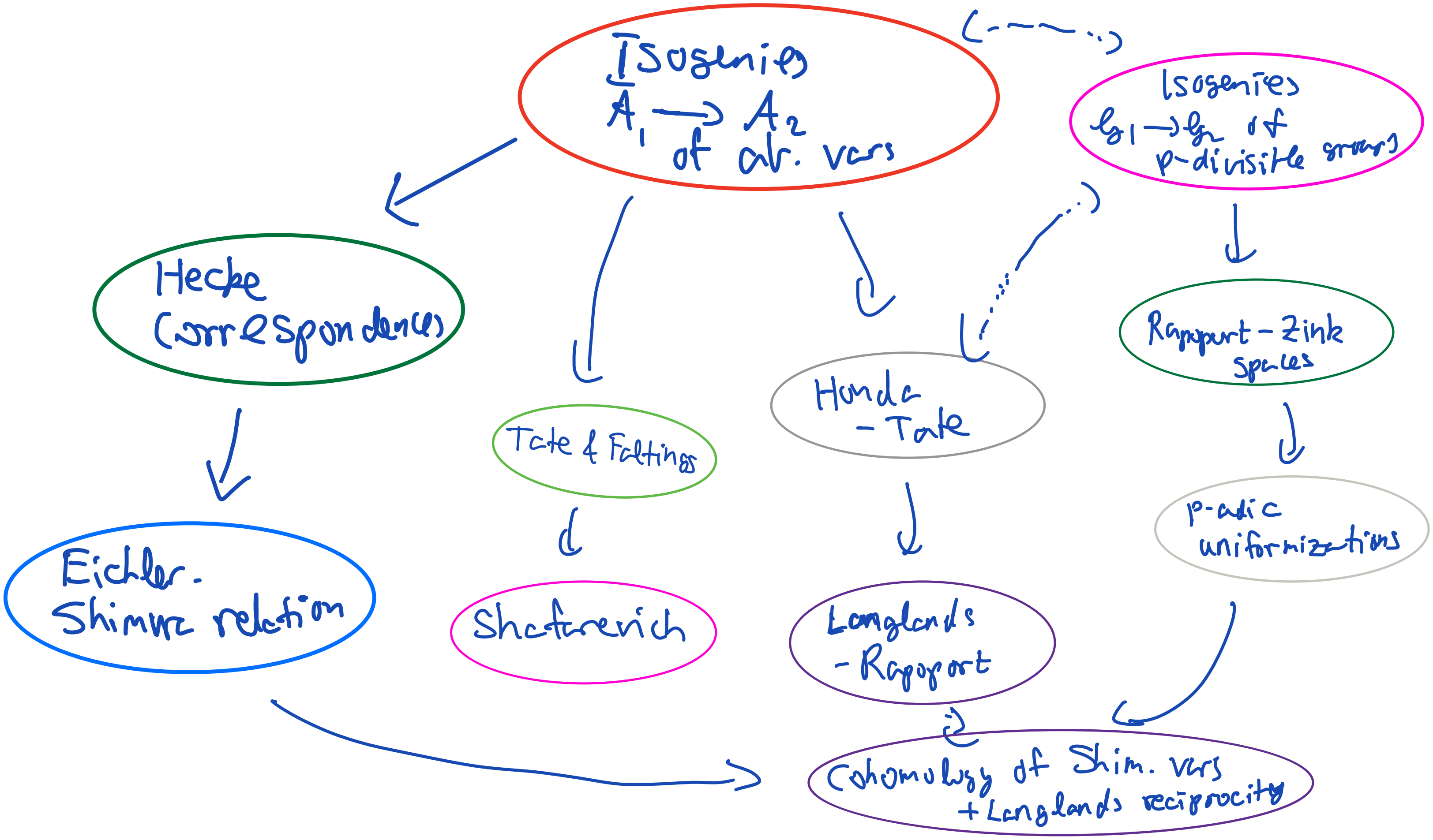
Isogenies, p-Hecke correspondences & Rapoport-Zink spaces

joint w.
Si Ying Lee



Isogenies $A_1 \longrightarrow A_2$ of ab. vars

Isogenies $\mathcal{G}_1 \to \mathcal{G}_2$ of p-divisible groups

Hecke correspondences

Tate & Faltings

Honda-Tate

Rapoport-Zink spaces

Eichler-Shimura relation

Shafarevich

Langlands-Rapoport

p-adic uniformizations

Cohomology of Shim. vars + Langlands reciprocity

Drawbacks of isogenies of ab. vars / p-divisible groups

① Geometry of p-isogenies in char p is complicated

② Functoriality for maps of groups is non-obvious

③ Requires moduli interpretation in terms of ab. vars/ p-divisible groups.

