OVERVIEW OF CONSTRUCTION OF SPECIAL CYCLES

These notes, written for a workshop at Darmstadt, presume some level of familiarity with Bhatt's lectures on F-gauges. See also §6 of Gardner-M.

1. F-gauges

Definition 1.0.1. An **F-gauge** over a p-complete ring R is a quasicoherent sheaf over the syntomification R^{syn}. It is **(almost) perfect** if the quasicoherent sheaf is a(n almost) perfect complex. Similarly for **vector bundle** F-gauges (implicitly always of finite rank).

An **F-gauge of level** n over a p-complete ring R is a quasicoherent sheaf over the mod-p^n syntomification $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. It is **(almost) perfect** if the quasicoherent sheaf is a(n almost) perfect complex. Similarly for **vector bundle** F-gauges of level n.

Remark 1.0.2 (Always ∞-categories)**.** Here we mean 'quasicoherent sheaf' in the sense of Lurie: For an affine scheme $\operatorname{Spec} R$, this would mean an object in the stable ∞-category $\mathcal{D}(R)$ of unbounded complexes. But for the purposes here, it is enough to work with the bounded above *pre*-stable ∞-category: this amounts to working with *almost connective* objects, with *connective* meaning 'cohomology supported in non-positive degrees' (non-negative degrees in the homological convention). For a general (pre-)stack X, the ∞-category $\mathrm{QCoh}(X)$ is obtained via right Kan extension from affine schemes: Concretely, giving an object $\mathcal{F}$ in $\mathrm{QCoh}(X)$ is equivalent to giving an object $\mathcal{F}_x \in \mathcal{D}(R)$ compatibly for each $x \in X(R)$. It is (almost) perfect if $\mathcal{F}_x$ is an (almost) perfect complex for all such x.[1]

Remark 1.0.3 (Perfectness and level)**.** One possibly subtle point with this level-n business is when one wants to talk about *perfect* F-gauges. Here, it is important to specify the level beforehand. This is simply because $\mathbb{Z}/p\mathbb{Z}$ is perfect as a complex of $\mathbb{Z}_p$-modules or of course $\mathbb{Z}/p\mathbb{Z}$-modules, but *not* as a complex of $\mathbb{Z}/p^n\mathbb{Z}$-modules for $n > 1$.

Remark 1.0.4 (Why ∞-categories)**.** When dealing with the syntomification, we are essentially forced to work with the derived ∞-category straight off the bat. The issue is that in general this is not going to be a classical stack, and so there is no natural t-structure we can use. This is already the case for something like the derived category of DG modules over a DG algebra with non-trivial cohomology in non-zero degrees.

Remark 1.0.5 (Animation for connective objects)**.** When R is a classical ring, the subcategory of connective objects, sometimes denoted $\mathrm{Mod}_R^{\mathrm{cn}}$, is equivalent via the Dold-Kan correspondence to the ∞-category obtained via localization from the simplicial category of simplicial R-modules. In supermodern terminology, this means that the assignment $R \mapsto \mathrm{Mod}_R^{\mathrm{cn}}$ is obtained via *animation* from the functor assigning to each finite dimensional polynomial $\mathbb{Z}$-algebra its category of locally free modules of finite rank.

Remark 1.0.6 (Quasi-geometricity of the syntomification)**.** The definition of $\mathrm{QCoh}(X)$ for a general prestack can be set-theoretically a bit problematic. Fortunately, R^{syn} has a more controlled theory, because it is what Lurie calls *quasi-geometric*. Essentially, it admits a flat cover by a (formal) algebraic stack, with the diagonal of the covering also algebraic. This lets us describe quasicoherent sheaves over it as sheaves over the covering equipped with certain descent data.

To begin, by the definition of R^{syn}, giving an F-gauge is the same as giving a quasicoherent sheaf over the filtered prismatization $R^{\mathcal{N}}$ along with an isomorphism of the pullbacks along the two open immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}} : R^{\Delta} \to R^{\mathcal{N}}$.

[1] Can feel free to ignore all the 'almosts'. 'Almost perfect' for complexes of modules is also sometimes referred to as 'pseudocoherent': For classical rings R, this is the same as saying that the complex can be represented by a bounded above complex of finite rank projectives. In general, they are complexes M such that some shift $M[j]$ is connective and the tautological truncations $\tau^{\leq k} M[j]$ are finitely presented (or compact) objects in the subcategory of k-truncated complexes. Note that, over a Noetherian ring, every finitely generated module can be viewed as an almost perfect complex.

Second, for most 'nice' R, we have a quasisyntomic cover $R \to R_\infty$ where R_∞ is semiperfectoid, and $R_\infty^{\mathrm{syn}} \to R^{\mathrm{syn}}$ is an fpqc cover. Moreover, all the higher (derived) tensor products $R_\infty^{\otimes_R m+1}$ are also semiperfectoid. This means that we can 'write down' F-gauges over R in terms of F-gauges over R_∞ equipped with certain descent data along the simplicial stack $(R_\infty^{\otimes \bullet+1})^{\mathrm{syn}}$.

Lastly, when R is semiperfectoid, R^{Δ} is *affine* and given by $\mathrm{Spf}\, \Delta_R$ where Δ_R is its absolute prismatic cohomology (which is miraculously an animated commutative ring). Moreover, $R^{\mathcal{N}}$ is given by the *Rees stack* associated with the *Nygaard filtration* $\mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_R$.

Concretely, a quasicoherent sheaf over $R^{\mathcal{N}}$ is simply a complex of filtered modules $\mathrm{Fil}^\bullet \mathsf{M}$ over the filtered ring $\mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_R$. The pullback along j_{dR} amounts to forgetting the filtration, while the pullback along j_{HT} amounts to noting that Frobenius takes $\mathrm{Fil}^i_{\mathcal{N}} \Delta_R$ to $\xi^i \Delta_R$, and so symbolically we look at the Δ_R-module

$$\sum_i \xi^{-i} \varphi^* \mathrm{Fil}^i \mathsf{M}.$$

Therefore, in the end (once again symbolically), the F-gauge structure amounts to writing down an isomorphism

$$\sum_i \xi^{-i} \varphi^* \mathrm{Fil}^i \mathsf{M} \xrightarrow{\simeq} \mathsf{M}.$$

This might be familiar (with ξ replaced by p) from the theory of Fontaine-Laffaille modules. When R is an $\mathbb{F}_p$-algebra, we actually have $\xi = p$.

Remark 1.0.7 (Rings with finite differentials)**.** The category of R admitting such quasisyntomic covers $R \to R_\infty$ includes all rings such that $\Omega^1_{\pi_0(R/pR)/\mathbb{F}_p}$ is a finitely generated $\pi_0(R)$-module. This encompasses all rings with $\pi_0(R/pR)$ of finite type over a field with finite p-basis or with $\pi_0(R/pR)$ semiperfect. We will assume this condition implicitly from now on.

Remark 1.0.8 (The quasisyntomic case)**.** When R is *quasisyntomic*, R_∞ above is what is called a *quasiregular semiperfectoid*. In this case, Δ_R is a *classical* ring and $\mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_R$ is a classical filtered ring (that is, the filtration is by *submodules*). In particular, all the stacks involved are firmly in the classical world, and so it makes sense to talk about coherent sheaves and quasicoherent sheaves in the usual sense without any derived intervention. This applies of course to $R = \mathbb{Z}_p$ or to $R = \mathscr{O}_K$ with $K/\mathbb{Q}_p$ finite. Still, the syntomifications of these 'simple' rings are quite complicated geometrically, and it's best to use indirect means to study their cohomology.

_f-gauges **Remark 1.0.9** (Explicit description for perfect rings)**.** Let's apply the 'explicit' description in terms of filtered modules to the case of a perfect field κ: We see that $\mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_\kappa$ is just $W(\kappa)$ with its p-adic filtration, and one can now translate everything to get Bhatt's description of F-gauges over κ as the Fontaine-Jannsen category (also known to Faltings) of systems of maps $M^i \underset{u}{\overset{t}{\leftrightarrows}} M^{i-1}$ of complexes of $W(\kappa)$-modules for $i \in \mathbb{Z}$, with $u \circ t = t \circ u = p$. The underlying complex is the colimit $M = M^{-\infty}$ under the t-maps, where we view $M^i = \mathrm{Fil}^i M$ as the i-th filtered part, with t giving the transition map $\mathrm{Fil}^i M \to \mathrm{Fil}^{i-1} M$. The map u then says that multiplication by p on $\mathrm{Fil}^i M$ factors via $\mathrm{Fil}^{i+1} M$, so that the filtration is indeed p-adic. Taking the colimit M^∞ along the u maps gives another $W(\kappa)$-complex, and the F-gauge structure now corresponds to an isomorphism $\varphi^* M^\infty \xrightarrow{\simeq} M$.

The same description is actually valid with κ replaced by any perfect ring in char p.

Construction 1.0.10 (The Hodge filtered de Rham realization)**.** Given an F-gauge $\mathcal{M}$ over R, we can obtain a filtered complex of R-modules $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M$, which can be viewed as the Hodge filtered de Rham realization of $\mathcal{M}$. Concretely, in the semiperfectoid case, this is obtained via base-change along the map of filtered rings $\mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_R \to \mathrm{Fil}^\bullet_{\mathrm{triv}} R$, where we view R as a filtered ring with trivial decreasing filtration—that is, the filtration with associated graded supported in degree 0 and equal to R itself.

Remark 1.0.11 (Perfectness from the de Rham realization)**.** It is possible to show that $\mathcal{M}$ corresponds to a *perfect* complex over $R^{\mathcal{N}}$ (or its mod-p^n counterpart) precisely when $\mathrm{gr}^\bullet_{\mathrm{Hdg}} M$ is a graded perfect complex of R-modules. Explicitly, this means that only finitely many Hodge-Tate weights can appear and $\mathrm{gr}^i_{\mathrm{Hdg}} M$ is a perfect complex over R for each i.

Similarly, $\mathcal{M}$ is a vector bundle F-gauge (of level n) when it has only finitely many Hodge-Tate weights and $\mathrm{gr}^i_{\mathrm{Hdg}} M$ is a vector bundle over R (over $R/^{\mathbb{L}}p^n$) for each i.

Definition 1.0.12. The **Hodge-Tate weights** of $\mathcal{M}$ are the integers j such that $\mathrm{gr}^{-j}_{\mathrm{Hdg}} M \neq 0$ (i.e. is not nullhomotopic).

Remark 1.0.13 (Sign convention)**.** The reason for this normalization is that we would like for geometric F-gauges obtained from the syntomic cohomology of smooth proper schemes to have negative Hodge-Tate weights (see Proposition 1.0.17 below). Another way of saying this is that via p-adic Hodge theory we would like the cyclotomic character to have Hodge-Tate weight 1 (and not -1).

Remark 1.0.14 (PD thickenings: F-gauges and crystals)**.** For every divided power thickening $R' \twoheadrightarrow R$, one can show that there is a canonical lift $\mathrm{Spf}\, R' \to R^{\mathbb{\Delta}}$ of the de Rham section $\mathrm{Spf}\, R \to R^{\mathbb{\Delta}}$ (for R semiperfectoid this just corresponds to the surjection $\mathbb{\Delta}_R \to R$). When R is a semiperfect $\mathbb{F}_p$-algebra, this is because $\mathbb{\Delta}_R = A_{\mathrm{crys}}(R)$ is the p-complete divided power envelope of the map $W(R^\flat) \to R$. In particular, every F-gauge gives a *crystal* in the (big) crystalline site of R. Of course, we only need a quasicoherent sheaf over $R^{\mathbb{\Delta}}$ (one can call this a *prismatic crystal*) for this.

