vee_\Phi)^* \mathcal{P}_{\mathcal{B}}^{-1};$$

$$I_\Phi = (c_\Phi \times c^\vee_\Phi)^* \mathcal{P}_{\mathcal{B}}^{-1}.$$

Then $I_{N,\Phi}$ is a $\mathbb{G}_m$-bi-extension of $\frac{1}{N}Y \times X$ over $\mathscr{P}_\Phi$ (cf. discussion before (1.2.2.3)), and I_Φ is a $\mathbb{G}_m$-bi-extension of $Y \times X$ over $\mathscr{P}_\Phi$ such that $(1 \times \lambda^{\text{ét}})^* I_\Phi$ is a symmetric $\mathbb{G}_m$-bi-extension of $Y \times Y$ (cf. discussion before (1.2.2.8)).

- We have the groups

$$\mathbf{B}_\Phi = \frac{1}{N}\mathbf{B}_{\lambda^{\text{ét}}}; \qquad \mathbf{S}_\Phi = \frac{1}{N}\mathbf{S}_{\lambda^{\text{ét}}},$$

as defined in (3.1.2). We also have the open convex cone

$$\mathbf{C}_\Phi = \mathbf{C}_{\lambda^{\text{ét}}} \subset \mathbf{B}_\Phi \otimes \mathbb{R}$$

as defined in (3.1.2.1.1).

- Let $\mathbf{E}_\Phi$ be the torus over $\mathscr{P}_\Phi$ with character group $\mathbf{S}_\Phi$. We then have the $\mathbf{E}_\Phi$-torsor $\mathbf{\Xi}_\Phi$ over $\mathscr{P}_\Phi$, whose points over any $\mathscr{P}_\Phi$-scheme C are given by:

$$\mathbf{\Xi}_\Phi(C) = \begin{pmatrix} \text{Trivializations } \tau_N : \mathbf{1}_{Y\times X} \xrightarrow{\simeq} I_{N,\Phi} \text{ over } C \\ \text{of } \mathbb{G}_m\text{-bi-extensions of } Y \times X \\ \text{inducing a symmetric trivialization of} \\ \text{the symmetric } \mathbb{G}_m\text{-bi-extension } (1\times\lambda^{\text{ét}})^* I_\Phi \text{ of } Y \times Y. \end{pmatrix}$$

- We have the group

$$\Gamma_\Phi = \left\{ (\gamma_Y, \gamma_X) \in \mathrm{GL}(Y)(N) \times \mathrm{GL}(X)(N) : \lambda^{\text{ét}} = \gamma_X \lambda^{\text{ét}} \gamma_Y \right\},$$

  where $\mathrm{GL}(Y)(N)$ (resp. $\mathrm{GL}(X)(N)$) is the group of automorphisms of Y (resp. X) that act trivially on Y/NY (resp. X/NX).

Definition 4.2.1.2. A **smooth, admissible, rational, polyhedral cone decomposition** Σ_Φ (cf. [Lan08, 6.1.1.14]) associated with a cusp label Φ is a collection $\{\sigma_\alpha\}_{\alpha\in\Pi}$ such that

1. For each $\alpha \in \Pi$, σ_α is a non-degenerate rational polyhedral cone in $\mathbf{C}_\Phi$, smooth with respect to the lattice $\mathbf{B}_\Phi$.
2. $\mathbf{C}_\Phi$ is the disjoint union of the σ_α. For each $\alpha \in \Pi$, the closure of σ_α in $\mathbf{C}_\Phi$ is a disjoint union of certain σ_β with $\beta \in \Pi$.
3. For any $g \in \Gamma_\Phi$ and any $\alpha \in \Pi$, $g\sigma_\alpha = \sigma_\beta$, for some $\beta \in \Pi$, and the action of Γ_Φ on $\{\sigma_\alpha\}$ has only finitely many orbits.

Given such a decomposition Σ_Φ associated with Φ and a cone σ_α within it, we can consider the monoid

$$\mathbf{S}_{\Phi,\sigma} = \sigma^\vee \cap \mathbf{S}_\Phi,$$

where

$$\sigma_\alpha^\vee = \{n \in \mathbf{S}_\Phi \otimes \mathbb{R} : \langle n, s\rangle \geq 0, \text{ for all } s \in \sigma\}.$$

Let $\mathbf{E}_{\Phi,\sigma} = \underline{\mathrm{Spec}}\mathscr{O}_{\mathscr{P}_\Phi}[\mathbf{S}_{\Phi,\sigma}]$: this gives us a torus embedding

$$\mathbf{E}_\Phi \hookrightarrow \mathbf{E}_{\Phi,\sigma}$$

over $\mathscr{P}_\Phi$. We set

$$\mathbf{\Xi}_{\Phi,\sigma} = \mathbf{\Xi}_\Phi \times^{\mathbf{E}_\Phi} \mathbf{E}_{\Phi,\sigma} :$$

This is a log scheme over $\mathscr{P}_\Phi$ in the evident way with the log structure induced by the divisor that is the complement of $\mathbf{\Xi}_{\Phi,\sigma}$. Moreover, the tautological trivialization τ of the

$\mathbb{G}_m$-bi-extension I_Φ of $Y \times X$ over $\mathbf{\Xi}_\Phi$ extends to a trivialization τ of the induced $\mathbb{G}_m^{\log}$-bi-extension $I_\Phi^{\log}$ over $\mathbf{\Xi}_{\Phi,\sigma}$ (cf. discussion above (1.2.2.3). This means that we have a tautological tuple

$$(\mathcal{B}, Y, X, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}}, c_\Phi, c_\Phi^\vee, \tau)$$

over $\mathbf{\Xi}_{\Phi,\sigma}$ satisfying the conditions of (1.2.2.8), and so we have a tautological polarized log 1-motif $(Q_{\Phi,\sigma}, \lambda_{\Phi,\sigma})$ over $\mathbf{\Xi}_{\Phi,\sigma}$. It is also evident from the construction that this polarized log 1-motif has a tautological level N structure $\alpha_{\Phi,\sigma}$ of type $(V_{\mathbb{Z}/N\mathbb{Z}}, \Psi_{N,r})$ (cf. 1.2.3.3). The stratification of $\mathbf{E}_{\Phi,\sigma}$ by orbits of $\mathbf{E}_\Phi$ gives rise to a stratification on $\mathbf{\Xi}_{\Phi,\sigma}$ as well. There is a unique closed stratum; let $\mathfrak{X}_{\Phi,\sigma}$ be the formal scheme obtained by completing $\mathbf{\Xi}_{\Phi,\sigma}$ along this closed stratum.

We now direct the reader to [Lan08, 6.3.3.4] for the notion of a **compatible choice** of smooth admissible rational polyhedral cone decomposition data Σ associated with $(V_\mathbb{Z}, \psi, K)$: this involves choosing enough cusp labels Φ for $(V_\mathbb{Z}, \psi, K)$, and choosing compatibly for each cusp label Φ a smooth admissible rational polyhedral cone decomposition Σ_Φ associated with Φ.