④ When defining Hecke correspondences, connection with rep. theoretic* understanding of spherical Hecke algebra is not clear

*Satake

Global results

Setup

$\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$

- (G,X) : Shimura datum w. reflex field E
- $K \subseteq G(\mathbb{A}_f)$ neat compact open
- Sh_K : canonical model for Shimura var. $/E$
- p prime s.t. $K \subseteq G(\mathbb{Q}_p)$ is hyperspecial
- $v|p$: place of E
- $T \subseteq G_{\mathbb{Q}_p}$: unramified quasi-split torus
- $X_*(T)_+$: Dominant cocharacters with Bruhat ordering $\preccurlyeq$

$\forall \lambda \in X_*(T)_+$, $\mathrm{Sh}_{K \cap \lambda(p)^{-1} K \lambda(p)} \xrightarrow[(\mathrm{can},\ \mathrm{int}(\lambda(p)))]{} \mathrm{Sh}_K \times \mathrm{Sh}_K$ Hecke correspondence

Theorem (M.-Youcis, Lee-M.) + PSTEG (André-Oort)

Suppose : ① (G, X) is of <u>pre-abelian</u> type

<u>or</u>

② G^{ad} is anisotropic + p is <u>large</u> enough

<u>Then</u>: Sh_K admits an integral canonical model $\mathscr{S}_K / \mathcal{O}_{E,(v)}$

and : ① $\forall \lambda \in X_*(T)_+$, $\exists$ Hecke correspondences

Hecke correspondence

$$\begin{tikzcd} \mathrm{Isog}_{\leq \lambda} \arrow[r, "(s,t)"] & \mathscr{S}_K \times \mathscr{S}_K \\ \mathrm{Isog}_{\lambda} \arrow[u, phantom, "\cup"] \arrow[ur, "(s,t)"'] & \end{tikzcd}$$

s.t. ⓐ $\mathrm{Isog}_{\leq \lambda}$ is proper with generic fiber

$$\bigsqcup_{\nu \leq \lambda} \mathrm{Sh}_{K \cap \nu(p)^{-1} K \nu(p)}$$

ⓑ Isog_{λ} is lci & flat $/ \mathcal{O}_{E,(v)}$ with generic fiber

$$\mathrm{Sh}_{K \cap \lambda(p)^{-1} K \lambda(p)}$$

c If $\lambda \lesssim \lambda'$, $\exists$ closed immersion

$$\begin{tikzcd}
\mathrm{Isog}_{\lesssim \lambda} \arrow[rr] \arrow[dr] & & \mathrm{Isog}_{\lesssim \lambda'} \arrow[dl] \\
& S_K \times S_K &
\end{tikzcd}$$

2 Every $x \in S_K(\overline{\mathbb{F}}_p)$ admits a CM lift up to isogeny

Honda-Tate

3 The Langlands-Rapoport-τ conjecture holds for S_K

Point counting

Remark These are properties of integral canonical models as defined in work with Youcis

Local results

Setup

$\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$
$\breve{\mathbb{Q}}_p = \breve{\mathbb{Z}}_p[1/p]$

- $\mathcal{G}$: reductive gr $/ \mathbb{Z}_p$
- $\{\mu\}$: conj. class of minuscule cochars of $\mathcal{G}$
- $b \in G(\breve{\mathbb{Q}}_p)$

$$X_{\{\mu\}}(b)(\overline{\mathbb{F}}_p) = \{ g \in G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) : g^{-1} b \sigma(g) \in \mathcal{G}(\breve{\mathbb{Z}}_p)\mu(p)\mathcal{G}(\breve{\mathbb{Z}}_p) \}$$

Affine Deligne-Lusztig set

Example: $G = GL_h$, $\mu = \mu_d = (\underbrace{1, \dots, 1}_{d}, \underbrace{0, \dots, 0}_{h-d})$

$X_{\{\mu\}}(b)(\overline{\mathbb{F}}_p) \neq \emptyset \Leftrightarrow b\sigma \Leftrightarrow$ Dieudonné F-isocrystal of height h, dim d

$\mathrm{Sht}^{\le\mu}_{G,b}$: Space of mixed char shtukas defined by Scholze-Weinstein

Diamond over $\mathrm{Spd}(\breve{\mathbb{Z}}_p)$

Thm. (Lee-M.)

There is a formal scheme $RZ^{G,\mu}_b$ over $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$ that is formally smooth & locally of finite type functorial in (G,μ) s.t.