Remark 1.0.15 (Strongly divisible filtered F-crystals)**.** When R is a smooth $\mathbb{F}_p$-algebra and R' is a (necessarily p-completely smooth) flat lift over $\mathbb{Z}_p$ with with $R'/pR' = R$, $R' \twoheadrightarrow R$ has canonical divided powers. Therefore, via the usual dictionary between crystals and vector bundles with connection, every prismatic crystal of vector bundles gives a vector bundle M over R' with topologically nilpotent integrable connection connection. If the crystal comes from a vector bundle F-gauge, and R' is equipped with a Frobenius lift φ, then the vector bundle is equipped with a p-adic filtration $\mathrm{Fil}^\bullet M$, along with an isomorphism

$$\sum_i p^{-i}\varphi^* \,\mathrm{Fil}^i M \xrightarrow{\simeq} M.$$

In other words, every vector bundle F-gauge over R gives a p-adically filtered divisible F-crystal over R'. Giving a lift of the F-gauge to one over R' refines $\mathrm{Fil}^\bullet M$ to a filtration $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M$ by vector subbundles such that

$$\mathrm{Fil}^i M = \sum_{j \le i} p^{i-j}\, \mathrm{Fil}^j_{\mathrm{Hdg}} M.$$

So we obtain a functor from F-gauges over R' to strongly divisible filtered F-crystals over R'. This is an equivalence in the Fontaine-Laffaille range where the Hodge-Tate weights are less than $p-1$ apart.

Remark 1.0.16 (F-gauges of geometric origin)**.** The way to get F-gauges geometrically is as follows: For any formal scheme X over $\mathrm{Spf}\, R$, we obtain a map of stacks $X^{\mathrm{syn}} \to R^{\mathrm{syn}}$. Under natural conditions on X (say X is formally of finite type), the (derived) pushforward of the structure sheaf on X^{syn} under this map gives an F-gauge over R: this is the relative syntomic cohomology of X over R.

The next result gives some idea of how one can use quasisyntomic descent to study F-gauges, but can be skipped.

er_smooth **Proposition 1.0.17.** *Let $\pi : X \to S$ be a proper smooth map of p-adic formal algebraic spaces of relative dimension d. If $\mathcal{M}$ is a perfect F-gauge over X of Hodge-Tate weights $\le m$ and Tor amplitude $[a,b]$, then $R\pi^{\mathrm{syn}}_*\mathcal{M}$ is a perfect F-gauge over S of Hodge-Tate weights $\le m$ and Tor amplitude $[a, b+2d]$.*

Proof. It is of course enough to prove it after replacing 'syn' with '$\mathcal{N}$'. One can reduce to the case where $S = \mathrm{Spec}\, R$ is in $\mathrm{CRing}^{p\text{-nilp}}$. By Noetherian approximation, we can assume that in fact we have $R = \mathrm{CRing}^{p\text{-nilp},f}$. Now, by quasisyntomic descent we can assume that R is semiperfectoid. In particular, $R^{\mathcal{N}}$ is canonically isomorphic to the formal Rees stack $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R)$. This means that perfect complexes over $R^{\mathcal{N}}$ are equivalent to filtered perfect complexes over the filtered animated commutative ring $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$.[2]

[2]We are using the fact that such objects are automatically derived (p, I_R)-complete.

Suppose that $\operatorname{Spec} A \to X$ is an affine quasisyntomic cover with A semiperfectoid, and consider the corresponding simplicial scheme $\operatorname{Spec} A^{(\bullet)}$ where

$$\operatorname{Spec} A^{(i)} = \underbrace{\operatorname{Spec} S \times_X \operatorname{Spec} S \times \cdots \times_X \operatorname{Spec} S}_{i\text{-times}}.$$

Then $A^{\mathcal{N}} \to X^{\mathcal{N}}$ is a flat cover, and we have

$$A^{(i),\mathcal{N}} \simeq \underbrace{A^{\mathcal{N}} \times_{X^{\mathcal{N}}} A^{\mathcal{N}} \times \cdots \times_{X^{\mathcal{N}}} A^{\mathcal{N}}}_{i\text{-times}}.$$

This shows that $R\pi_*^{\mathcal{N}}\mathcal{M}$ corresponds to the filtered complex $\operatorname{Tot}\left(\operatorname{Fil}^{\bigstar}_{\mathcal{N}} \mathcal{M}^{(\bullet)}\right)$, where $\operatorname{Fil}^{\bigstar}_{\mathcal{N}} \mathcal{M}^{(\bullet)}$ is the filtered perfect complex over $\operatorname{Fil}^{\bigstar}_{\mathcal{N}} A^{(\bullet)}$. To show that this is perfect with Tor amplitude in $[a, b+2d]$, it is enough to know that the associated graded $\operatorname{Tot}\left(\operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathcal{M}^{(\bullet)}\right)$ is a graded perfect complex over $\operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathbb{\Delta}_R$ with Tor amplitude in $[a, b+2d]$.

Let $\operatorname{gr}^{\heartsuit}_{\mathrm{Hdg}} M$ be the graded perfect complex over X obtained by pulling $\mathcal{M}$ back along the de Rham point, and let $\operatorname{gr}^{\heartsuit}_{\mathrm{Hdg}} M^{(i)}$ be the graded perfect complexes over $A^{(\bullet)}$ obtained by via restriction along $\operatorname{Spec} A^{(\bullet)} \to X$.

Then $\operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathcal{M}^{(\bullet)}$ admits a canonical finite 'weight' filtration whose i-th graded piece admits a canonical isomorphism

$$\operatorname{gr}^i_{\mathrm{wt}} \operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathcal{M}^{(\bullet)} \simeq \operatorname{gr}^i_{\mathrm{Hdg}} M^{(\bullet)} \otimes_{A^{(\bullet)}} \operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathbb{\Delta}_{A^{(\bullet)}}(-i),$$

where $(-i)$ denotes the $-i$-shift in grading.

This shows that $\operatorname{Tot}\left(\operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathcal{M}^{(\bullet)}\right)$ inherits a finite filtration with associated graded pieces

$$\operatorname{Tot}\left(\operatorname{gr}^i_{\mathrm{Hdg}} M^{(\bullet)} \otimes_{A^{(\bullet)}} \operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathbb{\Delta}_{A^{(\bullet)}}(-i)\right)$$

Now, the condition on Hodge-Tate weights implies that these pieces are non-zero precisely when $i \geq -m$. This reduces us to the following

Lemma 1.0.18. *For any perfect complex M over X restricting to a graded perfect complex $M^{(\bullet)}$ over $A^{(\bullet)}$ with Tor amplitude in $[a, b]$, the graded complex*

$$\operatorname{Tot}\left(M^{(\bullet)} \otimes_{A^{(\bullet)}} \operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathbb{\Delta}_{A^{(\bullet)}}\right)$$

is graded perfect over $\operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathbb{\Delta}_R$ with Tor amplitude $[a, b+2d]$, and supported in non-negative graded degrees.

Proof. Choose a map $R_0 \to R$ with R_0 perfectoid along with a generator $\xi \in \operatorname{Fil}^1_{\mathcal{N}} \mathbb{\Delta}_{R_0}$. For $i \geq 0$, this yields isomorphisms

$$\operatorname{gr}^i_{\mathcal{N}} \mathbb{\Delta}_{A^{(\bullet)}} \xrightarrow{\simeq} \operatorname{gr}^i_{\mathcal{N}} \varphi^* \mathbb{\Delta}_{A^{(\bullet)}/R_0} \xrightarrow{\simeq} \operatorname{Fil}^{\mathrm{conj}}_i \overline{\mathbb{\Delta}}_{A^{(\bullet)}/R_0},$$

and we have $\operatorname{gr}^{\mathrm{conj}}_i \overline{\mathbb{\Delta}}_{A^{(\bullet)}/R_0} \simeq \wedge^i \mathbb{L}_{A^{(\bullet)}/R_0}[-i]$.

Therefore, $\operatorname{Tot}\left(M^{(\bullet)} \otimes_{A^{(\bullet)}} \operatorname{gr}^{\bigstar}_{\mathcal{N}} \mathbb{\Delta}_{A^{(\bullet)}}\right)$ corresponds to a decreasingly filtered complex over $\operatorname{Fil}^{\mathrm{conj}}_{\bigstar} \overline{\mathbb{\Delta}}_{A^{(\bullet)}/R_0}$, and considering associated gradeds reduces us to knowing that the graded complex

$$\operatorname{Tot}\left(M^{(\bullet)} \otimes_{A^{(\bullet)}} \operatorname{gr}^{\mathrm{conj}}_{\bigstar} \overline{\mathbb{\Delta}}_{A^{(\bullet)}/R_0}\right) \simeq R\Gamma(X, M \otimes_{\mathscr{O}_X} \wedge^{\bigstar} \mathbb{L}_{X/R_0}[-\bigstar])$$

is graded perfect over $\operatorname{gr}^{\mathrm{conj}}_{\bigstar} \overline{\mathbb{\Delta}}_{R/R_0}$ with Tor amplitude in $[a, b+2d]$, and supported in non-negative graded degrees.

Now, $\wedge^i \mathbb{L}_{X/R_0}$ is canonically filtered with graded pieces isomorphic to $\wedge^k \mathbb{L}_{X/R} \otimes \wedge^l \mathbb{L}_{R/R_0}$, for $k + l = i$.

One can upgrade this to knowing that $\wedge^{\bigstar} \mathbb{L}_{X/R_0}[-\bigstar]$, as a complex over $\mathcal{X} = X \times_{\operatorname{Spec} R} \operatorname{Spec}(\operatorname{gr}^{\mathrm{conj}}_{\bigstar} \mathbb{\Delta}_{R/R_0})/\mathbb{G}_m$, is filtered with graded pieces isomorphic to $\wedge^k \mathbb{L}_{X/R}[-k] \otimes_R \mathscr{O}_{\mathcal{X}}$. Now we finally use our assumption that X is smooth over R of relative dimension d, which tells us that each of these graded pieces is a shifted vector bundle in degree k and vanishes if $k > d$.

Therefore, we are now reduced to knowing that the relative cohomology over $\operatorname{Spec}(\operatorname{gr}^{\mathrm{conj}}_{\bigstar} \overline{\mathbb{\Delta}}_{R/R_0})/\mathbb{G}_m$ of the restriction of a perfect complex M of Tor amplitude $[a, b]$ over the product $X \times_{\operatorname{Spec} R} \operatorname{Spec}(\operatorname{gr}^{\mathrm{conj}}_{\bigstar} \mathbb{\Delta}_{R/R_0})/\mathbb{G}_m$ is represented by a graded perfect complex with Tor amplitude in $[a, b+d]$. This is of course a standard fact about the coherent cohomology of proper morphisms of relative dimension d. $\square$

$\square$

Example 1.0.19 (Relative syntomic cohomology)**.** In particular, we see that $R\pi_*^{\mathrm{syn}}\mathcal{O}$ is perfect of Hodge-Tate weights ≤ 0 and Tor amplitude $[0, 2d]$.

Remark 1.0.20. There is a Poincaré duality on relative syntomic cohomology that would place further constraints on the Hodge-Tate weights. In the previous example, it would bound the weights between $-d$ and 0.

an_scheme **Example 1.0.21** (Syntomic cohomology of abelian schemes)**.** When X is the p-completion of an abelian scheme, one can show that we have

$$R\pi_*^{\mathrm{syn}}\mathcal{O} \simeq \bigoplus_{i=0}^{2d}(\wedge^i\mathcal{M})[-i],$$

where $\mathcal{M}$ is a vector bundle F-gauge of Hodge-Tate weights $-1, 0$. The idea is to reduce (using moduli of polarized abelian schemes for instance) to the case where R is p-completely smooth, in which case F-gauges admit a t-structure. Here, we can take $\mathcal{M} = R^1\pi_*^{\mathrm{syn}}\mathcal{O}$, and reduce the claim to standard facts about the coherent cohomology of abelian schemes.

sin_twist **Example 1.0.22** (The Breuil-Kisin twist)**.** Take X to be the formal projective line. Then we see that $R\pi^{\mathrm{syn}}\mathcal{O}$ is perfect of Tor amplitude $[0, 2]$. To see what this is, we can reduce to the case where $R = \mathbb{Z}_p$. Here $R^1\pi^{\mathrm{syn}}\mathcal{O}$ turns out to be a line bundle, the (inverse) **Breuil-Kisin twist** $\mathcal{O}\{-1\}$, which has Hodge-Tate weight -1. As usual, we can use this twist to shift Hodge-Tate weights of arbitrary F-gauges.