Theorem 4.2.1.3 (Faltings-Chai,Lan)**.** *For a compatible choice of admissible smooth rational polyhedral cone decomposition data Σ (cf* [Lan08, 6.3.3.4]*), there exists a proper smooth $\mathbb{Z}_{(p)}$-scheme $\overline{\mathscr{S}}_\Sigma$ into which $\mathscr{S}$ embeds as an open dense sub-scheme. It satisfies:*

1. *The complement $D_\Sigma = \overline{\mathscr{S}}_\Sigma \setminus \mathscr{S}$, viewed as a closed reduced sub-scheme of $\overline{\mathscr{S}}_\Sigma$, is a Cartier divisor with normal crossings. More precisely:*

   - *$\overline{\mathscr{S}}_\Sigma$ admits a decomposition*

   $$\overline{\mathscr{S}}_\Sigma = \bigsqcup_{[(\Phi,\sigma)]} Z_{[(\Phi,\sigma)]}$$

   into locally closed strata indexed by equivalence classes of pairs (Φ, σ) (under a certain equivalence relation; cf. [Lan08, 6.2.6.1]*: in particular, for fixed Φ, (Φ, σ) and (Φ, σ') can be equivalent if and only if there exists $g \in \Gamma_\Phi$ such that $g\sigma = \sigma'$), where Φ is a cusp label belonging to the compatible choice and σ is a cone in Σ_Φ.*
   - *There is an incidence relation between equivalence classes of pairs (Φ, σ), and $Z_{[(\Phi,\sigma)]}$ is in the closure of $Z_{[(\Phi',\sigma')]}$ if and only if $\left[(\Phi', \sigma')\right]$ is a face of $[(\Phi, \sigma)]$.*
   - *The formal completion of $\overline{\mathscr{S}}_\Sigma$ along the $[(\Phi, \sigma)]$-stratum is isomorphic to $\mathfrak{X}_{\Phi,\sigma}$, which was constructed just above; this does not depend on the choice of (Φ, σ) representing the equivalence class.*

2. *The universal polarized abelian scheme $\pi : (\mathcal{A}, \lambda) \to \mathscr{S}$ extends to a polarized semi-abelian scheme $\pi_\Sigma : (\mathcal{A}_\Sigma, \lambda) \to \overline{\mathscr{S}}_\Sigma$. For any equivalence class $[(\Phi, \sigma)]$ indexing a stratum $Z_{[(\Phi,\sigma)]}$ of $\overline{\mathscr{S}}_\Sigma$, the pull-back of $(\mathcal{A}, \lambda)$ over the complement of the boundary in $\mathfrak{X}_{\Phi,\sigma}$ corresponds to the tautological polarized log 1-motif $(Q_{\Phi,\sigma}, \lambda_{\Phi,\sigma})$ over $\mathfrak{X}_{\Phi,\sigma}$ via the equivalence in (1.2.4.2).*

3. *Let $s_0 \in \overline{\mathscr{S}}_\Sigma$ be a closed point, and let $Z_{[(\Phi,\sigma)]}$ be the minimal stratum in which it lies. Let $\mathscr{O}_{s_0} := \mathscr{O}_{\overline{\mathscr{S}}_\Sigma, s_0}$ be the complete local ring of $\overline{\mathscr{S}}_\sigma$ at s_0 equipped with the log structure induced from the boundary divisor. Let (Q_{s_0}, λ_{s_0}) be the reduction to $k(s_0)$ of the natural polarized positive log 1-motif over $\mathscr{O}_{s_0}$, induced from that on $\mathfrak{X}_{(\Phi,\sigma)}$. Here, we equip $k(s_0)$ with the log structure induced from $\mathscr{O}_{s_0}$. Then $\mathscr{O}_{s_0}$ is the universal deformation ring for the log 1-motif (Q_{s_0}, λ_{s_0}) over $k(s_0)$ (cf. (3.1.1.6),(3.1.3)).*

4. *Let $\mathscr{O}_L$ be a complete discrete valuation $\mathbb{Z}_{(p)}$-algebra with residue field l and fraction field L. Let (A, λ) be a polarized abelian variety over L arising from a point in $\tilde{x} : \mathscr{S}(L)$. Let $x \in \overline{\mathscr{S}}_\Sigma(\mathscr{O}_L)$ be the point obtained from $\tilde{x}$ via properness of $\overline{\mathscr{S}}_\Sigma$, and let $x_0 \in \overline{\mathscr{S}}_\Sigma(l)$ be its reduction. The minimal stratum $Z_{[(\Phi,\sigma)]}$ of $\overline{\mathscr{S}}_\Sigma$ containing x_0 is determined as follows: By (1.2.4.2), (A, λ) corresponds to a tuple*

$$(B, Y, X, c, c^\vee, \lambda^{\mathrm{ab}}, \lambda^{\text{ét}}, \tau).$$

This tuple, along with the level N structure on A, determines a cusp label

$$\Phi = (Y, X, \lambda^{\text{ét}}, \Psi_{N,r}, \phi_N^{\text{ét}}, \phi_N^{\mathrm{mult}}, \delta)$$

that is isomorphic to a cusp label in the compatible choice of admissible cone decomposition data. Once Φ is determined, the cone σ is determined to be the minimal cone in the decomposition Σ_Φ such that the monodromy N_A (cf. 2.3) of (A, λ) lies in σ.

5. *π can be extended to a compactification $\overline{\pi} : \overline{\mathcal{A}} \to \overline{\mathscr{S}}_\Sigma$ such that the complement of $\mathcal{A}$ in $\overline{\mathcal{A}}$ is a relative normal crossings divisor over $\overline{\mathscr{S}}_\Sigma$.*

Proof. (1), (2) and (4) follow from [Lan08, 6.4.1.1], and (5) follows from [Lan10a, 2.15]; cf. also [FC90, VI.1.1]. As for (3), it can be directly deduced by comparing the construction of $\mathfrak{X}_{(\Phi,\sigma)}$ with that of the deformation space for (Q_{s_0}, λ_{s_0}) in §3.1: cf. especially (3.1.3.4). □

4.2.2

Let us start with a Shimura datum (G, X). Suppose that G is **unramified** at p: this means that $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified extension. This is also equivalent to saying that G has a reductive model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. Fix such a model.

Definition 4.2.2.1. An embedding $i : (G, X) \hookrightarrow (G', X')$ of Shimura data is said to be **p-integral** if there exists a reductive model $G'_{\mathbb{Z}_{(p)}}$ of G' over $\mathbb{Z}_{(p)}$, and if the embedding of groups $G \hookrightarrow G'$ underlying i is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow G'_{\mathbb{Z}_{(p)}}$.