- $RZ^{G,\mu}_b(\overline{\mathbb{F}}_p) = X_{\{\mu\}}(b)(\overline{\mathbb{F}}_p)$
- $(RZ^{G,\mu}_b)^\diamond = \mathrm{Sht}^{\le\mu}_{G,b}$

Moreover: If • $(G, \mu) = (GL_n, \mu_d)$

• $\exists$ p-div group $\mathbb{G}_0 / \breve{\mathbb{Z}}_p$ of height h & dim d

$$\begin{tikzcd}
D(\mathbb{G}_0)[\frac{1}{p}] \arrow[r, "F"] \arrow[d, leftrightarrow, "\cong"] & D(\mathbb{G}_0)[1/p] \arrow[d, leftrightarrow, "\cong"] \\
\breve{\mathbb{Q}}_p^d \arrow[r, "b\sigma"'] & \breve{\mathbb{Q}}_p^d
\end{tikzcd}$$

Then: $RZ_b^{GL_n, \mu_d}$ is the "classical" Rapoport-Zink space parametrizing quasi-isogenies

$$R \otimes_{\breve{\mathbb{Z}}_p} \mathbb{G}_0 \dashrightarrow \mathbb{G}$$

Local-to-global principle

$$\begin{tikzcd} s^*\mathrm{Isog} \arrow[dr] & & t^*\mathrm{Isog} \arrow[dl] & & \mathrm{Isog} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ & \hat{S} \arrow[rrr, "x \mapsto A_x{[p^\infty]}"] & & & p\text{-div} \end{tikzcd}$$

Isogenies $\mathcal{G}_s \longrightarrow \mathcal{G}_t$ of p-div gps

$A \longrightarrow \hat{S}$ abelian scheme

$$\begin{tikzcd} p\text{-}\mathrm{Isog} = \left\{ (x,y,\xi) : \begin{array}{l} x,y \in \hat{S},\ \xi : A_x[p^\infty] \\ \longrightarrow A_y[p^\infty] \end{array} \right\} \arrow[r] \arrow[dr] & s^*\mathrm{Isog} = \left\{ (x,\xi) : \begin{array}{l} x \in \hat{S} \\ \xi : A_x[p^\infty] \longrightarrow \mathcal{G}_t \end{array} \right\} \\ & t^*\mathrm{Isog} = \left\{ (y,\xi) : \begin{array}{l} y \in \hat{S} \\ \xi : \mathcal{G}_s \longrightarrow A_y[p^\infty] \end{array} \right\} \end{tikzcd}$$

If $\hat{S}$ is suitably <u>universal</u>, both maps are isomorphisms.

Reinterpreting isogenies

M_1, M_2 : vector bundles / S : flat/$\mathbb{Z}_p$

$$\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[dr, "p^n"'] & M_2 \arrow[d, shift right] \arrow[dr, "p^n"] & \\
& M_1 \arrow[u, shift right, "f"'] \arrow[r, "f"'] & M_2
\end{tikzcd}$$

$\operatorname{rank} M_1 = \operatorname{rank} M_2 = h$

:

p-isogeny of $\deg \leq p^n$

||

$$X(n) = \left\{ (A, B) \in M_{h \times h} \times M_{h \times h} : AB = BA = p^n I_h \right\}$$

$\circlearrowright$ $GL_n \times GL_h$

$(g_1, g_2)(A, B) = (g_1 A g_2^{-1}, g_2 B g_1^{-1})$

$X(n)[\frac{1}{p}] \xrightarrow{\sim} GL_h$

$(A, B) \longmapsto A$

:

Sections of $X(n)(M_1, M_2)$

"twist of $X(n)$ by $GL_h \times GL_h$-bundle (M_1, M_2)"

Isogenies between G-bundles

$$\begin{matrix} X / \operatorname{Spec} \mathbb{Z}_p \\ \circlearrowleft \\ G \times G \end{matrix} \quad + \quad \begin{matrix} X[1/p] \xrightarrow{\simeq} G[1/p] \\ G \times G\text{-equivariant} \end{matrix}$$

"Isogeny model"

$Q_1, Q_2 \longrightarrow S$: $X(Q_1, Q_2)$

two G-bundles

"twist of X by $G \times G$-torsor (Q_1, Q_2)"