Construction 1.0.23 (The étale realization)**.** Suppose that R is semiperfectoid. Then every perfect F-gauge of level n over R gives a perfect complex M over $\Delta_R[1/\xi] \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with an isomorphism $\varphi^*M \xrightarrow{\simeq} M$. Following Katz and Bhatt-Scholze, one can use Artin-Schreier and almost purity arguments to show that such objects are equivalent to perfect complexes of locally constant $\mathbb{Z}/p^n\mathbb{Z}$-sheaves over $R[1/p]_{\text{ét}}$.[3] This gives the p-adic étale realization functor $T_{\text{ét}}$.

Remark 1.0.24 (Comparison with étale cohomology)**.** One can show that, in the context of Proposition 1.0.17, the étale realization of the relative syntomic cohomology yields the relative p-adic étale cohomology of the adic generic fiber.

Remark 1.0.25 (p-adic comparison for abelian schemes)**.** In particular, if $\mathcal{M}$ is as in Remark 1.0.21, then its étale realization is the dual of the Tate module of the abelian scheme A.

2. Representability of syntomic cohomology

The content of this section can be gleaned from Section 8 of Gardner-M. We will take R to be a p-nilpotent ring.

Theorem 2.0.1. *Suppose that $\mathcal{M}$ is a(n almost) perfect F-gauge of level n over R with Hodge-Tate weights bounded below by -1. Then the functor*[4]

$$\Gamma_{\mathrm{syn}}(\mathcal{M}) : C \mapsto \tau^{\leq 0}\mathrm{RHom}_{\mathcal{O}_{C^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}}(\mathcal{M}, \mathcal{O})$$

is representable by a(n almost) finitely presented derived algebraic stack over R. Moreover, the deformation theory of this stack is controlled by the complex $\mathrm{Fil}^1_{\mathrm{Hdg}} M$; that is, for all square-zero (or even nilpotent PD) thickenings $C' \twoheadrightarrow C$ of p-nilpotent R-algebras with fiber I, we have a fiber sequence

$$\Gamma_{\mathrm{syn}}(\mathcal{M})(C') \to \Gamma_{\mathrm{syn}}(\mathcal{M})(C) \to \mathrm{Map}_{\mathrm{Mod}_R}(\mathrm{Fil}^1_{\mathrm{Hdg}} M, I).$$

[3] Note that $\Delta_R[1/\xi] \otimes \mathbb{Z}/p^n\mathbb{Z}$ vanishes if $R[1/p] = 0$!

[4] This definition clashes with the current definition in the arXiv version of G.-M., but fits better with Grothendieck's definition of $\mathbf{V}(M)$ for an R-module M, and is the correct one for almost perfect F-gauges.

Remark 2.0.2. Explicitly, we have

$$\Gamma_{\mathrm{syn}}(\mathcal{M})(C) = \tau^{\leq 0}\,\mathrm{hker}\left(\mathrm{RHom}_{\mathcal{O}_{C^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}}}(\mathcal{M},\mathcal{O}) \xrightarrow{j^*_{\mathrm{dR}}-j^*_{\mathrm{HT}}} \mathrm{RHom}_{\mathcal{O}_{C^{\mathbb{A}}\otimes\mathbb{Z}/p^n\mathbb{Z}}}(\mathcal{M},\mathcal{O})\right)$$

ect_field **Remark 2.0.3** (Over perfect fields)**.** In terms of the explicit description of F-gauges (of level 1) over perfect fields in Remark 1.0.9, $\mathcal{M}$ is given by a system $M^i \underset{u}{\overset{t}{\leftrightarrows}} M^{i-1}$ of perfect complexes of κ-modules with $u \circ t = t \circ u = 0$. The restriction on the Hodge-Tate weights means that $t : M^i \to M^{i-1}$ is the zero map and $u : M^{i-1} \to M^i$ is an isomorphism for $i \geq 2$. In particular, we have $M^1 = M^\infty$, and the F-gauge structure is given by an isomorphism $\varphi^* M^1 \xrightarrow{\simeq} M^{-\infty}$.

The structure sheaf of level 1 is given by $N^i = \kappa$ for all i with

$$u = \begin{cases} \mathrm{id} & \text{if } i \geq 1 \\ 0 & \text{otherwise.} \end{cases} \; ; \; t = \begin{cases} 0 & \text{if } i \geq 1 \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

The F-gauge structure is given simply by the identity on κ.

From this, one finds that

$$\mathrm{RHom}_{\mathcal{O}_{\kappa^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}}}(\mathcal{M},\mathcal{O}) = \mathrm{RHom}_\kappa(M^1,\kappa)\; ;\; \mathrm{RHom}_{\mathcal{O}_{C^{\mathbb{A}}\otimes\mathbb{Z}/p^n\mathbb{Z}}}(\mathcal{M},\mathcal{O}) = \mathrm{RHom}_\kappa(M^{-\infty},\kappa).$$

The map j^*_{dR} is given as the limit of precomposition with the maps $M^{-i} \xrightarrow{u^{i+1}} M^1$ for $i \geq 0$, and the map j^*_{HT} is via precomposition with the inverse of the semilinear isomorphism $M^1 \xrightarrow{\simeq} M^{-\infty}$ giving the F-gauge structure.

Note in particular, that syntomic cohomology is computing the (homotopy) kernel of the difference between a linear and a semilinear map. This is a fundamental feature.

Remark 2.0.4 (The adic generic fiber)**.** If R is p-completely flat, then we can still define a *formal* algebraic stack $\Gamma_{\mathrm{syn}}(\mathcal{M})$ over $\mathrm{Spf}\, R$ by taking the limit of the corresponding stacks over $\mathrm{Spec}\, R/p^nR$. We can also take the adic generic fiber $\Gamma_{\mathrm{syn}}(\mathcal{M})^{\mathrm{rig}}$ over $(\mathrm{Spf}\, R)^{\mathrm{rig}}$. If $T_{\text{ét}}(\mathcal{M})$ is the étale realization of $\mathcal{M}$, then we get a canonical map

$$\Gamma_{\mathrm{syn}}(\mathcal{M})^{\mathrm{rig}} \to \underline{\mathrm{RHom}}(T_{\text{ét}}(\mathcal{M}), \mathbb{Z}/p^n\mathbb{Z})$$

where the right hand side is the internal RHom in the category of perfect complexes of $\mathbb{Z}/p^n\mathbb{Z}$-sheaves over $(\mathrm{Spf}\, R)^{\mathrm{rig}}$.

Remark 2.0.5 (The case of vector bundles)**.** The main application of this result is in the case where $\mathcal{M} = \mathcal{F}^\vee$ is the dual of a vector bundle of Hodge-Tate weights ≤ 1. In this case, we are looking at the functor

$$C \mapsto \tau^{\leq 0} R\Gamma(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{F})$$

of 'global sections' of $\mathcal{F}$. In this case, $\Gamma_{\mathrm{syn}}(\mathcal{F}^\vee)(C)$ is just an abelian group for classical C, and the deformation theory is given by the fact that we have

$$\Gamma_{\mathrm{syn}}(\mathcal{F}^\vee)(C') = \ker(\Gamma_{\mathrm{syn}}(\mathcal{F}^\vee)(C) \to \mathrm{gr}^{-1}_{\mathrm{Hdg}} F \otimes_R I).$$

Note that $\mathrm{gr}^{-1}_{\mathrm{Hdg}} F$ is the 'bottom' degree associated graded for $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} F$.

vdim_zero **Remark 2.0.6** (Quasi-smooth spaces of sections)**.** When $\mathcal{M}$ is a vector bundle, $\mathrm{Fil}^1_{\mathrm{Hdg}} M$ is a vector bundle over $R/^{\mathbb{L}}p$, and so a perfect complex of Tor amplitude $[-1, 0]$ over R of virtual rank 0. The deformation theory is telling us that its pullback over $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is the *cotangent complex* for this (derived) scheme. This tells us that this derived scheme is *quasismooth* or *derived lci* over R of virtual codimension 0: That is, étale locally on the source, $\Gamma_{\mathrm{syn}}(\mathcal{M})$ can be presented as the derived vanishing locus of n functions on n-dimensional affine space over R.

Remark 2.0.7 (Endomorphisms of syntomic cohomology of abelian schemes)**.** One way to get such a vector bundle F-gauge is from Example 1.0.21: Take a vector bundle F-gauge $\mathcal{N}$ of Hodge-Tate weights $-1, 0$ obtained from the syntomic cohomology of an abelian scheme A, and consider the endomorphism F-gauge $\mathcal{F} = \mathcal{N}^\vee \otimes \mathcal{N}$: this has HT weights $-1, 0, 1$. As we will see below, the global sections of its mod-p^n quotients are exactly giving (for at least for classical inputs) the endomorphism scheme of the truncated BT group scheme $A[p^n]$. In general, the underlying classical scheme is far from being lci, so the F-gauge construction is providing a sort of derived resolution.

Remark 2.0.8 (Sub-bundles of endomorphisms)**.** We can also take any vector subbundle of an endomorphism F-gauge as above. If it is cut out by 'PEL' type conditions, then the result is just giving us (derived resolutions of) endomorphism schemes respecting these additional structures. But one can have non-PEL situations where such an interpretation is less clear. This happens of course in the basic situation of orthogonal Shimura varieties associated with quadratic spaces over $\mathbb{Q}$.

The rest of this section is about the proof of the key result and can be skipped at a first reading.

Remark 2.0.9. By standard dévissage, one can reduce the proof of the theorem immediately to the case where $n = 1$, so we will assume from here on that this is the case.

Remark 2.0.10 (Derived descent)**.** One can actually reduce to the case where R is an $\mathbb{F}_p$-algebra; this is the content of §8.9 of G.-M. This involves the remarkable notion of derived descent along the map $\mathbb{Z}_p \to \mathbb{F}_p$. The basic point is that the map $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spf}\mathbb{Z}_p$ satisfies descent in the following sense: Giving a p-complete $\mathbb{Z}_p$-module is the *same* as giving its (derived) mod-p quotient, along with the data of descent for the cosimplicial animated ring given by the derived tensor products $\mathbb{F}_p^{\otimes_{\mathbb{Z}_p}(i+1)}$. See this nice answer on MO: `https://mathoverflow.net/questions/430129/basic-example-of-derived-descent`.

Remark 2.0.11 (Syntomification from conjugate and Hodge filtrations)**.** The proof involves understanding the stack $R^{\mathrm{syn}} \otimes \mathbb{F}_p$ in a different way. The starting point is that $R^{\mathcal{N}}$ lives over the mod-p fiber of that of $\mathbb{Z}_p$ equipped with its p-adic filtration. The latter can be explicitly written as $[\operatorname{Spf}\mathbb{Z}_p[u,t]/(ut-p)/\mathbb{G}_m]$, where $z \cdot t = zt$ and $z \cdot u = z^{-1}u$, and is in fact canonically isomorphic to $\mathbb{F}_p^{\mathcal{N}}$. Its mod-$p$ fiber contains the closed substacks

$$(\mathbb{F}_p^{\mathcal{N}})_{(u=0)} = [\operatorname{Spec}\mathbb{Z}_p[t]/\mathbb{G}_m] = \mathbb{A}^1/\mathbb{G}_m \ ; \ (\mathbb{F}_p^{\mathcal{N}})_{(u=0)} = \mathbb{A}^1/\mathbb{G}_m,$$

where the weights of the $\mathbb{G}_m$-action are opposite. The mod-p fiber $R^{\mathcal{N}} \otimes \mathbb{F}_p$ is obtained by gluing these two closed substacks along their common closed sub-locus $(u = t = 0)$.

Therefore, for any R, $R^{\mathcal{N}} \otimes \mathbb{F}_p$ is obtained by gluing together the two closed substacks $R^{\mathcal{N}}_{(u=0)}$ and $R^{\mathcal{N}}_{(t=0)}$ along their common closed substack $R^{\mathcal{N}}_{(u=t=0)}$.

When R is semiperfect we can describe everything in terms of the Nygaard filtration. The map t is giving the transition maps $\operatorname{Fil}^i_{\mathcal{N}} \Delta_R \to \operatorname{Fil}^{i-1}_{\mathcal{N}} \Delta_R$ and u is giving the factoring of the multiplication by p map through $\operatorname{Fil}^{i-1}_{\mathcal{N}} \Delta_R \to \operatorname{Fil}^i_{\mathcal{N}} \Delta_R$.