Lemma 4.2.2.2. *Suppose (G, X) is a Shimura datum of Hodge type such that G is unramified at p with reductive model $G_{\mathbb{Z}_{(p)}}$.*

1. *There exists a p-integral embedding of Shimura data $i : (G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$ into a Siegel Shimura datum.*

2. *Suppose that the embedding $G \hookrightarrow \mathrm{GSp}$ arises from an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSp}_{\mathbb{Z}_{(p)}} = \mathrm{GSp}(V_{\mathbb{Z}_{(p)}}, \psi)$, for a symplectic $\mathbb{Z}_{(p)}$-lattice $(V_{\mathbb{Z}_{(p)}}, \psi)$. Then there exist tensors $\{s_\alpha\} \subset V^{\otimes}_{\mathbb{Z}_{(p)}}$ such that $G_{\mathbb{Z}_{(p)}} \subset \mathrm{GSp}_{\mathbb{Z}_{(p)}}$ is the pointwise stabilizer of $V_{\mathbb{Z}_{(p)}}$.*

Proof. For (1), choose any embedding $i' : (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm})$ of Shimura data. By [Kis10, 2.3.1], there exist a $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V$ and an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ that induces i' over $\mathbb{Q}$. The problem is that ψ might not induce a perfect $\mathbb{Z}_{(p)}$-pairing on $V_{\mathbb{Z}_{(p)}}$. To take care of this, we apply Zarhin's trick, which tells us that there exists a perfect pairing ψ' on $V'_{\mathbb{Z}_{(p)}} = (V_{\mathbb{Z}_{(p)}} \times V^{\vee}_{\mathbb{Z}_{(p)}})^4$ and an embedding $\mathrm{GSp}(V, \psi) \hookrightarrow \mathrm{GSp}(V', \psi')$ induced by the natural diagonal embedding $V \hookrightarrow (V \times V^{\vee})^4$ (here we use the polarization ψ to identify V with $V^{\vee}$). This also induces an embedding of the corresponding Shimura data. We can then check that the induced embedding $(G, X) \hookrightarrow (\mathrm{GSp}(V', \psi'), S^{\pm})$ arises from an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GSp}(V'_{\mathbb{Z}_{(p)}}, \psi')$ and is thus p-integral.

(2) follows from [Kis10, 1.3.2]. □

Let $K \subset G(\mathbb{A}_f)$ be a neat compact open sub-group such that $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Choose some p-integral embedding

$$(G, X) \subset (\mathrm{GSp}, S^{\pm}),$$

so that $K_p = K'_p \cap G(\mathbb{Q}_p)$, where $K'_p = \mathrm{GSp}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. By (4.1.1.2), we can find a compact open $K' = K'^{p} K'_p \subset \mathrm{GSp}(\mathbb{A}_f)$ such that $K \subset K'$, and the map

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$$

is a closed embedding defined over $E(G, X)$. By replacing K'^{p} with a finite index sub-group containing K^p, if necessary, we can assume that K' is also neat.

Choose a place $v|p$ of $E = E(G, X)$, let $\mathscr{O}_v$ be the completion of $\mathscr{O}_E$ at v, and let $\mathscr{S}' = \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})_{\mathscr{O}_v}$ be the base change of $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$ from $\mathbb{Z}_{(p)}$ to $\mathscr{O}_v$. Choose also a compatible choice of admissible smooth rational polyhedral cone decomposition data Σ and let $\overline{\mathscr{S}}'_{\Sigma}$ be the compactification of $\mathscr{S}'$ from (4.2.1.3). Let $\mathscr{D}_{\Sigma}$ be the complement $\overline{\mathscr{S}}'_{\Sigma} \setminus \mathscr{S}'$ equipped with its reduced closed sub-scheme structure. Let $\overline{\mathscr{S}}$ be the Zariski closure of $\mathrm{Sh}_K(G, X)$ in $\overline{\mathscr{S}}'_{\Sigma}$; let $s_0 \in \overline{\mathscr{S}}(l)$ be a closed point valued in a finite field $l/\mathbb{F}_p$; and let $\mathscr{O}'$ (resp. $\mathscr{O}$) be the complete local ring of $\overline{\mathscr{S}}'_{\Sigma}$ (resp. $\overline{\mathscr{S}}$) at k. Over $\mathrm{Spec}\, \mathscr{O}' \setminus \mathscr{D}_{\Sigma}$ we have the pull-back $\mathcal{A}$ of the universal abelian scheme over $\mathscr{S}'$. By (4.2.1.3)(3), we can identify $\mathscr{O}'$ with a log deformation ring R of polarized log 1-motives as described in § 3.1.

It is known by the functoriality of analytic toroidal compactifications [Har89], and the compatibility between analytic and arithmetic compactifications [Lan], that $\mathscr{D}_{\Sigma}\left[\frac{1}{p}\right]$ intersects $\overline{\mathscr{S}}\left[\frac{1}{p}\right]$ transversally. In particular, $\overline{\mathscr{S}} \setminus \mathscr{D}_{\Sigma}$ is open and dense in $\overline{\mathscr{S}}$. Since $\overline{\mathscr{S}}$ is

flat over $\mathscr{O}_v$ by construction, we can find a finite extension $L/\mathscr{O}_v$ with residue field l and some lift $s \in \overline{\mathscr{S}}(\mathscr{O}_L)$ of s_0 such that we have a factoring:

$$\begin{array}{ccc} \operatorname{Spec} \mathscr{O}_L & \xrightarrow{s} & \overline{\mathscr{S}s} \\ \uparrow & & \uparrow \\ \operatorname{Spec} L & \xrightarrow{s|_{\operatorname{Spec} L}} & \operatorname{Sh}_K(G,X)_{E_v} \end{array}$$

So s is associated with a polarized semi-stable abelian variety (A_s, λ_s) over L. Fix some algebraic closure $\overline{E}_v$ of E_v and an embedding $L \subset \overline{E}_v$. Since s arises from a point of $\operatorname{Sh}_K(G,X)(L)$, we find from the theory of (4.1.3) Galois-invariant tensors $\{s_{\alpha,s,\text{ét}}\} \subset H^1_{\text{ét}}(A_{s,\overline{E}_v}, \mathbb{Z}_p)$ associated with the Hodge tensors $\{s_\alpha\}$ defining the embedding $G_{\mathbb{Z}_{(p)}} \subset \mathrm{GSp}_{\mathbb{Z}_{(p)}}$. The theory of § 3.2 now allows us to construct an explicit model for R (cf. (3.2.2) to see how the tensors $\{s_{\alpha,s,\text{ét}}\}$ are used); and the theory of § 3.3 gives us a continuous map $R \to R_G$ of log W-algebras that is the normalization of a surjection.

Proposition 4.2.2.3. *Let $\mathscr{O}(s)$ be the quotient domain of R corresponding to the irreducible component of* $\operatorname{Spec} R$ *through which s factors. Then $\mathscr{O}(s)$ is strongly adapted to G (cf. 3.3.4.8).*

Proof. We have to check that conditions (1)-(3) and (5) from (3.3.3.4), and conditions (4'), and (6) from (3.3.4.8), hold for the quotient $\mathscr{O}(s)$. Conditions (1) and (2) are clear from construction, and condition (3) is true, since $\dim \operatorname{Sh}_K(G,X) = \operatorname{rk}_W \operatorname{Lie} U^{\mathrm{op}}_{F,G}$, as can be seen from the analytic uniformization of $\operatorname{Sh}_K(G,X)$. Since the original lift $s : R \to \mathscr{O}_L$ factors through $\mathscr{O}(s)$ by construction, condition (5) holds. The validity of condition (6) follows from algebraicity of $\mathscr{O}(s)$ via (3.3.4.9).