Sections of $X(Q_1, Q_2)$ are isogenies from Q_1 to Q_2 bounded by X

Vinberg monoid V_G

$$\begin{tikzcd}
G_{enh} = G \times^{Z_G} T \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V_G \arrow[d] \\
T_{ad} = \prod_{\alpha\in\Delta} \mathbb{G}_m \arrow[r, hook] & \prod_{\alpha\in\Delta} \mathbb{A}^1 \overset{\uparrow}{=} \overline{T}_{ad}
\end{tikzcd}$$

V_G : monoid scheme $/\mathbb{Z}_p$

$G \times G$ -invariant →

geometrically

$\lambda \in X_*(T)_+ \rightsquigarrow \lambda(p) \in \overline{T}_{ad}(\mathbb{Z}_p) \cap T_{ad}(\mathbb{Q}_p) \subseteq T_{ad}(\mathbb{Q}_p)$

$$V_{G,\lambda} = V_G \times_{\overline{T}_{ad}} \{-w_0\lambda(p)\} \subseteq V_G$$

$$V_{G,\lambda} = V_G \times_{T_{ad}} \{-w_0\lambda(p)\} \subseteq V_G$$

Then:

① $V_{G,\lambda} \circlearrowleft G \times G$

② $G[1/p] \xrightarrow{\simeq} V_{G,\lambda}[1/p]$

$g \longmapsto [(g, (-w_0\lambda)(p))]$

③ $V_{G,\lambda}(\mathbb{Z}_p) = \bigsqcup_{\nu \leq \lambda} G(\mathbb{Z}_p)\nu(p)G(\mathbb{Z}_p)$

$\subseteq G(\mathbb{Q}_p)$

④ If $\lambda_1 \leq \lambda_2 \in X_*(T)_+$

$V_{G,\lambda_1} \longrightarrow V_{G,\lambda_2}$ $G\times G$-equivariant

Isogenies bounded by λ

Q_1, Q_2 : G-bundles / S

$\mathrm{Isog}_{\leq \lambda}(Q_1, Q_2) =$ sections of $V_{G,\lambda}(Q_1, Q_2)$

:

Isogenies from Q_1 to Q_2 bounded by λ

Vinberg monoid can be used to set robust theory of isogenies between G-torsors with <u>clear</u> connection with Satake basis

Connection to p-divisible groups

Thm. (Anschütz-LeBras, Gardner-M.-Mathew)

X : p-adic formal scheme

$$\begin{tikzcd} X \arrow[d, squiggly] \\ X^{syn} \end{tikzcd}$$

X^{syn} : syntomification

There exists canonical fully faithful functor

$$p\text{-div}(X)_{\simeq} \hookrightarrow \mathrm{Vect}(X^{syn})_{\simeq} \simeq BGL(X^{syn})$$

realizing source as objects on RHS with Hodge-Tate wts 0, 1

Apertures

- A (G,μ)-aperture over X is an object in $BG(X^{syn})$ "banded by μ"

Thm (Gardner-M.)

(G,μ)-apertures are parameterized by a formal pro-algebraic stack (formally smooth) $BT_\infty^{G,\mu}$ over $\mathrm{Spf}\ \mathcal{O}$ ← reflex ring for $\{\mu\}$

Remark

$\exists$ canonical functor $BT_\infty^{G,\mu} \longrightarrow G\text{-}F\text{-}\mathrm{Isoc}$

$X_{\{\mu\}}(b)(\bar{\mathbb{F}}_p) \neq \emptyset \iff \exists\, Q_b \in BT_\infty^{G,\mu}(\bar{\mathbb{F}}_p)$ of type $b\sigma$

Isogenies between apertures

$Q_1, Q_2 \in BG(X^{syn})$

$V_{G,\lambda}(Q_1, Q_2) \longrightarrow X^{syn}$

:

$\mathrm{Isog}_{\le\lambda}(Q_1, Q_2)(X)$

$= \left\{ \text{sections } X^{syn} \longrightarrow V_{G,\lambda}(Q_1, Q_2) \right\}$

Thm

$$\mathrm{Isog}_{\le\lambda} \xrightarrow{(s,t)} BT_\infty^{h,\mu} \times BT_\infty^{h,\mu}$$

is <u>proper</u> & <u>finitely presented</u>

over each projection

$$Q\mathrm{Isog} = \underset{\substack{\longrightarrow \\ \lambda}}{\mathrm{colim}}\ \mathrm{Isog}_{\le\lambda}$$