Setting $u = 0$ is basically giving the associated graded $\operatorname{Fil}^i_{\mathcal{N}} \Delta_R / p \operatorname{Fil}^{i-1}_{\mathcal{N}} \Delta_R$, and this can be mapped to $\overline{\Delta}_R = \Delta_R / p\Delta_R$ (all quotients are understood to be derived). The result is the i-th *Hodge* filtered piece $\operatorname{Fil}^i_{\mathrm{Hdg}} \overline{\Delta}_R$, which gives the decreasingly filtered ring $\operatorname{Fil}^\bullet_{\mathrm{Hdg}} \overline{\Delta}_R$. In fact, $R^{\mathcal{N}}_{(u=0)}$ is just the Rees stack for this ring.

Setting $t = 0$ is giving the associated graded $\operatorname{gr}^i_{\mathcal{N}} \Delta_R$: This maps via the divided Frobenius to $\overline{\Delta}_R$ giving the i-th *conjugate* filtered piece $\operatorname{Fil}^{\mathrm{conj}}_i \overline{\Delta}_R$. This in turn makes $\overline{\Delta}_R$ an *increasingly* filtered ring, and $R^{\mathcal{N}}_{(t=0)}$ is the corresponding Rees stack.

The gluing along $u = t = 0$ amounts to the observation that these two filtered rings have (by construction) the same associated gradeds up to sign change.

To get the mod-p syntomification one now glues the two closed substacks further along the common open loci $R^{\mathcal{N}}_{(u=0,t\neq 0)}$ and $R^{\mathcal{N}}_{(t=0,u\neq 0)}$, both of which are isomorphic to $\operatorname{Spf}\Delta_R$.

One can now reverse the order of gluing:

(1) First glue the two substacks $R^{\mathcal{N}}_{(u=0)}$ and $R^{\mathcal{N}}_{(u=0)}$ along the common open locus: A vector bundle over this glued stack is the same as a vector bundle F over $\overline{\Delta}_R$ equipped with two filtrations, $\operatorname{Fil}^\bullet_{\mathrm{Hdg}} \mathsf{F}$ (corresponding to the $u = 0$ locus) and $\operatorname{Fil}^{\mathrm{conj}}_\bullet \mathsf{F}$ (corresponding to the $t = 0$ locus).
(2) Then glue the resulting stack along the commong closed locus $u = t = 0$.

This shows that a vector bundle over $R^{\mathrm{syn}} \otimes \mathbb{F}_p$ is the same as a doubly filtered vector bundle M over $\overline{\Delta}_R$ equipped with an isomorphism $\operatorname{gr}^\bullet_{\mathrm{Hdg}} \mathsf{F} \xrightarrow{\simeq} \operatorname{gr}^{\mathrm{conj}}_{-\bullet} \mathsf{F}$.

In particular, the syntomic cohomology of this vector bundle F-gauge can be computed in terms of a complex

$$\operatorname{Fil}^0_{\mathrm{Hdg}} \mathsf{F} \times_{\mathsf{F}} \operatorname{Fil}^{\mathrm{conj}}_0 \mathsf{F} \to \operatorname{gr}^0_{\mathrm{Hdg}} \mathsf{F},$$

where the map is the difference between the two projections onto the common quotient $\mathrm{gr}^0_{\mathrm{Hdg}}\,\mathsf{F}$.

Remark 2.0.12 (The F-zip stack)**.** There is a simpler stack obtained via the same method. It is the ***F*-zip stack** $R^{F\mathrm{Zip}}$, obtained as follows:

(1) Take two copies of $\mathbb{A}^1_R$, one with the standard $\mathbb{G}_m$-action, and the other with the action twisted by the automorphism $z \mapsto z^{-1}$ of $\mathbb{G}_m$: We can also view these as the Rees stacks associated with the decreasing trivial filtration $\mathrm{Fil}^{\bullet}_{\mathrm{triv}}\,R$ and the increasing trivial filtration $\mathrm{Fil}^{\mathrm{triv}}_{\bullet}\,R$.
(2) Glue these two copies along the automorphism of $\mathbb{G}_{m,R}$ to get a (non-separated!) $\mathbb{G}_m$-equivariant scheme X: $\mathbb{G}_m$-equivariant vector bundles over X are the same as vector bundles F over R equipped with two filtrations, one decreasing $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}}\,F$ and the other increasing $\mathrm{Fil}^{\mathrm{conj}}_{\bullet}\,F$, which we think of as the Hodge and conjugate filtrations.
(3) Next, glue the zero section $\mathrm{Spec}\,R$ of the first copy with that of the second copy via Frobenius to get another $\mathbb{G}_m$-equivariant scheme Y: $\mathbb{G}_m$-equivariant vector bundles over Y are now doubly filtered modules as above equipped further with an isomorphism $\varphi^*\,\mathrm{gr}^{\bullet}_{\mathrm{Hdg}}\,F \xrightarrow{\simeq} \mathrm{gr}^{\mathrm{conj}}_{-\bullet}\,F$.

The F-zip stack is now defined by $R^{F\mathrm{Zip}} = Y/\mathbb{G}_m$. By construction vector bundles over it are the same as F-zips, as defined by Pink-Wedhorn-Ziegler. The cohomology of such a vector bundle (which can be termed 'F-zip cohomology') can be computed in terms of a complex

$$\mathrm{Fil}^0_{\mathrm{Hdg}}\,F \times_F \mathrm{Fil}^{\mathrm{conj}}_0\,F \to \mathrm{gr}^{\mathrm{conj}}_0\,F$$

where the map is the difference between the natural projection from $\mathrm{Fil}^{\mathrm{conj}}_0\,F$ and the Frobenius twisted one from $\mathrm{Fil}^0_{\mathrm{Hdg}}\,F$.

To make this more compatible with syntomic cohomology, let us rewrite it as the quasiisomorphic complex

$$\mathrm{Fil}^0_{\mathrm{Hdg}}\,F \times_F (\mathrm{Fil}^{\mathrm{conj}}_0\,F \times_{\mathrm{gr}^{\mathrm{conj}}_0\,F} \mathrm{gr}^0_{\mathrm{Hdg}}\,F) \to \mathrm{gr}^0_{\mathrm{Hdg}}\,F,$$

where we are now taking the difference between the two natural projections

$$\mathrm{Fil}^0_{\mathrm{Hdg}}\,F \to \mathrm{gr}^0_{\mathrm{Hdg}}\,F\ ;\ \mathrm{Fil}^{\mathrm{conj}}_0\,F \times_{\mathrm{gr}^{\mathrm{conj}}_0\,F} \mathrm{gr}^0_{\mathrm{Hdg}}\,F \to \mathrm{gr}^0_{\mathrm{Hdg}}\,F.$$

Remark 2.0.13 (From F-gauges to F-zips)**.** There is a canonical map $R^{F\mathrm{Zip}} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ that can be understood explicitly in the semiperfect case in terms of maps of filtered rings. In particular, we have a canonical map from syntomic cohomology to F-zip cohomology.

se_change **Remark 2.0.14.** A basic but fundamental observation about graded modules that will be used in the proof is the following: Suppose that $B_\bullet$ is a non-positively graded (animated commutative) ring and that $M_\bullet$ is a graded $B_\bullet$-module (or complex of such modules) with $M_i = 0$ for $i \geq a$ (for some integer a). Let $\overline{M}_\bullet$ be the (derived) graded base-change of $M_\bullet$ over B_0. Then:

(1) $M_a \xrightarrow{\simeq} \overline{M}_a$;
(2) There is a canonical fiber sequence

$$\overline{M}_a \otimes_{B_0} B_{-1} \to M_{a-1} \to \overline{M}_{a-1}.$$

up scheme **Remark 2.0.15** (Height 1 group schemes)**.** Associated with any vector bundle N with a φ-semilinear endomorphism ψ is a canonical finite flat height 1 group scheme[5] $G(N,\psi)$ whose Cartier dual is given by the kernel of the map

$$\mathbf{V}(N) \xrightarrow{\psi^\vee - \varphi^*} \mathbf{V}(\varphi^* N).$$

When $N = R$ and $\psi = \mathrm{id}$, the Artin-Schreier sequence combined with Cartier duality ends up giving us μ_p. When $\psi = 0$, we get the self-dual group α_p.

[5] This is a commutative p-torsion finite flat group scheme with Frobenius acting by 0.

Sketch of proof. As observed before, we can assume that $n = 1$ and R is an $\mathbb{F}_p$-algebra. Assume for simplicity that $\mathcal{M} = \mathcal{F}^\vee$ for a vector bundle $\mathcal{F}$ with HT weights bounded above by 1. What one shows is that $\Gamma_{\mathrm{syn}}(\mathcal{F}^\vee)$ is fibered over the functor $\Gamma_{F\mathrm{Zip}}(\mathcal{F}^\vee)$ computing F-zip cohomology, which is manifestly representable being given by maps between vector bundles. Moreover, the kernel is a certain finite flat height 1 group scheme.

To begin, by Remark 2.0.14, the condition on Hodge-Tate weights ensures that we have

$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} \mathsf{F} \xrightarrow{\simeq} \mathrm{gr}^{-1}_{\mathrm{Hdg}} F.$$

It also tells us that we have a fiber sequence

$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} F \otimes_R \mathrm{gr}^{1}_{\mathrm{Hdg}} \overline{\mathbbl{\Delta}}_R \to \mathrm{gr}^{0}_{\mathrm{Hdg}} \mathsf{F} \to \mathrm{gr}^{0}_{\mathrm{Hdg}} F.$$

This is in fact the only (though *essential*) utility of the condition on the Hodge-Tate weights. Another way of phrasing the first isomorphism is that we have

cartesian

$$\mathrm{Fil}^0_{\mathrm{Hdg}} \mathsf{F} = \mathsf{F} \times_F \mathrm{Fil}^0_{\mathrm{Hdg}} F. \tag{2.0.15.1}$$

And another way of phrasing the fiber sequence is:

_sequence

$$\mathrm{gr}_0^{\mathrm{conj}} F \otimes_R \mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_R \to \mathrm{gr}_0^{\mathrm{conj}} \mathsf{F} \to \mathrm{gr}^0_{\mathrm{Hdg}} F. \tag{2.0.15.2}$$

This means that the homotopy kernel (i.e. shifted cone) of

$$\mathrm{Fil}^0_{\mathrm{Hdg}} \mathsf{F} \times_{\mathsf{F}} \mathrm{Fil}_0^{\mathrm{conj}} \mathsf{F} \to \mathrm{Fil}^0_{\mathrm{Hdg}} F \times_F (\mathrm{Fil}_0^{\mathrm{conj}} F \times_{\mathrm{gr}_0^{\mathrm{conj}} F} \mathrm{gr}^0_{\mathrm{Hdg}} F)$$

is the same as that of $\mathrm{Fil}_0^{\mathrm{conj}} \mathsf{F} \to \mathrm{Fil}_0^{\mathrm{conj}} F \times_{\mathrm{gr}_0^{\mathrm{conj}} F} \mathrm{gr}^0_{\mathrm{Hdg}} F$. One checks that this is isomorphic to

$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} F \otimes_R \mathrm{hker}(\mathrm{Fil}_1^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_R \to R).$$

In sum the fiber of the map from syntomic cohomology to F-zip cohomology is computed by a complex

e_complex

$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} F \otimes_R \mathrm{hker}(\mathrm{Fil}_1^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_R \to R) \to \mathrm{gr}^{-1}_{\mathrm{Hdg}} F \otimes_R \mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_R. \tag{2.0.15.3}$$

This is the kind of thing studied in §7 of G.-M. and is a prismatic spin on a classical construction of Artin-Milne. Note that we have a canonical semilinear endomorphism of $\mathrm{gr}^{-1}_{\mathrm{Hdg}} F$ given by

$$\varphi^* \mathrm{gr}^{-1}_{\mathrm{Hdg}} F \xrightarrow{\simeq} \mathrm{Fil}_1^{\mathrm{conj}} F \to F \to \mathrm{gr}^{-1}_{\mathrm{Hdg}} F.$$

By Remark 2.0.15, this gives us a height 1 group scheme G over R. It turns out that the complex 2.0.15.3 is computing the fppf cohomology of G. When R is a smooth $\mathbb{F}_p$-algebra, this is literally what Artin-Milne prove (after unwinding definitions). Note that in this case, we have

$$\mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_R \xrightarrow{\simeq} \mathbb{L}_{R/\mathbb{F}_p}[-1] \simeq \Omega^1_{R/\mathbb{F}_p}[-1],$$

and

$$\mathrm{hker}(\mathrm{Fil}_1^{\mathrm{conj}} \overline{\mathbbl{\Delta}}_R \to R) \xrightarrow{\simeq} \mathrm{hker}(\tau^{\leq 1}\Omega^\bullet_{R/\mathbb{F}_p} \to R) \xrightarrow{\simeq} Z^1(\Omega^\bullet_{R/\mathbb{F}_p})[-1].$$

In particular, the kernel of the map (2.0.15.3) is simply computing the values of the group scheme G. This completes the sketch of the proof, except perhaps for the part about the deformation theory. While one can piece that together from the proof, the cleaner way is to make systematic use of frames and what one can call the 'Lau-Zink unique lifting principle'. This is explained in §8.9 of G.-M. □

3. Apertures and p-divisible groups

More stuff from G.-M. We fix a reductive group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ and a minuscule cocharacter μ defined over an unramified ring of integers $\mathcal{O}$. This gives in particular a $\mathcal{G}$-torsor $\mathcal{P}_\mu$ over $B\mathbb{G}_m \times \mathrm{Spf}\,\mathcal{O}$ classified by the map $B\mu$.