The only thing that remains to be checked is condition (4'). So choose a lift $\tilde{s} : \mathscr{O}(s) \to \mathscr{O}_{\tilde{L}}$ of s_0, for some finite extension $\tilde{L}/L$ within $\overline{E}_v$, and suppose $\tilde{s}$ corresponds to a polarized semi-stable abelian variety $A_{\tilde{s}}$ over $\tilde{L}$. We have Galois-invariant tensors $\{s_{\alpha,\tilde{s},\text{ét}}\} \subset H^1_{\text{ét}}(A_{\tilde{s},\overline{E}_v}, \mathbb{Z}_p)$ arising from the Hodge tensors $\{s_\alpha\}$, and corresponding de Rham tensors $\{s_{\alpha,\tilde{s},\mathrm{dR}}\} \subset H^1_{\mathrm{dR}}(A_{\tilde{s}})$. By (4.1.3.1) and (3.3.4.7), to check that $\tilde{s}$ is strongly G-admissible, it is enough to show that the parallel transport isomorphism

$$\eta_{s,\tilde{s}} : H^1_{\mathrm{dR}}(A_s) \otimes_L \tilde{L}^{\log} \xrightarrow{\simeq} H^1_{\mathrm{dR}}(A_{\tilde{s}}) \otimes_{\tilde{L}} \tilde{L}^{\log}$$

carries $s_{\alpha,s,\mathrm{dR}} \otimes 1$ to $s_{\alpha,\tilde{s},\mathrm{dR}} \otimes 1$, for all α.

Let $\mathscr{Z}(s)$ be the irreducible component of $\overline{\mathscr{S}}$ through which s factors, and let $Z = (\mathscr{Z}(s) \setminus \mathscr{D}_\Sigma)\left[\frac{1}{p}\right]$. Then Z is a smooth, connected E_v-scheme, and there is a polarized abelian scheme $\mathcal{A}$ over Z specializing to A_s and $A_{\tilde{s}}$. Moreover, the tensors $\{s_\alpha\}$ give rise to parallel tensors over the relative de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{A}/Z)$ specializing to the tensors $\{s_{\alpha,s,\mathrm{dR}}\}$ at s and $\{s_{\alpha,\tilde{s},\mathrm{dR}}\}$ at $\tilde{s}$. By (1.2.4.2)(2), $H^1_{\mathrm{dR}}(\mathcal{A}/Z)$ is naturally identified

with $\mathbb{D}(A)(R)|_Z$ as a vector bundle with flat connection. Since the tensors $\{s_\alpha\}$ are rational over a number field, p-adic parallel transport of these tensors must agree with archimedean parallel transport. This means precisely that $\eta_{s,\tilde{s}}$ must carry $s_{\alpha,s,\mathrm{dR}}$ to $s_{\alpha,\tilde{s},\mathrm{dR}}$, for all α. □

Let L/E_v be a finite extension and let $s \in \mathrm{Sh}_K(G,X)(L) \subset \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})(L)$ be a point giving rise to a semi-stable polarized abelian variety (A,λ) over L.

Lemma 4.2.2.4. *Let (A,λ) and $s \in \mathrm{Sh}'_K(\mathrm{GSp}, S^{\pm})(L)$ be as above. Then we can find a compatible choice of smooth admissible rational polyhedral cone decomposition data Σ such that the minimal stratum $Z_{[(\Phi,\sigma)]}$ of $\overline{\mathscr{S}}'_\Sigma$ containing the specialization of s (cf. (4.2.1.3)(4)) satisfies the following property: The cone σ has maximal dimension in $\mathbf{C}_\Phi$; that is, we have* $\dim \sigma = \mathrm{rank}\, \mathbf{B}_\Phi$.

Proof. First of all, it is simple to check that this property does not depend on the choice of representative (Φ,σ) of the equivalence class of $[(\Phi,\sigma)]$. Next, since the level structure and degeneration data associated with (A,λ) determine the equivalence class of Φ entirely, the issue of finding a good compatible choice comes down to an entirely combinatorial question:

Given N in the interior of $\mathbf{C}_\Phi$ (in our specific situation, this N will be the monodromy N_A of A), can we find a smooth admissible rational polyhedral cone decomposition for $\mathbf{C}_\Phi$ such that the minimal cone in the decomposition containing N has the maximal possible dimension in $\mathbf{C}_\Phi$?

The case where $\mathbf{C}_\Phi$ has dimension 1 is trivial. In the other cases, the dimension of $\mathbf{C}_\Phi$ is at least 3, and we find from [KKMSD73, Ch. III] that the answer to our question is indeed 'yes'.[2] □

Let (A,λ) and $s \in \mathrm{Sh}_K(G,X)(L)$ still be as above. Let (Q,λ) be the polarized log 1-motif over $\mathscr{O}_L$ associated with (A,λ) via (1.2.4.2). Suppose that (Q,λ) corresponds to the tuple $(B,Y,X,c,c^\vee,\lambda^{\mathrm{ab}},\lambda^{\text{ét}},\tau)$ over $\mathscr{O}_L$ (cf. 1.2.2.8). Let $\Lambda = H^1_{\text{ét}}(A_{\overline{L}},\mathbb{Z}_p)$ and let $W_\bullet\Lambda$ be the weight filtration so that we have

$$W_0\Lambda = \mathrm{Hom}(Y,\mathbb{Z}_p); \quad \mathrm{gr}^W_2\,\Lambda = X \otimes \mathbb{Z}_p(-1).$$

Let $U^{-2}_{\mathrm{wt},\mathbb{Z}_p} \subset \mathrm{GSp}(\Lambda)$ be the sub-group preserving $W_\bullet\Lambda$ and acting trivially on $W_1\Lambda$. Let $\mathbf{B}_{\lambda^{\text{ét}}}$ be as in (3.1.2). By the argument in (3.2.1.1), we have

$$\mathbf{B}_{\lambda^{\text{ét}}} \otimes \mathbb{Z}_p = \mathrm{Lie}\, U^{-2}_{\mathrm{wt},\mathbb{Z}_p}.$$

Corollary 4.2.2.5. *With the notation above, $\mathbf{B}_{\lambda^{\text{ét}}} \cap \mathrm{Lie}\, G_{\mathbb{Z}_p}$ generates*

$$\mathrm{Lie}\, U^{-2}_{\mathrm{wt},G,\mathbb{Z}_p} = \mathrm{Lie}\, U^{-2}_{\mathrm{wt},\mathbb{Z}_p} \cap \mathrm{Lie}\, G_{\mathbb{Z}_p}$$

as a $\mathbb{Z}_p$-module.