: colimit of closed immersions of formal stacks

Rapoport-Zink spaces

$X_{\{\mu\}}(b)(\overline{\mathbb{F}}_p) \neq \emptyset$

$\Downarrow$

$\mathcal{G}_0 \in BT^{G,\mu}_{\nu}(\overline{\mathbb{F}}_p)$ of type $b_0 \rightsquigarrow \mathcal{Q} \in BT^{G,\mu}_{\nu}(\breve{\mathbb{Z}}_p)$

$$RZ^{G,\mu}_{b}(X) = \mathrm{QIsog}(\mathcal{Q}|_X, -)$$

Finiteness results of Viehmann-Hamacher show this is a formal sche

Global application

$$\begin{tikzcd} \hat{\mathcal{S}}_K \arrow[r, "\text{``Serre-Tate''}", "\uparrow \text{ formally étale}"'] & BT_\infty^{G,\mu} \end{tikzcd}$$

$\uparrow$

integral canonical model of M.-Youcis

$s^* \mathcal{I}sog_{\leq\lambda} \xrightarrow{\simeq} t^* \mathcal{I}sog_{\leq\lambda}$ yielding space of isogenies over $\mathcal{S}_K \times \mathcal{S}_K$

Igusa stack interpretation

$$\begin{tikzcd}
p\text{-}\mathrm{QIsog} \arrow[r] \arrow[d, shift right] \arrow[d, shift left, "\text{pull back Groupoid}"] & \mathrm{QIsog} \arrow[d, shift right] \arrow[d, shift left, "\text{Groupoid}"] \\
\mathrm{Sh}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{BT}^{G,\mu}_{\nabla} \arrow[d, "\text{quotient}"] \\
\mathrm{Ig}_K \arrow[r] & (G\text{-}F\text{-}\mathrm{Isoc})_{\mu\text{-adm}} \\
\text{Igusa stack} \arrow[u] &
\end{tikzcd}$$

Dieudonné theory
(reformulated)

Thm (Dieudonné - Manin / Cartier / Fontaine)

k: perfect field in char p

$\exists$ Equivalence

$$\left\{ p\text{-divisible groups} / k \right\} \xleftrightarrow{\simeq} \left\{ (M, F) : \begin{array}{l} M: \text{finite free } W(k)\text{-module} \\ F: \varphi^* M \longrightarrow M \text{ s.t.} \\ p \cdot (M / F(\varphi^* M)) = 0 \end{array} \right\}$$

$$\Big\updownarrow \simeq$$

$$\left\{ (M, F, V) : M: \text{finite free} / W(k) \quad \begin{tikzcd} M \arrow[r, "V"] \arrow[rr, bend right, "p"'] & \varphi^* M \arrow[r, "F"] & M \end{tikzcd} \right\}$$

finite free / $W(k)$

$$\left\{ \begin{tikzcd} {} \arrow[r, shift left, "t"] & M^{-1} \arrow[l, shift left, "u"] \arrow[r, shift left, "t"] \arrow[d, equal] & M^0 \arrow[l, shift left, "u"] \arrow[r, shift left, "t"] \arrow[d, equal] & M^1 \arrow[l, shift left, "u"] \arrow[r, shift left, "t"] \arrow[d, equal] & M^2 \arrow[l, shift left, "u"] \arrow[r, shift left] \arrow[d, equal] & \cdots \arrow[l, shift left] \\ {} \arrow[r, shift left, "p"] & M \arrow[l, shift left, "1"] \arrow[r, shift left, "p"] & M \arrow[l, shift left, "1"] \arrow[r, shift left, "V"] & \varphi^* M \arrow[l, shift left, "F"] \arrow[r, shift left, "1"] & \varphi^* M \arrow[l, shift left, "p"] \arrow[r, shift left, "1"] & \cdots \arrow[l, shift left, "p"] \end{tikzcd} \right\} \xleftrightarrow{\simeq}$$

Dieudonné theory & F-gauges

$$\begin{tikzcd} \cdots \arrow[r, shift left] & M^{-1} \arrow[l, shift left] \arrow[r, shift left, "t"] & M^{0} \arrow[l, shift left, "u"] \arrow[r, shift left, "t"] & M^{1} \arrow[l, shift left, "u"] \arrow[r, shift left, "t"] & M^{2} \arrow[l, shift left, "u"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \end{tikzcd}$$

s.t. ① $ut = tu = p$

② If $M^{-\infty} = \operatorname{colim}_t M^i$

$M^{\infty} = \operatorname{colim}_u M^i$

then $\varphi^* M^{\infty} \xrightarrow{\sim} M^{-\infty}$

③ $\bigoplus_{i \in \mathbb{Z}} M^i$ is finite free / $W(k)[u,t]/(ut-p)$ & generated in degs $0, 1$

deg $u = -1$

deg $t = 1$

F-gauge / k of Hodge-Tate weights $0, 1$