Definition 3.0.1. A(n n-truncated) $(\mathcal{G}, \mu)$-**aperture** over a p-complete $\mathcal{O}$-algebra R is a $\mathcal{G}$-torsor over R^{syn} ($R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$) whose restriction to $R^{\mathcal{N}}$ ($R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$) is p-completely flat locally on $\mathrm{Spf}\,R$ is isomorphic to $\mathcal{P}_\mu$.

Remark 3.0.2. There is a canonical Hodge point $B\mathbb{G}_m \times \mathrm{Spf}\,R \to R^{\mathcal{N}}$. It suffices to check that this is flat locally on R isomorphic to $\mathcal{P}_\mu$. In fact, assuming $\mathrm{Spf}\,R$ connected, it suffices to check this after restriction to $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ for some algebraically closed field κ over R.

Remark 3.0.3 (The adjoint F-gauge)**.** Given an n-truncated $\mathcal{G}$-aperture $\mathfrak{Q}[n]$ over R, we can twist the Lie algebra $\mathrm{Lie}\,\mathcal{G}$ by $\mathfrak{Q}$ to obtain a vector bundle F-gauge $(\mathrm{Lie}\,G)_{\mathfrak{Q}[n]}$ over R of level n. The minuscule condition on μ ensures that this has Hodge-Tate weights $-1,0,1$. In particular, we have the Hodge filtered de Rham realization $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_{\mathfrak{Q}[n]}(\mathrm{Lie}\,G)$ associated with this F-gauge, which is a filtered vector bundle over $R/^{\mathbb{L}}p^n$ supported in graded degrees $-1,0,1$, and so can be viewed as a filtered perfect complex over R of Tor amplitude $[-1,0]$.

Remark 3.0.4 (The case of GL_h)**.** When $\mathcal{G}=\mathrm{GL}_h$, μ is conjugate to a cocharacter μ_d of type $(1,\dots,1,0,\dots,0)$ with d 1s, a (GL_h,μ_d)-aperture is the same as a vector bundle F-gauge $\mathcal{F}$ of Hodge-Tate weights $0,1$ and rank h with $\mathrm{gr}^{-1}_{\mathrm{Hdg}}\,F$ of rank d.

Theorem 3.0.5. *(1) n-truncated $(\mathcal{G},\mu)$-apertures are parameterized by a smooth formal Artin stack $\mathrm{BT}_n^{\mathcal{G},\mu}$ over $\mathrm{Spf}\,\mathcal{O}$ of virtual dimension 0 and with affine diagonal.*

(2) The deformation theory of $\mathrm{BT}_n^{\mathcal{G},\mu}$ is governed by $\mathrm{gr}^{-1}_{\mathrm{Hdg}}\,\mathbf{dR}_{\mathfrak{Q}[n]}(\mathrm{Lie}\,G)[-1]$, where $\mathfrak{Q}[n]$ is the tautological aperture. More precisely, this is the tangent *complex for $\mathrm{BT}_n^{\mathcal{G},\mu}$ over $\mathcal{O}$.*

(3) The transition maps $\mathrm{BT}_{n+1}^{\mathcal{G},\mu}\to\mathrm{BT}_n^{\mathcal{G},\mu}$ are smooth and surjective.

(4) $\mathrm{BT}_n^{\mathrm{GL}_h,\mu_d}$ is canonically equivalent to the stack of n-truncated Barsotti-Tate groups.

Remark 3.0.6. The map from $\mathrm{BT}_n^{\mathrm{GL}_h,\mu_d}$ to the stack of BT group schemes is easy to define: It's just given by the global sections functor $\mathcal{F}\mapsto\Gamma_{\mathrm{syn}}(\mathcal{F}^\vee)$ from the previous section. The rest of the proof uses the smoothness of the stack of truncated BTs (a famous theorem of Grothendieck), and an inverse functor for qrsp algebras constructed by Anschütz-Le Bras and Mondal.

Remark 3.0.7. The proof of (1) in the theorem is a non-linear version of the argument from the previous section. The dévissage from level 1 to level $n\ge 1$ happens by noting that the deformation theory from level $n-1$ to level n is governed by the syntomic cohomology of the level-1 perfect F-gauge $(\mathrm{Lie}\,G)_{\mathfrak{Q}}[-1]\otimes\mathbb{F}_p$ of Hodge-Tate weights $-1,0,1$. Derived descent works again to reduce to the case of $\mathbb{F}_p$-algebras. Here one shows that $\mathrm{BT}_1^{\mathcal{G},\mu}$ is presented over the (smooth Artin) stack of G-zips of type μ as a *gerbe* banded by a certain height 1 group scheme called the *Lau group scheme*. This amounts to studying deformations of $\mathcal{G}$-bundles from the F-zip stack to the syntomification.

Remark 3.0.8 (The Hodge type case)**.** When we have a faithful representation $(\mathcal{G},\mu)\to(\mathrm{GL}_h,\mu_d)$ (the *Hodge type* situation), we can interpret $\mathrm{BT}_n^{\mathcal{G},\mu}$ as parameterizing truncated BT groups with additional structure, though already in the case of GSp_{2g} with its standard representation (and $\mu_d=\mu_g$), the precise interpretation of what this additional structure is can be subtle. See the discussion in §11.6 of G.-M.

Remark 3.0.9 (The étale realization for apertures)**.** One can use the construction used for the étale realization of F-gauges, or argue using Tannakian reconstruction, to see that every $(\mathcal{G},\mu)$-aperture $\mathfrak{Q}$ over R of level n gives a $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-local system $T_{\text{ét}}(\mathfrak{Q})$ over $R[1/p]$. For (GL_h,μ_d)-apertures, this is a repackaging of the p-adic Tate module of the associated p-divisible group.

4. The syntomic realization for Shimura varieties

Suppose now that $\mathcal{G}$ is a reductive model over $\mathbb{Z}_p$ for a reductive group G over $\mathbb{Q}$ underlyiing a Shimura datum (G,X). Let $v|p$ be a place of the reflex field $E=E(G,X)$, and suppose that the level $K\subset G(\mathbb{A}_f)$ has been chosen such that $K=K_pK^p$ with $K_p=\mathcal{G}(\mathbb{Z}_p)$. If $\{\mu\}$ is the conjugacy class of Shimura cocharacters associated with X, then we can find a representative μ_v defined over E_v and extending to a minuscule cocharacter of $\mathcal{G}$ over $\mathcal{O}=\mathscr{O}_{E_v}$. In particular, we can consider the formal pro-Artin stack

$$\mathrm{BT}_\infty^{\mathcal{G},\mu_v^{-1}}=\varprojlim_n\mathrm{BT}_n^{\mathcal{G},\mu_v^{-1}}.$$

Theorem 4.0.1. *Suppose that (G,X) is of abelian type and that Z_G does not admit an $\mathbb{R}$-split subtorus that is not $\mathbb{Q}$-split*[6]*, and let $\mathscr{S}_K$ be Kisin's integral canonical model over $\mathscr{O}_E[1/D]$ where D is the product of all primes at which*

[6]This is a technical 'cuspidality' condition that exists only because we are afraid of stacks with infinite discrete stabilizers.

the level K is not hyperspecial, and let $\mathscr{S}^{\mathfrak{F}}_{K,v}$ be the associated formal scheme over $\operatorname{Spf}\mathcal{O}$*. There is a canonical* surjective *formally étale syntomic realization map*

$$\mathscr{S}^{\mathfrak{F}}_{K,v} \to \mathrm{BT}^{\mathcal{G},\mu_v^{-1}}_{\infty}.$$

When $\mathcal{G} = \mathrm{GSp}_{2g}$, this is the map associating with every principally polarized abelian scheme (up to prime-to-p isogeny) its principally polarized p-divisible group. Moreover, the étale realization of the syntomic realization is the canonical $\mathcal{G}(\mathbb{Z}_p)$-local system over the generic fiber.

Remark 4.0.2. When $p > 2$, this is due to Imai-Kato-Youcis, who use a detour via prismatic and crystalline cohomology and Ito's theory of prismatic G-displays.

Remark 4.0.3 (Characterization of the integral canonical model)**.** As observed by I-K-Y, the integral model $\mathscr{S}_{K,v}$ over $\mathcal{O}$ is actually *characterized* by the existence of this formally étale map and the fact that the tube around its special fiber in rigid space $\mathrm{Sh}^{\mathrm{rig}}_{K,E_v}$ consists exactly of the *crystalline* points.

Remark 4.0.4 (BST models)**.** One can show that the exotic exceptional integral canonical models constructed by Bakker-Shankar-Tsimerman also support such realizations. This essentially comes down to their exhibiting Fontaine-Laffaille modules associated with representations of $\mathcal{G}$, and of course only works for large enough primes p.

One key input into the proof of the theorem is the following:

Theorem 4.0.5 (Tate's full faithfulness for apertures)**.** *Suppose that R is a p-complete normal Noetherian domain. Then the functor $T_{\text{ét}}$ is a fully faithful functor from $\mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R)$ to pro-étale $\mathcal{G}(\mathbb{Z}_p)$-local systems over $R[1/p]$.*

Sketch. The main point is that the diagonal of $\mathrm{BT}^{\mathcal{G},\mu}_{\infty}$ is affine, so a section of it over $R[1/p]$ extends to one over R if and only if it does so after completion at every height 1 prime of R. This reduces us to the case of p-complete DVRs where we can use full faithfulness results of Bhatt-Scholze (or Guo-Reinecke).

There is one other key point: We have to first algebraize $\mathrm{BT}^{\mathcal{G},\mu}_{n}$ to a stack over $\operatorname{Spec} R$ by gluing (in the sense of Beauville-Laszlo) the formal stack with the stack of $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-bundles over $R[1/p]$ via the étale realization functor. ☐

Remark 4.0.6 (Tate's theorem for p-divisible groups)**.** In the case of GL_h, this is just Tate's full faithfulness theorem for p-divisible groups.

Sketch of construction of the syntomic realization for abelian type cases. We want to show that the canonical $\mathcal{G}(\mathbb{Z}_p)$-torsor over the generic fiber underlies a $(\mathcal{G},\mu)$-aperture over $\mathscr{S}^{\mathfrak{F}}_{K,v}$. The full faithfulness above lets us prove this flat locally: the descent data is obtained for free. So using standard methods in the business (with some refinements by Lovering), one reduces to the case where (G,X) is of Hodge type or of CM type. The second case can be dealt with Lubin-Tate theory.