2. The answer is not always 'yes' for cone decompositions of 2-dimensional spaces.

Proof. To begin, we make a compatible choice Σ of admissible decomposition data as in (4.2.2.4), and let $\overline{\mathscr{S}}'_\Sigma$ be the corresponding compactification of $\mathscr{S}'$. Let $s_0 \in \overline{\mathscr{S}}'_\Sigma$ be the specialization of s. The completion of $\mathscr{S}'_\Sigma$ along the minimal stratum $Z_{[(\Phi,\sigma)]}$ containing s_0 will be of the form $\mathfrak{X}_{\Phi,\sigma}$: this is the completion along the closed stratum of a torus embedding of a torus torsor over $\mathcal{P}_\Phi$ (cf. 4.2.1.3)(1). By our choice of Σ, σ has the maximal possible dimension in $\mathbf{C}_\Phi$; in other words, $\{s_0\}$ is the unique closed stratum of $\mathfrak{X}_{\Phi,\sigma}$.

If we further complete along s_0, then we get the deformation ring $\mathscr{O}_{s_0}$ for the polarized log 1-motif over $k(s_0)$ (where $k(s_0)$ has the induced log structure; cf. (4.2.1.3)(4)). We can again use the point s and the theory of §3.2 to build an explicit model R for $\mathscr{O}_{s_0}$, along with the polarized log 1-motif over it. Let $\mathscr{O}(s)$ be the irreducible quotient of R associated with s and $\mathscr{S}_K(G,X)$ as in (4.2.2.3).

Let $M_0 = \mathfrak{M}(\Lambda)/u\mathfrak{M}(\Lambda)$ be as in (2.2.4.3); then, by *loc. cit.*, we have a reductive sub-group $G_W \subset \mathrm{GSp}(M_0)$ (here $W = W(l)$), a G_W-split weight filtration $W_\bullet M_0$, and an isomorphism

$$\Lambda \otimes_{\mathbb{Z}_p} W(\bar{l}) \xrightarrow{\simeq} M_0 \otimes_W W(\bar{l})$$

identifying $G_{\mathbb{Z}_p} \otimes W$ with $G_W \otimes W(\bar{l})$. Let $U^{-2}_{\mathrm{wt}} \subset \mathrm{GSp}(M_0)$ be the unipotent sub-group preserving $W_\bullet M_0$ and acting trivially on $W_1 M_0$. Then by (3.2.1.1) we have a natural identification

$$\mathbf{B}_{\lambda^{\text{ét}}} \otimes W = \mathrm{Lie}\, U^{-2}_{\mathrm{wt}}.$$

By (4.2.2.3), $\mathscr{O}(s)$ is a strongly adapted to G. So it follows from (3.3.3.6) that $\mathbf{B}_{\lambda^{\text{ét}}} \cap \mathrm{Lie}\, G_W$ generates

$$\mathrm{Lie}\, U^{-2}_{\mathrm{wt},G} = \mathrm{Lie}\, U^{-2}_{\mathrm{wt}} \cap \mathrm{Lie}\, G_W.$$

This is where we need our assumption on Σ, since we need σ to be of maximal dimension to apply *loc. cit.*

Now, consider Fontaine's comparison isomorphism

$$\Lambda \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \xrightarrow{\simeq} M_0 \otimes_W B_{\mathrm{dR}} :$$

by functoriality, it preserves weight filtrations and therefore carries $\mathrm{Lie}\, U^{-2}_{\mathrm{wt},\mathbb{Z}_p} \otimes B_{\mathrm{dR}}$ onto $\mathrm{Lie}\, U^{-2}_{\mathrm{wt}} \otimes B_{\mathrm{dR}}$. Again, by functoriality, it carries $G_{\mathbb{Z}_p} \otimes B_{\mathrm{dR}}$ onto $G_W \otimes B_{\mathrm{dR}}$, and $\mathbf{B}_{\lambda^{\text{ét}}} \subset \mathrm{Lie}\, U^{-2}_{\mathrm{wt},\mathbb{Z}_p}$ onto $\mathbf{B}_{\lambda^{\text{ét}}} \subset \mathrm{Lie}\, U^{-2}_{\mathrm{wt}}$. In particular, it takes

$$\mathbf{B}_{\lambda^{\text{ét}}} \cap \mathrm{Lie}\, G_{\mathbb{Z}_p} = \mathbf{B}_{\lambda^{\text{ét}}} \cap (\mathrm{Lie}\, G_{\mathbb{Z}_p} \otimes B_{\mathrm{dR}})$$

onto

$$\mathbf{B}_{\lambda^{\text{ét}}} \cap \mathrm{Lie}\, G_W = \mathbf{B}_{\lambda^{\text{ét}}} \cap (\mathrm{Lie}\, G_W \otimes B_{\mathrm{dR}}).$$

This shows that the rank of $\mathbf{B}_{\lambda^{\text{ét}}} \cap \mathrm{Lie}\, G_{\mathbb{Z}_p}$ must equal the rank of $\mathrm{Lie}\, U^{-2}_{\mathrm{wt},G,\mathbb{Z}_p}$, and so finishes the proof of the corollary. □

Theorem 4.2.3.1. *With the notation as above, the integral canonical model $\mathscr{S}_K(G,X)$ over $\mathscr{O}_v$ for $\mathrm{Sh}_K(G,X)$ constructed in* [Kis10] *admits a compactification $\overline{\mathscr{S}}_K(G,X)$ such that:*

1. *The boundary $\mathscr{D} = \overline{\mathscr{S}}_K(G,X) \setminus \mathscr{S}_K(G,X)$ is an effective Cartier divisor relative to $\mathscr{O}_v$.*
2. *$\overline{\mathscr{S}}_K(G,X)$ is normal with at most toroidal singularities along the boundary. In particular, it is log smooth with respect to the log structure induced from the boundary divisor.*
3. *The vector bundle with flat connection $\mathcal{V}^\circ$ over $\mathscr{S}_K(G,X)$ obtained from the relative de Rham cohomology of the family of abelian varieties (cf.* [Kis10, 2.3.9]*) extends to a vector bundle $\overline{\mathcal{V}}^\circ$ over $\overline{\mathscr{S}}_K(G,X)$ with regular singularities along the boundary divisor $\mathscr{D}$. Moreover, the parallel sections $\{s_{\alpha,\mathrm{dR}}\}$ of $\mathcal{V}^\otimes$ over $\mathrm{Sh}_K(G,X)$ extend to parallel sections of $\overline{\mathcal{V}}^\circ$.*

Proof. Choose a p-integral embedding $(G,X) \hookrightarrow (\mathrm{GSp}, S^\pm)$, $K' \subset \mathrm{GSp}(\mathbb{A}_f)$ neat such that $K'_p = \mathrm{GSp}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ and such that we have a closed embedding

$$\mathrm{Sh}_K(G,X) \hookrightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}, S^\pm).$$

Choose some smooth toroidal compactification $\overline{\mathscr{S}}'_\Sigma$ for $\mathscr{S}' := \mathscr{S}_{K'}(\mathrm{GSp}, S^\pm)_{\mathscr{O}_v}$. Let $\overline{\mathscr{S}}$ be the Zariski closure of $\mathrm{Sh}_K(G,X)$ in $\overline{\mathscr{S}}'_\Sigma$, and let $\overline{\mathscr{S}}_K(G,X)$ be the normalization of $\overline{\mathscr{S}}$. We will show that this has the desired properties. Fix some point $s_0 \in \overline{\mathscr{S}}(l)$ valued in some finite extension $l/\mathbb{F}_p$; let $\mathscr{O}'$ (resp. $\mathscr{O}$) be the completion of $\overline{\mathscr{S}}'_\Sigma$ (resp. $\overline{\mathscr{S}}$) at s_0. We fix some lift $s \in \overline{\mathscr{S}}(\mathscr{O}_L)$ of s_0 corresponding to a semi-stable abelian variety A_s over a finite extension L/E_v, and we use the tensors $\{s_{\alpha,s,\text{ét}}\} \subset H^1_{\text{ét}}(A_{s,\overline{L}}, \mathbb{Z}_p)^\otimes$ to build our explicit model R for $\mathscr{O}'$. We see from (4.2.2.3) that the quotient $\mathscr{O}(s)$ of R (the irreducible component of $\mathscr{O}$ through which s factors) is strongly adapted to G. Moreover, by (4.2.2.5) above, the rationality assumption (3.3.3.5) is valid. Let $\mathscr{O}(s)^{\mathrm{norm}}$ be the normalization of $\mathscr{O}(s)$. It follows from (3.3.4.10) that the map $R \to \mathscr{O}(s)^{\mathrm{norm}}$ can be identified with the explicit map $R \to R_G$. This immediately implies assertions (1) and (2).

Assertion (3) follows from the argument used in [Kis10, 2.3.9]. Over R we have the log crystal $\mathbb{D}(A)$ associated with the family of degenerating abelian varieties over R, and over $\overline{\mathscr{S}}'_\Sigma$, we have the vector bundle with regular singularities given by

$$\overline{\mathcal{V}}^\circ = \mathbf{R}^1\pi_*(\Omega^\bullet_{\overline{\mathcal{A}}/\overline{\mathscr{S}}'_\Sigma}(\log)),$$

the first de Rham cohomology of $\overline{\mathcal{A}}$, the relative compactification of the family $\mathcal{A}$ of abelian varieties over $\mathscr{S}'$, with logarithmic singularities along the complement of $\mathcal{A}$. The re-

striction of $\mathcal{V}^\circ$ to $\operatorname{Spec} R$ is $\mathbb{D}(A)(R)$(cf (1.2.4.2)(2)). In particular, over $\operatorname{Spec} R_G\left[\frac{1}{p}\right] = \operatorname{Spec} \mathscr{O}^{\mathrm{norm}}\left[\frac{1}{p}\right]$, we have parallel tensors $\{s_{\alpha,\mathrm{dR}}\}$ in $\overline{\mathcal{V}}^\circ$.

By construction of R_G, we also have parallel sections $\{s_{\alpha,0} \otimes 1\} \subset \mathbb{D}(A)^\otimes|_{\operatorname{Spf} R_G}$. The argument in (4.2.2.3) shows that the specializations of these sections at a dense set of points of $(\operatorname{Spf} R_G)^{\mathrm{an}} = (\operatorname{Spf} \mathscr{O}(s)^{\mathrm{norm}})^{\mathrm{an}}$ agrees with the specialization of the de Rham tensors $\{s_{\alpha,\mathrm{dR}}\}$. In other words, the tensors $\{s_{\alpha,0} \otimes 1\}$ give us an arithmetic parallel extension (even over the boundary) of the tensors $\{s_{\alpha,\mathrm{dR}}\}$. □

4.2.4

We end by listing some immediate corollaries of our construction.

Corollary 4.2.4.1. *The geometric special fiber of the integral canonical model $\mathscr{S}_K(G,X)$ over $\mathscr{O}_v$ has the same number of connected components as $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$.*

Proof. This follows from Zariski's Main Theorem, since we have a normal compactification $\overline{\mathscr{S}}_K(G,X)$ of $\mathscr{S}_K(G,X)$. See [FC90, 5.10]. □

Corollary 4.2.4.2. *Suppose $G/Z(G)$ is anisotropic; then $\mathscr{S}_K(G,X)$ is proper over $\mathscr{O}_v$.*

Proof. The hypothesis implies that $\mathrm{Sh}_K(G,X)$ is proper over E; cf. [BB66]. Consider the compactification $\overline{\mathscr{S}}_K(G,X)$: the complement of $\mathscr{S}_K(G,X)$ in it is the boundary divisor $\mathscr{D}$, which is a Cartier divisor and is in particular flat over $\mathscr{O}_v$. Since $\mathrm{Sh}_K(G,X)$ is proper over E, it follows that the generic fiber of $\mathscr{D}$ is trivial; by flatness, this implies that $\mathscr{D}$ is itself trivial. The corollary follows. □

Theorem 4.2.4.3. *Suppose A is an abelian variety defined over a number field F, and suppose its Mumford-Tate group G is anisotropic modulo its center. Then, for every $p > 2$ such that G has a reductive model over $\mathbb{Z}_p$ and for every finite place $v|p$ of F, A has potentially good reduction over F_v.*

Proof. Fix some embedding $F \hookrightarrow \mathbb{C}$, and let $V = H^1(A(\mathbb{C}), \mathbb{Q})$ be the rational Hodge structure of weights $(0,1),(1,0)$ associated with A. If necessary, we can replace A by $A^4 \times (A^\vee)^4$, and assume via Zarhin's trick that A is principally polarized; this will not affect the truth of the statement of the theorem. Now, G is the Mumford-Tate group associated with the rational Hodge structure V. Let X be the $G(\mathbb{R})$-conjugacy class of the map $h : \mathbf{S} \to G_{\mathbb{R}}$ classifying the Hodge structure on $V_{\mathbb{R}}$. For each p where G is unramified, we can find some neat compact open sub-group $K \subset G(\mathbb{A}_f)$ such that K_p is hyperspecial and such that A corresponds to a point $s \in \mathrm{Sh}_K(G,X)(F')$, for some finite extension $F'/FE(G,X)$. For each such $p > 2$ and each place $v|p$ of $E = E(G,X)$, we know from (4.2.4.2) above that $\mathrm{Sh}_K(G,X)$ extends to a smooth proper scheme $\mathscr{S}_K(G,X)$ over $\mathscr{O}_{E,v}$. In particular, the point s extends to a point $\tilde{s} \in \mathscr{S}_K(G,X)(\mathscr{O}_{F',w})$, for some place $w|v$ of F'. This tells us that A must have good reduction over F'_w. □

REFERENCES

[And90] Yves André, *p-adic Betti lattices*, p-adic analysis (Trento, 1989), Lecture Notes in Math., vol. 1454, Springer, Berlin, 1990, pp. 23–63, DOI 10.1007/BFb0091133, (to appear in print). MR1094846 (92c:14015)