In the first case, one wants to give a reduction of structure group of the $(\mathrm{GSp}_{2g},\mu_g)$-aperture associated with a principally polarized p-divisible group. One reduces to exhibiting this for the complete local rings of $\mathscr{S}^{\mathfrak{F}}_{K,v}$, which are deformation rings constructed by a method of Faltings. It turns out that these rings are also deformation rings for $\mathrm{BT}^{\mathcal{G},\mu}_{\infty}$, and this does the job.[7]

Now, the map $\mathscr{S}^{\mathfrak{F}}_{K,v} \to \mathrm{BT}^{\mathcal{G},\mu}_{n}$ is smooth and so has open image. To show that it is surjective, it is enough to show that the image is closed and that the target is connected. The former can be shown by showing that $\mathscr{S}^{\mathfrak{F}}_{K,v} \to \mathrm{BT}^{\mathcal{G},\mu}_{\infty}$ satisfies the relative valuative criterion for properness, which amounts in the end to the Nerón-Ogg-Shafarevich criterion for good reduction. The connectedness can be verified mod-p and with $n = 1$, where it comes down to the fact that the stack of $\mathcal{G}$-zips of type μ is connected (since $\mathcal{G}$ is itself connected). ☐

Remark 4.0.7 (Non-emptiness of strata)**.** The surjectivity can be used to easily deduce that all Newton and Ekedahl-Oort strata are non-empty. For the former, one needs a result of Wintenberger saying that the hyperspecial ADL $X_\mu(b)$ is non-empty if and only if $b \in B(G,\mu)$.

[7]The same method also works for the BST models, where a F-L module associated with a faithful representation plays the role of the $(\mathrm{GSp}_{2g},\mu_g)$-aperture.

5. Formal special cycles

Now suppose that W is a representation of G on which μ acts via weights $-1,0,1$. We'd like to associate certain special cycles on $\mathscr{S}_{K,v}$ associated with W. For this, we fix a lattice $W_{\mathbb{Z}} \subset W$ such that $W_{\widehat{\mathbb{Z}}}$ is K-stable. In particular, $W_p = W_{\mathbb{Z}_p} \subset W_{\mathbb{Q}_p}$ is a K_p-stable lattice and so corresponds to an algebraic $\mathcal{G}$-representation over $\mathbb{Z}_p$. Here, we will construct the *formal* completions of the special cycles along their special fibers.

Example 5.0.1 (Siegel type)**.** Suppose that H_1, H_2 are two representations of G of *Siegel type*. This means that the associated variations of Hodge structures on $\mathrm{Sh}_K(\mathbb{C})$ are homological realizations of abelian schemes $\mathcal{A}_1, \mathcal{A}_2$ up to isogeny over Sh_K. Then $\mathrm{Hom}(H_1, H_2)$ satisfies the constraints on weights.

Construction 5.0.2 (Local cycles)**.** Let $\mathfrak{Q}$ be the canonical $(\mathcal{G}, \mu_v^{-1})$-aperture over $\mathscr{S}^{\mathfrak{F}}_{K,v}$. For every lattice $W_p \subset W_{\mathbb{Q}_p}$ as above, we can twist its dual by $\mathfrak{Q}$ to obtain a vector bundle F-gauge $\mathbf{Syn}_{\mathfrak{Q}}(W_p^\vee)$ of Hodge-Tate weights $-1,0,1$. Therefore, we obtain, for every $n \geq 1$, a formally quasi-smooth of virtual codimension 0 derived algebraic scheme

$$\mathscr{Z}^{\mathrm{syn}}_{K,v,n}(W_p) = \Gamma_{\mathrm{syn}}\left(\mathbf{Syn}_{\mathfrak{Q}}(W_p^\vee) \otimes \mathbb{Z}/p^n\mathbb{Z}\right)$$

Also set

$$\mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p) = \varprojlim_n \mathscr{Z}^{\mathrm{syn}}_{K,v,n}(W_p).$$

The tangent complex of this stack over $\mathscr{S}^{\mathfrak{F}}_{K,v}$ is given by the pullback of $\mathrm{gr}^{-1}_{\mathrm{Hdg}}\,\mathbf{dR}_K(W_p)[-1]$.

Example 5.0.3 (Siegel type local cycles)**.** When $W = \mathrm{Hom}(H_1, H_2)$ is of Siegel type and W_p is of the form $\mathrm{Hom}(H_{1,p}, H_{2,p})$ for K_p-stable lattices $H_{i,p} \subset H_i$, $\mathbf{Syn}_{\mathfrak{Q}}(H_{i,p})$ is an F-gauge of Hodge-Tate weights $0,1$ associated with the p-divisible group $\mathcal{A}_i[p^\infty]$, and one sees that $\mathscr{Z}^{\mathrm{syn}}_{K,v,n}(W_p)$ (resp. $\mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p)$) is a derived resolution of the scheme of homomorphisms $\underline{\mathrm{Hom}}(\mathcal{A}_1[p^n], \mathcal{A}_2[p^n])$ (resp. $\underline{\mathrm{Hom}}(\mathcal{A}_1[p^\infty], \mathcal{A}_2[p^\infty])$).

Remark 5.0.4 (Points of local cycles)**.** Suppose that $z \in \mathscr{S}_{K,v}(\kappa)$ is a geometric point. Then $\mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p)(z)$ is a finite free $\mathbb{Z}_p$-module (which is course compatible with the previous example). Indeed, the description from Remark 2.0.3 shows that it is the kernel of a map

$$M^1 \xrightarrow{F-\iota} M^{-\infty}$$

of finite free $W(\kappa)$-modules of the same rank, where F is φ-semilinear, and ι is an inclusion as a submodule. After inverting p, we are getting the finite dimensional $\mathbb{Q}_p$-vector space of invariants in an F-isocrystal (generating the slope 0 part), and it's not hard to see from this that the kernel is already free of finite rank over $\mathbb{Z}_p$ before inverting p.

Remark 5.0.5 (Basic idea behind formal special cycles)**.** As the previous example shows, a purely local construction cannot give us the right construction of cycles: indeed, we would like to have spaces of homomorphisms between *abelian schemes* (perhaps up to prime-to-p isogeny), which is at worst a $\mathbb{Z}_{(p)}$-linear object. What we have here is a $\mathbb{Z}_p$-linear object of homomorphisms between their p-divisible groups. Now, Serre-Tate theory tells us that the map from the locally of finite type scheme $\underline{\mathrm{Hom}}(\mathcal{A}_1, \mathcal{A}_2)$ to $\underline{\mathrm{Hom}}(\mathcal{A}_1[p^\infty], \mathcal{A}_2[p^\infty])$ is formally étale, so we essentially only have to know how to pick out the correct set of $\overline{\mathbb{F}}_p$-points to pin down the correct finite type object. Ideally, for a given representation W, we would know exactly what the associated family of *motives* over the Shimura variety is, and picking out these points is the same as picking out the sections of $\mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p)(z)$ that are actually motivic. Of course, we are far from such an idyll. Still, we can exploit as much as possible the motivic realizations of abelian schemes to pick out a canonical $\mathbb{Q}$-subspace of the finite dimensional $\mathbb{Q}_p$-vector space

$$\mathscr{Z}^{\mathrm{syn}}_{K,v}(W)(z) = \mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p)(z)[1/p].$$

Assumption 5.0.6. Suppose that there exist H_1, H_2 of Siegel type and an equivariant embedding $W \subset E \overset{\mathrm{defn}}{=} \mathrm{Hom}(H_1, H_2)$.[8]

[8] One actually only needs these auxiliary representations to exist for some cover of G.

Remark 5.0.7. The subcategory of representations satisfying this assumption is closed under direct sums and subquotients, and also contains the trivial representation.

structure **Construction 5.0.8** (A possibly non-canonical rational structure)**.** In this situation, we set

$$\mathscr{Z}_K(W_p)(z) = \mathscr{Z}^{\mathrm{syn}}_{K,v}(W)(z) \cap \mathrm{Hom}^\circ(\mathcal{A}_{1,z}, \mathcal{A}_{2,z}) \subset \mathscr{Z}^{\mathrm{syn}}_{K,v}(E)(z) = \mathrm{Hom}^\circ(\mathcal{A}_{1,z}[p^\infty], \mathcal{A}_{2,z}[p^\infty]).$$

This is a finite free $\mathbb{Z}_{(p)}$-module.

Remark 5.0.9. Here, $\mathcal{A}_{1,z}$ and $\mathcal{A}_{2,z}$ are determined up to isogeny by the representations H_1, H_2, and Hom° means homomorphisms in the isogeny category.

Theorem 5.0.10. *There is a canonical locally finite unramified quasismooth map $\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v \to \mathscr{S}^{\mathfrak{F}}_{K,v}$ characterized by the following properties:*

(1) It admits a formally étale map $\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v \to \mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p)$ over $\mathscr{S}^{\mathfrak{F}}_{K,v}$.

(2) For any choice of H_1, H_2 as in the assumption above, and any geometric point $z \in \mathscr{S}_{K,v}(\kappa)$, $\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v(z)$ agrees with the $\mathbb{Z}_{(p)}$-module from Construction 5.0.8.

Sketch of construction. The main point is to construct the map first in the situation where $W_p = \mathrm{Hom}(H_{1,p}, H_{2,p})$ where $H_{i,p} \subset H_i$ is a K_p-stable lattice. At least on classical points, $\mathscr{Z}_K(W_p)(z)$ should just be the space of maps $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_1, \mathcal{A}_2$ are now abelian schemes up to prime-to-p isogeny associated canonically with the chosen lattices. The question is now: How do we thicken this up into a quasi-smooth derived scheme? There aren't really that many choices since we are requiring that we have a formally étale map

$$\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v \to \mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p),$$

so there is at most one way of doing this 'thickening up'.

The Hom scheme $\underline{\mathrm{Hom}}(\mathcal{A}_1, \mathcal{A}_2)$ can be made into a derived scheme[9] as follows: A classical rigidity theorem says that the scheme of homomorphisms is the same as the scheme of morphisms that respect the zero section. This makes perfect sense for animated inputs as well, and gives us the desired derived thickening $\tilde{\mathbb{H}}(\mathcal{A}_1, \mathcal{A}_2)$. However, this isn't the 'correct' object: Standard arguments show that its deformation theory is governed by $\mathrm{Lie}(\mathcal{A}_2) \otimes \tau^{\geq 1} R\Gamma(\mathcal{A}_1, \mathscr{O}_{A_1})$, and unless $\mathcal{A}_1$ is an elliptic curve, this tangent complex sits in too many degrees for us to get something quasi-smooth: an unramified quasi-smooth map must have tangent complex concentrated in cohomological degree 1. We'd like to get rid of the contributions from the higher degrees, so we can get something with tangent complex $\mathrm{Lie}(\mathcal{A}_2) \otimes H^1(\mathcal{A}_1, \mathscr{O}_{A_1})[-1]$.

Now, let $\mathcal{M}_i$ be the F-gauge of HT weights $-1, 0$ over $\mathscr{S}^{\mathfrak{F}}_{K,v}$ associated with $\mathcal{A}_i$ as in Example 1.0.21. Then $\mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p)$ is the functor of sections associated with (the dual of) $\mathcal{M}_1^\vee \otimes \mathcal{M}_2$. One sees that its tangent complex is given by (the pullback of) $\mathrm{Lie}(\mathcal{A}_2) \otimes H^1(\mathcal{A}_1, \mathscr{O}_{A_1})[-1]$.

But we can also look at the functor associated with $\left(\oplus_{i\geq 1} \wedge^i \mathcal{M}_1[-i]\right)^\vee \otimes \mathcal{M}_2$: this still has HT weights bounded above by 1, and so the theorem from § 2 still gives a derived scheme, but now its tangent complex is the pullback of

$$\mathrm{Lie}(\mathcal{A}_2) \otimes \tau^{\geq 1} R\Gamma(\mathcal{A}_1, \mathscr{O}_{A_1})$$

. There is a syntomic realization map from $\tilde{\mathbb{H}}(\mathcal{A}_1, \mathcal{A}_2)$ to this scheme, and Kodaira-Spencer theory shows that it is actually formally étale. Therefore, we obtain $\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v$ as the pullback of $\mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p)$ along this syntomic realization map.

In general, we can always find lattice $H_{i,p}$ such that $W_p \subset E_p = \mathrm{Hom}(H_{1,p}, H_{2,p})$, and we can set

$$\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v = \mathscr{Z}_{K,v}(E_p)^{\mathfrak{F}} \times_{\mathscr{Z}^{\mathrm{syn}}_{K,v}(E_p)} \mathscr{Z}^{\mathrm{syn}}_{K,v}(W_p).$$

To see that this is independent of choices, one has to work a bit harder. The key is to prove a refinement of Tate's theorem on homomorphisms of abelian schemes. Namely, we need to know that when z is defined over a finite field, the natural map

$$\mathscr{Z}_K(W)(z) \otimes \mathbb{Q}_p \to \mathscr{Z}^{\mathrm{syn}}_{K,v}(W)(z)$$

[9] In general, any classical scheme can be lifted into the derived world in infinitely many ways. Indeed, for affine schemes, we are asking for animated commutative rings R with a given π_0.

is an isomorphism. This can be shown using the strong rationality results from Kisin's paper on the Langlands-Rapoport conjecture. □

Remark 5.0.11 (Products of formal cycles)**.** One can now easily show that if W_1, W_2 are two representations satisfying the standing assumption and $W_{i,p} \subset W_i$ are K_p-stable lattices, then we have a canonical isomorphism

$$\mathscr{Z}_K(W_{1,p})^{\mathfrak{F}}_v \times_{\mathscr{S}^{\mathfrak{F}}_{K,v}} \mathscr{Z}_K(W_{2,p})^{\mathfrak{F}}_v \xrightarrow{\simeq} \mathscr{Z}_K(W_{1,p} \oplus W_{2,p})^{\mathfrak{F}}_v.$$

Here, on the left, we are of course taking the derived fiber product.

Remark 5.0.12 (Formal cycles and étale realizations)**.** For every prime ℓ, we have an étale realization $\mathbf{Et}_{K,\ell}(W)$ over $\mathscr{S}_K[\ell^{-1}]$ associated with W (standard constructions for Shimura varieties). Using the ℓ-adic realization for homomorphisms of abelian schemes and the refined Tate conjecture mentioned above, one can show that, for $\ell \neq p$, we have a canonical map

$$\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v \to \mathbf{Et}_{K,\ell}(W).$$

For any K^p-stable lattice $W^p \subset W_{\mathbb{A}_f^p}$, we obtain $\mathbb{Z}_\ell$-lattices

$$\mathbf{Et}_{K,\ell}(W^p) \subset \mathbf{Et}_{K,\ell}(W),$$

and their common pre-image in $\mathscr{Z}_K(W_p)^{\mathfrak{F}}_v$ gives a *finite* unramified quasi-smooth map $\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v$ associated with the $\mathbb{Z}$-lattice

$$W_{\mathbb{Z}} = W \cap W_p \cap W^p \subset W_{\mathbb{A}_f}.$$

o_section **Remark 5.0.13** (Structure over the zero section)**.** Consider the zero section $0 : \mathscr{S}^{\mathfrak{F}}_{K,v} \to \mathscr{Z}_K(W_{\mathbb{Z}})$: this lands in an open and closed substack $\mathscr{Z}_K(W_{\mathbb{Z}}, 0) \subset \mathscr{Z}_K(W_{\mathbb{Z}})$ that is still quasi-smooth over $\mathscr{S}^{\mathfrak{F}}_{K,v}$ with tangent complex given by the pullback of $\boldsymbol{co}_{W_p}[-1] \stackrel{\text{defn}}{=} \mathrm{gr}^{-1}_{\mathrm{Hdg}}\,\mathbf{dR}_K(W_p)[-1]$. Moreover, the underlying classical scheme of this stack is $\mathscr{S}^{\mathfrak{F}}_{K,v}$. By a general principle, any such quasi-smooth map is obtained in the following way: Given any vector bundle M over a scheme X, we can take the derived self-intersection of the zero section

$$X \times_{0,\mathbf{V}(M^\vee),0} X$$

and view it as a derived scheme over X via either projection. Concretely, this is just X as a topological space, but with its structure sheaf replaced by the Koszul complex of the zero cosection of $M^\vee$, which is just $\bigoplus_i \wedge^i M^\vee[i]$. Observe that the class of this complex in $K_0(X)$ is precisely the top Chern class of M and maps to the r-th Adams eigenspace $K_0(X)^{(r)}_{\mathbb{Q}}$ where $r = \mathrm{rank}(M)$ (here we assume that X is regular).

Remark 5.0.14 (Classicality criterion)**.** The following are equivalent for a connected component $\mathscr{Z}^{\mathfrak{F}}_v$ of $\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v$:

(1) $\mathscr{Z}^{\mathfrak{F}}_v$ is a classical formal stack;
(2) The classical truncation of $\mathscr{Z}^{\mathfrak{F}}_v \otimes k(v)$ is lci and unramified over $\mathscr{S}_K \otimes k(v)$ of codimension $\mathrm{rank}\boldsymbol{co}_{W_p}$;
(3) The classical truncation in (2) is equidimensional of dimension $\dim \mathrm{Sh}_K - \mathrm{rank}\boldsymbol{co}_{W_p}$.

The point is that the quasi-smooth formal stack $\mathscr{Z}^{\mathfrak{F}}_v$ is étale locally (on the source) cut out as a *derived* formal scheme by $d = \mathrm{rank}\boldsymbol{co}_{W_p}$ equations in $\mathscr{S}^{\mathfrak{F}}_{K,v}$, and is classical if and only if the d equations form a regular sequence. Since $\mathscr{S}_K \otimes k(v)$ is regular, and in particular Cohen-Macaulay, any classical subscheme cut out by d equations has codimension at most d, and has codimension exactly d if and only if the equations form a regular sequence.

m:adjoint **Remark 5.0.15** (The adjoint representation)**.** A somewhat interesting observation (made to me by Rapoport a couple of years ago) is that, for any abelian type Shimura datum (G, X), there is a canonical representation W that one can look at: The adjoint representation! The cycles in this case have codimension $\dim \mathrm{Sh}_K$, and so are virtual 1-cycles on $\mathscr{S}_K$. These seem related to the very big CM cycles considered by Wei Zhang in his proof of the AFL.

6. Properties of formal special cycles: the classical orthogonal case

Example 6.0.1 (Quadratic spaces: special divisors)**.** The ur-example is the Shimura variety associated with a quadratic space V over $\mathbb{Q}$ of signaure $(n,2)$. Here, we have $G = \mathrm{SO}(V)$[10], and we can take $W = V$ and $\mathcal{G} = \mathrm{SO}(V_p)$ for some self-dual lattice $V_p \subset V_{\mathbb{Q}_p}$. The Kuga-Satake construction tells us that our Siegel type assumption is satisfied (via the GSpin cover). The space $\mathscr{Z}_K(V_{\mathbb{Z}})^{\mathfrak{F}}_v$ is the formal completion of the stack of *special endomorphisms* (of the Kuga-Satake abelian scheme, but could be of a power, or a summand, or indeed a space of homomorphisms from one summand to another: the particular abelian schemes involved are not important, and are not in any sense God-given.).

The deformation theory here is governed by $\mathrm{gr}^{-1}_{\mathrm{Hdg}}\,\mathbf{dR}_K(V_p)$, which is a line bundle. This means that we are getting quasi-smooth maps of virtual codimension 1. In fact, it turns out that as long as we are dealing with non-zero endomorphisms, we always get genuine (generalized) Cartier divisors, which are the so-called *special divisors*. In fact, the quadratic form on V induces a canonical map[11]

$$\mathscr{Z}_K(V_{\mathbb{Z}})^{\mathfrak{F}}_v \to \underline{\mathbb{Q}}_{\geq 0},$$

and taking the pre-image of $m \in \mathbb{Q}_{\geq 0}$ gives a finite unramified quasi-smooth map $\mathscr{Z}_K(V_{\mathbb{Z}}, m)^{\mathfrak{F}}_v \to \mathscr{S}^{\mathfrak{F}}_{K,v}$.

When $m \neq 0$, this is a classical object and can be constructed and studied without any derived nonsense. The classicality can be checked for instance using the criterion from the end of the previous section: One just has to know that its special fiber does not contain any connected components of $\mathscr{S}_K \otimes k(v)$.

Remark 6.0.2 (Linear invariance for special divisors)**.** Note that we can replace $V_{\mathbb{Z}}$ by $N \cdot V_{\mathbb{Z}}$ for any rational number $N \neq 0$. This won't change the cycle $\mathscr{Z}_K(V_{\mathbb{Z}})^{\mathfrak{F}}_v$, but it *will* change the induced quadratic form, so that

$$\mathscr{Z}_K(N \cdot V_{\mathbb{Z}}, m)^{\mathfrak{F}}_v \simeq \mathscr{Z}_K(V_{\mathbb{Z}}, N^2 m)^{\mathfrak{F}}_v$$

This is an easy instance of what Howard calls *linear invariance*.

Remark 6.0.3 (The case $m = 0$)**.** The locus over the zero section $\mathscr{Z}_K(V_{\mathbb{Z}}, 0)^{\mathfrak{F}}_v$ is, by Remark 5.0.13, a derived square zero thickening of $\mathscr{S}^{\mathfrak{F}}_{K,v}$ by $\boldsymbol{\omega}_{V_p}[1]$, where $\boldsymbol{\omega}_{V_p} \overset{\mathrm{defn}}{=} \boldsymbol{co}^{\vee}_{V_p} = \mathrm{Fil}^1_{\mathrm{Hdg}}\,\mathbf{dR}_K(V_p)$[12] This is an explanation for why the tautological bundle $\boldsymbol{\omega}_{V_p}$ (or rather its inverse) shows up as the degree 0 term for the generating series of special divisors on the orthogonal Shimura variety.

Example 6.0.4 (Higher codimension cycles)**.** In the previous example, we can replace V with $W = V^n$ for some $n \geq 2$. In this case, there are many more interesting K_p-stable lattices $W_p \subset W_{\mathbb{Q}_p}$ (not just ones of the for $V^n_{\mathbb{Z}}$ necessarily), and, if we fix the lattices away from p, each such choice will give a quasi-smooth morphism of virtual codimension n, $\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v \to \mathscr{S}^{\mathfrak{F}}_{K,v}$. The quadratic form on V will now induce a map

$$\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v \to \underline{\mathrm{Sym}^n(\mathbb{Q})_{\geq 0}}$$

where the right hand side is the locally constant sheaf valued in positive semi-definite symmetric matrices over $\mathbb{Q}$. Therefore, we can take the pre-image of any $T \in \mathrm{Sym}^n(\mathbb{Q})_{\geq 0}$ to get a cycle

$$\mathscr{Z}_K(W_{\mathbb{Z}}, T)^{\mathfrak{F}}_v \to \mathscr{S}^{\mathfrak{F}}_{K,v}.$$

Remark 6.0.5 (Relationship with Howard-M.)**.** If $W_{\mathbb{Z}} = V^n_{\mathbb{Z}}$, and $m_1, \ldots, m_n$ are the diagonal entries of T, then we immediately see that $\mathscr{Z}_K(W_{\mathbb{Z}}, T)^{\mathfrak{F}}_v$ is the open and closed substack of the derived fiber product

$$\mathscr{Z}_K(V_{\mathbb{Z}}, m_1)^{\mathfrak{F}}_{K,v} \times_{\mathscr{S}^{\mathfrak{F}}_{K,v}} \mathscr{Z}_K(V_{\mathbb{Z}}, m_2)^{\mathfrak{F}}_v \times \cdots \times_{\mathscr{S}^{\mathfrak{F}}_{K,v}} \mathscr{Z}_K(V_{\mathbb{Z}}, m_n)^{\mathfrak{F}}_v$$

supported on the locus where the moment matrix of the m-tuple of special endomorphisms is exactly T. In this way, we see that the structure sheaf of $\mathscr{Z}_K(V^n_{\mathbb{Z}}, T)^{\mathfrak{F}}_v$ recovers (formally locally), the more *ad hoc* constructions of Howard-M.

---

[10]There is the GSpin variant that's actually of Hodge type, but its only real role is to reassure us that there is in fact an abelian scheme floating around somewhere.

[11]The positivity here is a consequence of the positivity of the Rosati involution.