[ABV05] Fabrizio Andreatta and Luca Barbieri-Viale, *Crystalline realizations of 1-motives*, Math. Ann. **331** (2005), no. 1, 111–172, DOI 10.1007/s00208-004-0576-4. MR2107442 (2005h:14047)

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010. With the collaboration of Peter Scholze. MR2590897

[BB66] Walter L. Baily Jr. and Armand Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528. MR0216035 (35 #6870)

[Ber04] Laurent Berger, *An introduction to the theory of p-adic representations*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 255–292. MR2023292 (2005h:11265)

[BCC04] Alessandra Bertapelle, Maurizio Candilera, and Valentino Cristante, *Monodromy of logarithmic Barsotti-Tate groups attached to 1-motives*, J. Reine Angew. Math. **573** (2004), 211–234, DOI 10.1515/crll.2004.062. MR2084588 (2006f:14051)

[BBM82] Pierre Berthelot, Lawrence Breen, and William Messing, *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982 (French). MR667344 (85k:14023)

[BO78] Pierre Berthelot and Arthur Ogus, *Notes on crystalline cohomology*, Princeton University Press, Princeton, N.J., 1978. MR0491705 (58 #10908)

[BO83] P. Berthelot and A. Ogus, *F-isocrystals and de Rham cohomology. I*, Invent. Math. **72** (1983), no. 2, 159–199, DOI 10.1007/BF01389319. MR700767 (85e:14025)

[Bla94] Don Blasius, *A p-adic property of Hodge classes on abelian varieties*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 293–308. MR1265557 (95j:14022)

[Bor91] Armand Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR1102012 (92d:20001)

[Bre98] Christophe Breuil, *Schemas en Groupes et Corps des Normes*, Unpublished (1998).

[Bre00] Christophe Breuil, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549. MR1804530 (2001k:14087)

[Bre02] Christophe Breuil, *Integral p-adic Hodge theory*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, Tokyo, 2002, pp. 51–80. MR1971512 (2004e:11135)

[Bry83] Jean-Luc Brylinski, "1-*motifs" et formes automorphes (théorie arithmétique des domaines de Siegel)*, Conference on automorphic theory (Dijon, 1981), Publ. Math. Univ. Paris VII, vol. 15, Univ. Paris VII, Paris, 1983, pp. 43–106 (French). MR723182 (85g:11047)

[CC95] Maurizio Candilera and Valentino Cristante, *Periods and duality of p-adic Barsotti-Tate groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1995), no. 4, 545–593. MR1375313 (97d:14065)

[CI99] Robert Coleman and Adrian Iovita, *The Frobenius and monodromy operators for curves and abelian varieties*, Duke Math. J. **97** (1999), no. 1, 171–215, DOI 10.1215/S0012-7094-99-09708-9. MR1682268 (2000e:14023)

[CF00] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43, DOI 10.1007/s002220000042 (French). MR1779803 (2001g:11184)

[dJ95] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Inst. Hautes Études Sci. Publ. Math. **82** (1995), 5–96 (1996). MR1383213 (97f:14047)

[dJ96] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 51–93. MR1423020 (98e:14011)

[Del70] Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970 (French). MR0417174 (54 #5232)

[Del72] Pierre Deligne, *La conjecture de Weil pour les surfaces $K3$*, Invent. Math. **15** (1972), 206–226 (French). MR0296076 (45 #5137)

[Del71] Pierre Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, Springer, Berlin, 1971, pp. 123–165. Lecture Notes in Math., Vol. 244 (French). MR0498581 (58 #16675)

[Del74] Pierre Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77 (French). MR0498552 (58 #16653b)

[Del79] Pierre Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289 (French). MR546620 (81i:10032)

[Del80] Pierre Deligne, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252 (French). MR601520 (83c:14017)

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. MR654325 (84m:14046)

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind, *Non-Archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157, DOI 10.1515/CRELLE.2006.097. MR2289207 (2007k:14038)

[Fal89] Gerd Faltings, *Crystalline cohomology and p-adic Galois-representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80. MR1463696 (98k:14025)

[Fal99] Gerd Faltings, *Integral crystalline cohomology over very ramified valuation rings*, J. Amer. Math. Soc. **12** (1999), no. 1, 117–144, DOI 10.1090/S0894-0347-99-00273-8. MR1618483 (99e:14022)

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990. With an appendix by David Mumford. MR1083353 (92d:14036)

[Fon82] Jean-Marc Fontaine, *Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate*, Ann. of Math. (2) **115** (1982), no. 3, 529–577, DOI 10.2307/2007012 (French). MR657238 (84d:14010)

[Fon90] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 249–309 (French). MR1106901 (92i:11125)

[Fon94a] Jean-Marc Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59–111 (French). With an appendix by Pierre Colmez; Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293971 (95k:11086)

[Fon94b] Jean-Marc Fontaine, *Représentations p-adiques semi-stables*, Astérisque **223** (1994), 113–184 (French). With an appendix by Pierre Colmez; Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293972 (95g:14024)

[FO09] Jean-Marc Fontaine and Ouyang Ou, *p-adic Hodge Theory*, Springer-Verlag, 2009.

[FK88] Eberhard Freitag and Reinhardt Kiehl, *Étale cohomology and the Weil conjecture*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 13, Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse; With an historical introduction by J. A. Dieudonné. MR926276 (89f:14017)

[EGAIII] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 167 (French). MR0163910 (29 #1209)

[SGA1] Alexander Grothendieck et al, *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, 2003 (French). MR2017446 (2004g:14017)

[SGA3] A. Grothendieck, M. Artin, J. E. Bertin, M. Demazure, P. Gabriel, M. Raynaud, and J.-P. Serre, *Schémas en groupes. Fasc. 2b: Exposés 7a et 7b*, Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques, vol. 1963/64, Institut des Hautes Études Scientifiques, Paris, 1965 (French). MR0207704 (34 #7519)

[Gub07] Walter Gubler, *Tropical varieties for non-Archimedean analytic spaces*, Invent. Math. **169** (2007), no. 2, 321–376, DOI 10.1007/s00222-007-0048-z. MR2318559 (2008k:14085)

[Har89] Michael Harris, *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc. (3) **59** (1989), no. 1, 1–22, DOI 10.1112/plms/s3-59.1.1. MR997249 (90h:11048)

[Hum75] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21. MR0396773 (53 #633)

[HK94] Osamu Hyodo and Kazuya Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, Astérisque **223** (1994), 221–268. Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293974 (95k:14034)

[KKN08a] Takeshi Kajiwara, Kazuya Kato, and Chikara Nakayama, *Logarithmic abelian varieties. I. Complex analytic theory*, J. Math. Sci. Univ. Tokyo **15** (2008), no. 1, 69–193. MR2422590 (2009f:14086)

[KKN08b] Takeshi Kajiwara, Kazuya Kato, and Chikara Nakayama, *Logarithmic abelian varieties*, Nagoya Math. J. **189** (2008), 63–138. MR2396584 (2009d:14061)

[Kat89] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224. MR1463703 (99b:14020)

[Kata] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie II*, Unpublished.

[Katb] Kazuya Kato, *Logarithmic degeneration and Dieudonné theory*, Unpublished.