[12]Here, we are actually using the self-duality of V_p. If we were working more canonically, we would have to replace it with the dual $V_p^{\vee}$.

Remark 6.0.6 (Linear invariance in higher codimensions)**.** There is an action of GL_n on V^n that commutes with that of G. Once again, this action doesn't affect the cycles $\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v$, but it *does* change the moment matrices. That is, for any $g \in \mathrm{GL}_n(\mathbb{Q})$, we have a canonical isomorphism

$$\mathscr{Z}_K(gW_{\mathbb{Z}}, T)^{\mathfrak{F}}_v \xrightarrow{\simeq} \mathscr{Z}_K(W_{\mathbb{Z}}, {}^t gTg).$$

When $W_{\mathbb{Z}} = V^n_{\mathbb{Z}}$ and $g \in \mathrm{GL}_n(\mathbb{Z})$, this recovers the key *linear invariance* result from Howard-M.

her_codim **Remark 6.0.7** (Zero locus in higher codimensions)**.** Remark 5.0.13 shows that we have

$$\mathscr{Z}_K(W_{\mathbb{Z}}, 0)^{\mathfrak{F}}_v \xrightarrow{\simeq} \mathscr{S}^{\mathfrak{F}}_{K,v} \times_{0,\mathbf{V}(\boldsymbol{\omega}_{W_p}),0} \mathscr{S}^{\mathfrak{F}}_{K,v}$$

This shows that it is just $\mathscr{S}^{\mathfrak{F}}_{K,v}$ equipped with the structure sheaf given by the Koszul complex associated with the zero cosection of $\boldsymbol{\omega}_{W_p}$. In the global setting, the class in K-theory of this structure sheaf will represent the top Chern class of $\boldsymbol{co}_{W_p}$. When $W_p = V^n_p$, this is nothing but $(-1)^n c_1(\boldsymbol{\omega}_{V_p})^n$.

Remark 6.0.8 (Product formula with moment matrices)**.** If $W_{1,\mathbb{Z}}$ and $W_{2,\mathbb{Z}}$ are lattices in V^{n_1} and V^{n_2}, then we have (once again all fiber products are derived):

$$\mathscr{Z}_K(W_{1,\mathbb{Z}}, T_1)^{\mathfrak{F}}_v \times_{\mathscr{S}^{\mathfrak{F}}_{K,v}} \mathscr{Z}_K(W_{2,\mathbb{Z}}, T_2)^{\mathfrak{F}}_v \xrightarrow{\simeq} \bigsqcup_{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix}} \mathscr{Z}_K(W_{1,\mathbb{Z}} \oplus W_{2,\mathbb{Z}}, T)^{\mathfrak{F}}_v.$$

Example 6.0.9 (Product formula with one zero matrix)**.** Suppose that $T_2 = 0$. In this case, combining the previous remark with Remark 6.0.7 shows that we have

$$\mathscr{Z}_K(W_{1,\mathbb{Z}}, T_1)^{\mathfrak{F}}_v \times_{0,\mathbf{V}(\boldsymbol{\omega}_{W_{2,p}}),0} \mathscr{S}^{\mathfrak{F}}_{K,v} \simeq \mathscr{Z}_K(W_{1,\mathbb{Z}}, T_1)^{\mathfrak{F}}_v \times_{\mathscr{S}^{\mathfrak{F}}_{K,v}} \mathscr{Z}_K(W_{2,\mathbb{Z}}, 0)^{\mathfrak{F}}_v \simeq \mathscr{Z}_K(W_{1,\mathbb{Z}} \oplus W_{2,\mathbb{Z}}, \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix})^{\mathfrak{F}}_v.$$

Here, we are using the fact that there is only one semi-definite symmetric matrix with T_1 and 0 along the block diagonal.

Example 6.0.10 (Improper intersection)**.** Suppose that $\mathscr{Z}_K(W_{\mathbb{Z}}, T)$ is *classical* (can be checked using the criterion from the end of the previous section). Taking $W_{1,\mathbb{Z}} = W_{2,\mathbb{Z}} = W_{\mathbb{Z}}$ and $T_1 = T_2 = T$, and applying Remark 5.0.13, we see that

$$\mathscr{Z}_K(W_{\mathbb{Z}}, T)^{\mathfrak{F}}_v \times_{\mathscr{S}^{\mathfrak{F}}_{K,v}} \mathscr{Z}_K(W_{\mathbb{Z}}, T)^{\mathfrak{F}}_v \simeq \mathscr{Z}_K(W_{\mathbb{Z}}, T)^{\mathfrak{F}}_v \times_{0,\mathbf{V}(\boldsymbol{\omega}_{W_p}),0} \mathscr{S}^{\mathfrak{F}}_{K,v} \simeq \mathscr{Z}_K(W_{\mathbb{Z}}, T)^{\mathfrak{F}}_v \times_{\mathscr{S}^{\mathfrak{F}}_{K,v}} \mathscr{Z}_K(W_{\mathbb{Z}}, 0)^{\mathfrak{F}}_v$$

alization **Remark 6.0.11** (Generalization)**.** Everything here generalizes *mutatis mutandis* to the situation where V is a ι-Hermitian space over an associative $\mathbb{Q}$-algebra D equipped with a positive involution ι, with $G = \mathrm{U}_D(V)$ the associated unitary group, and with the Shimura cocharacter splitting an isotropic subspace of $V_{\mathbb{R}}$. In this case, the cycles one obtains are indexed by positive semi-definite ι-Hermitian matrices with coefficients in D, and the minimal (virtual) codimension one sees is the dimension d_+ of the isotropic subspace. A basic case is where $D = E$ is a CM field with totally real maximal subfield F, in which case the Shimura cocharacter is given by a tuple (p_τ, q_τ) of signatures indexed by places $\tau : F \to \mathbb{R}$. Here, the minimal codiension is $\sum_\tau \min\{p_\tau, q_\tau\}$.

7. Algebraization

Let the setup be as in §5. Here's a quick explanation of how one algebraizes the cycles $\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v$ into quasi-smooth maps $\mathscr{Z}_K(W_{\mathbb{Z}}) \to \mathscr{S}_K$.

Remark 7.0.1 (Beauville-Laszlo gluing)**.** The maps $\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v \to \mathscr{S}^{\mathfrak{F}}_{K,v}$ are *finite*. Therefore, if we choose an affine étale cover $\operatorname{Spec} R \to \mathscr{S}_K$, the restriction of $\mathscr{Z}_K(W_{\mathbb{Z}})^{\mathfrak{F}}_v$ over $\operatorname{Spf} \hat{R}_v$ (here $\hat{R}_v$ is the v-adic completion of R) algebraizes, in the sense that it is represented by the spectrum of a finite[13] animated commutative $\hat{R}_v$-algebra, call it $\hat{B}_v$. Derived Beauville-Laszlo gluing (as explained for instance by Bhatt) now says that algebraizing $\hat{B}_v$ to a finite (animated commutative) R-algebra B is the same as finding a finite $R[1/p]$-algebra $B[1/p]$ and an isomorphism

[13]For us, this means that the complex of $\hat{R}_v$-modules underlying $\hat{B}_v$ is perfect.

$\hat{R}_v \otimes_R B[1/p] \xrightarrow{\simeq} \hat{B}_v[1/p]$ of finite $\hat{R}_v[1/p]$-algebras. In other words, to get $\mathscr{Z}_K(W_{\mathbb{Z}})$, we have to write down the generic fiber $Z_K(W_{\mathbb{Z}}) \to \mathrm{Sh}_K$ of $\mathscr{Z}_K(W_{\mathbb{Z}})$, and show that its restriction to p-adic affinoid animated $\mathbb{Q}_p$-algebras is given by the rigid fiber of $\mathscr{Z}_K(W_{\mathbb{Z}})_v^{\mathfrak{F}}$.[14]

Remark 7.0.2 (Saved by abelian varieties)**.** This sort of gluing condition is a little bit involved to check directly, so we will take a shortcut, and fall back to the space $\widetilde{\mathbb{H}}(\mathcal{A}_1, \mathcal{A}_2)$ of derived morphisms that exists globally. Now, we just have to cut out $\mathscr{Z}_K(W_{\mathbb{Z}})$ within this existing derived scheme. It turns out we can characterize this derived subscheme by knowing its points over $\overline{\mathbb{F}}_p$ and $\mathbb{C}$, and by knowing what its tangent complex should be. Therefore, once we have the construction of the generic fiber and of the formal cycle, the gluing data necessary for Beauville-Laszlo will be obtained automatically.

So all said and done, it remains to describe the generic fiber $Z_K(W_{\mathbb{Z}})$.

Construction 7.0.3 (Derived special cycles over the complex fiber)**.** Over the complex fiber $\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$, we can consider the local system

$$G(\mathbb{Q})\backslash X \times Y(W_{\mathbb{Z}})/K \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where $Y(W_{\mathbb{Z}}) = \{(w, g) \in W \times G(\mathbb{A}_f) : \ w \in gW_{\hat{\mathbb{Z}}}\}$. This admits a map to the vector bundle $\mathrm{gr}^{-1}_{\mathrm{Hdg}}\, \mathbf{dR}_K(W) = \boldsymbol{co}_W$ where $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}}\, \mathbf{dR}_K(W)$ is the filtered de Rham realization associated with W obtained in the complex fiber from the associated filtered vector bundle over the compact dual $\check{X}$. The *derived* pullback of the zero section of $\mathrm{gr}^{-1}_{\mathrm{Hdg}}\, \mathbf{dR}_K(W)$ now gives a locally finite unramified derived lci (in the sense of derived complex geometry, whatever that is) map $Z_K(W_{\mathbb{Z}})^{\mathrm{an}} \to \mathrm{Sh}_K(\mathbb{C})$.

Remark 7.0.4 (Noether-Lefschetz loci)**.** The underlying classical cycle of $Z_K(W)(\mathbb{C})$ is a *Noether-Lefschetz locus* where the associated variation of $\mathbb{Z}$-Hodge structures is picking up additional Hodge cycles: Indeed, the image of $(w, g) \in W$ in $\mathrm{gr}^{-1}_{\mathrm{Hdg}}\, \mathbf{dR}_K(W)$ vanishes precisely when it lies in $\mathrm{Fil}^0_{\mathrm{Hdg}}\, \mathbf{dR}_K(W)$.

Construction 7.0.5 (Algebraization and descent)**.** The theory of Shimura varieties gives an immediate algebraization of the classical analytic space $Z_K(W_{\mathbb{Z}})(\mathbb{C})$ underlying $Z_K(W_{\mathbb{Z}})^{\mathrm{an}}$, and we can give an algebraization of its derived structure sheaf by working étale locally, where it is given by a Koszul complex associated with a cosection of $\boldsymbol{co}_W$. The theory of canonical models now gives a canonical descent over the reflex field E for the classical truncation, and a similar noodling around with Koszul complexes also gives the descent for the derived scheme.

Remark 7.0.6 (Alternate construction using infinitesimal cohomology)**.** In the arXiv version of the derived cycles paper, there is a more elaborate construction of the generic fiber that once again uses the Siegel type assumption, and uses infinitesimal cohomology of abelian schemes to cut out $Z_K(W_{\mathbb{Z}})$ inside a derived Hom scheme $\tilde{\mathbb{H}}(\mathcal{A}_1, \mathcal{A}_2)$. This agrees with the construction using canonical models, but only works under the Siegel type assumption.

Remark 7.0.7 (Virtual fundamental classes)**.** Once we have the quasi-smooth cycles $\mathscr{Z}_K(W_{\mathbb{Z}}) \to \mathscr{S}_K$, in the situation of Remark 6.0.11, we can obtain for every $T \in \mathrm{Herm}_n(D)_{\geq 0}$, virtual fundamental classes $\mathcal{C}_K(W_{\mathbb{Z}}, T) \in \mathrm{CH}^{nd_+}(\mathscr{S}_K)_{\mathbb{Q}}$. This is constructed using K-theoretic methods and is explained in Appendix H of the arXiv version of the derived cycles paper. Many properties of these cycle classes fall out immediately from the properties of the underlying geometric cycles explained in the formal situation in §6.

[14] There is recent work of Achinger-Youcis that pushes this kind of gluing through for arbitrary algebraic spaces, though only in the classical context.