[Katc] Kazuya Kato, *Logarithmic Dieudonné theory*, Unpublished.

[KT03] Kazuya Kato and Fabien Trihan, *On the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$*, Invent. Math. **153** (2003), no. 3, 537–592, DOI 10.1007/s00222-003-0299-2. MR2000469 (2004h:11058)

[KU09] Kazuya Kato and Sampei Usui, *Classifying spaces of degenerating polarized Hodge structures*, Annals of Mathematics Studies, vol. 169, Princeton University Press, Princeton, NJ, 2009. MR2465224 (2009m:14012)

[Kat81] N. Katz, *Serre-Tate local moduli*, Algebraic surfaces (Orsay, 1976), Lecture Notes in Math., vol. 868, Springer, Berlin, 1981, pp. 138–202. MR638600 (83k:14039b)

[KM74] Nicholas M. Katz and William Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77. MR0332791 (48 #11117)

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973. MR0335518 (49 #299)

[Kis06] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 459–496. MR2263197 (2007j:11163)

[Kis09] Mark Kisin, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180, DOI 10.4007/annals.2009.170.1085. MR2600871

[Kis10] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. (2010).

[Kol48] E. R. Kolchin, *On certain concepts in the theory of algebraic matric groups*, Ann. of Math. (2) **49** (1948), 774–789. MR0026996 (10,232d)

[Kot84] Robert E. Kottwitz, *Shimura varieties and twisted orbital integrals*, Math. Ann. **269** (1984), no. 3, 287–300, DOI 10.1007/BF01450697. MR761308 (87b:11047)

[Kot90] Robert E. Kottwitz, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 161–209. MR1044820 (92b:11038)

[Kot92] Robert E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444, DOI 10.2307/2152772. MR1124982 (93a:11053)

[KS67] Michio Kuga and Ichirô Satake, *Abelian varieties attached to polarized K_3-surfaces*, Math. Ann. **169** (1967), 239–242. MR0210717 (35 #1603)

[Lan08] Kai-Wen Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Harvard University. MR2711676

[Lan] Kai-Wen Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, J. Reine Angew. Math., to appear.

[Lan10a] Kai-Wen Lan, *Toroidal compactifications of PEL-type Kuga families* (2010). preprint.

[Lan10b] Kai-Wen Lan, *Elevators for degenerations of PEL structures* (2010). preprint.

[Lan76] Robert P. Langlands, *Some contemporary problems with origins in the Jugendtraum*, Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), Amer. Math. Soc., Providence, R. I., 1976, pp. 401–418. MR0437500 (55 #10426)

[Lar92] Michael J. Larsen, *Arithmetic compactification of some Shimura surfaces*, The zeta functions of Picard modular surfaces, Univ. Montréal, Montreal, QC, 1992, pp. 31–45. MR1155225 (93d:14037)

[Liu08] Tong Liu, *On lattices in semi-stable representations: a proof of a conjecture of Breuil*, Compos. Math. **144** (2008), no. 1, 61–88. MR2388556 (2009c:14087)

[Mes72] William Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, Vol. 264, Springer-Verlag, Berlin, 1972. MR0347836 (50 #337)

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR559531 (81j:14002)

[Mil90] James S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 283–414. MR1044823 (91a:11027)

[Mil92] James S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surfaces, Univ. Montréal, Montreal, QC, 1992, pp. 151–253. MR1155229 (94g:11041)

[Mil94] J. S. Milne, *Shimura varieties and motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 447–523. MR1265562 (95c:11076)

[Moo98] Ben Moonen, *Models of Shimura varieties in mixed characteristics*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 267–350. MR1696489 (2000e:11077)

[Mor10] Sophie Morel, *On the cohomology of certain noncompact Shimura varieties*, Annals of Mathematics Studies, vol. 173, Princeton University Press, Princeton, NJ, 2010. With an appendix by Robert Kottwitz. MR2567740

[Mor75] Yasuo Morita, *On potential good reduction of abelian varieties*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **22** (1975), no. 3, 437–447. MR0404269 (53 #8072)

[Mum69] David Mumford, *Bi-extensions of formal groups*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 307–322. MR0257089 (41 #1743)

[Nak97] Chikara Nakayama, *Logarithmic étale cohomology*, Math. Ann. **308** (1997), no. 3, 365–404. MR1457738 (98j:14022)

[Nam80] Yukihiko Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Mathematics, vol. 812, Springer, Berlin, 1980. MR584625 (82a:32034)

[Niz08] Wiesława Nizioł, *K-theory of log-schemes. I*, Doc. Math. **13** (2008), 505–551. MR2452875

[Ogu94] Arthur Ogus, *F-crystals, Griffiths transversality, and the Hodge decomposition*, Astérisque **221** (1994), ii+183. MR1280543 (95g:14025)

[Pau04] Frédéric Paugam, *Galois representations, Mumford-Tate groups and good reduction of abelian varieties*, Math. Ann. **329** (2004), no. 1, 119–160, DOI 10.1007/s00208-004-0514-5. MR2052871 (2005e:11072)

[Pin90] Richard Pink, *Arithmetical compactification of mixed Shimura varieties*, Bonner Mathematische Schriften [Bonn Mathematical Publications], 209, Universität Bonn Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989. MR1128753 (92h:11054)

[Rap78] M. Rapoport, *Compactifications de l'espace de modules de Hilbert-Blumenthal*, Compositio Math. **36** (1978), no. 3, 255–335 (French). MR515050 (80j:14009)

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR1393439 (97f:14023)

[Ray71] Michel Raynaud, *Variétés abéliennes et géométrie rigide*, Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 473–477. MR0427326 (55 #360)

[Ray94] Michel Raynaud, *1-motifs et monodromie géométrique*, Astérisque **223** (1994), 295–319 (French). Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293976 (95h:14020)

[SR72] Neantro Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, Vol. 265, Springer-Verlag, Berlin, 1972 (French). MR0338002 (49 #2769)

[Shi10] Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math., to appear (2010).

[Str10] Benoît Stroh, *Compactification de variétés de Siegel aux places de mauvaise réduction*, Bull. Soc. Math. France **138** (2010), no. 2, 259–315 (French, with English and French summaries). MR2679041

[Tak77] Mitsuhiro Takeuchi, *Formal schemes over fields*, Comm. Algebra **5** (1977), no. 14, 1483–1528. MR0498540 (58 #16645)

[Vas99] Adrian Vasiu, *Integral canonical models of Shimura varieties of preabelian type*, Asian J. Math. **3** (1999), no. 2, 401–518. MR1796512 (2002b:11087)

[Vas08] Adrian Vasiu, *Projective integral models of Shimura varieties of Hodge type with compact factors*, J. Reine Angew. Math. **618** (2008), 51–75, DOI 10.1515/CRELLE.2008.033. MR2404746 (2009c:11091)

[Vol03] Vadim Vologodsky, *Hodge structure on the fundamental group and its application to p-adic integration*, Mosc. Math. J. **3** (2003), no. 1, 205–247, 260 (English, with English and Russian summaries). MR1996809 (2004h:14019)