ON THE IRREDUCIBILITY OF THE MODULI OF POLARIZED K3 SURFACES IN FINITE CHARACTERISTIC

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Abstract. We study p-Hecke correspondences on the μ-ordinary locus of the mod-p fiber of a Shimura variety of Hodge type. We also study the p-adic monodromy over the μ-ordinary locus using ideas of Chai and Hida. Applying these ideas to certain orthogonal Shimura varieties attached to quadratic lattices, and using the Kuga-Satake period map, we conclude that the moduli stack of primitively polarized K3 surfaces of any fixed degree is geometrically irreducible in characteristic $p > 0$.

Introduction

Fix a positive integer $d \geq 1$, and consider the moduli stack M°_{2d} over $\mathbb{Z}$ that parameterizes primitively polarized K3 surfaces of degree $2d$. In this note, we prove:

Theorem 1. *For every prime p, the fiber $\mathsf{M}^{\circ}_{2d,\mathbb{F}_p}$ is geometrically irreducible.*

When p is odd and $p^2 \nmid d$, this result was shown in [25]. The point here is to prove this unconditionally, for which we follow the spirit of a paper of de Jong [5] and reduce to the known cases, by relating the ordinary locus of $\mathsf{M}^{\circ}_{2d/p^2}$ with that of M°_{2d} using p-ordinary Hecke correspondences.

A Hodge theoretic analogue of the essential idea is easy to explain: Suppose that (X, ξ) is a primitively polarized K3 surface over $\mathbb{C}$ of degree $2d$. Then the Betti cohomology $H^2(X, \mathbb{Z})$ is a pure Hodge structure of weight 2, and the Poincaré pairing endows it with the structure of a quadratic space over $\mathbb{Z}$ that is isometric to $U = \boldsymbol{H}^{\oplus 3} \oplus E_8^{\oplus 2}$. Here, $\boldsymbol{H}$ is the hyperbolic plane, and E_8 is the root lattice corresponding to its eponymous Dynkin type.

We will distinguish one hyperbolic plane $\boldsymbol{H} \subset U$, and choose a hyperbolic basis e, f for it satisfying $e^2 = f^2 = 0, [e, f] = 1$. Let $U' = \boldsymbol{H}^{\perp} \subset U$ be its orthogonal complement, so that we can write U as the orthogonal direct sum

$$U = \boldsymbol{H} \perp U'.$$

We can choose the isometry

$$H^2(X, \mathbb{Z}) \xrightarrow{\simeq} U$$

so that the Chern class of ξ maps to the element $e + df \in \boldsymbol{H}$. Within $H^2(X, \mathbb{Q})$ we have a lattice corresponding on the right hand side to the subspace

$$\langle p^{-1}e, pf \rangle \perp U' \subset U_{\mathbb{Q}}.$$

The basic point is that this lattice corresponds to an 'isogenous' K3 surface X' equipped with an isometry

$$H^2(X', \mathbb{Z}) \xrightarrow{\simeq} \langle pe, p^{-1}f \rangle \perp U',$$

and that it admits a canonical primitive polarization ξ' of degree $2p^2 d$, whose Chern class maps under the above isometry to the element $pe + p^2 \cdot (p^{-1}f)$.

If one does this carefully in families, one finds essentially a Hecke correspondence between the moduli of primitively polarized K3 surfaces of degrees $2d$ and $2p^2 d$, and this allows the direct comparison between their connected components.

In characteristic p of course, Hodge theoretic methods are no longer valid. However, an analogue does work if we restrict to the *ordinary* locus. It gives us a p-adic Hecke correspondence between the ordinary loci of the moduli spaces above. Roughly speaking, we obtain a morphism

$$\mathsf{M}^{\circ,\mathrm{ord}}_{2p^2d,\mathbb{F}_p} \to \mathsf{M}^{\circ,\mathrm{ord}}_{2d,\mathbb{F}_p}$$

between the ordinary loci, which factors as a purely inseparable map followed by a finite étale cover obtained from the so-called Igusa tower over the ordinary locus. To finish, we show that the monodromy of the Igusa tower can be sufficiently controlled so as to see that the finite étale cover involved above induces a bijection on geometric connectec components.

This is of course an impressionistic sketch: none of this is possible in so direct a fashion. Instead, we have to work with the associated GSpin Shimura varieties and their integral models, considered in [24]. As always is the case with these spaces, the lack of a moduli interpretation necessitates the use of more group theoretic methods, and appeals (implicitly) to the theory of motives with absolute Hodge cycles and (explicitly) to integral p-adic Hodge theory.

The methods used here can also be used to prove irreducibility results for Noether-Lefschetz loci in the moduli of polarized K3 surfaces or of cubic fourfolds. They will also be employed in ongoing work of the author with B. Howard concerning the modularity of generating series of higher codimension cycles on orthogonal Shimura varieties.

Also, since there is little additional effort to working more generally, we have chosen to include a study of the μ-ordinary locus of a general Shimura variety of Hodge type at a place of good reduction, as well as the Igusa tower over it. Among other things, this should find applications towards the Blasius-Rogawski congruence conjecture, as in [27], [33].

Here is a brief summary of the contents of this paper. In § 1, we introduce in abstract form ideas due to Chai and Hida concerning certain irreducibility results for an abstract version of the Igusa towers arising in this theory. The methods are quite straightforward, and, in applications, require only a good understanding of the set of connected components of the fibers of the Shimura variety, as well as the existence of certain so-called hypersymmetric points.

In § 2, we lay out the study of the μ-ordinary locus of a Shimura variety of Hodge type at a place of good reduction. This mainly involves collecting results of [18],[16],[34] and [32]. We apply these results in § 3 to construct the Igusa tower over the μ-ordinary locus and to study its relation with the p-adic tower over the generic fiber. In § 4, we look at the space of p-isogenies over the ordinary locus, and show that it has the expected properties.

In the final section, we apply these general results to the case of GSpin Shimura varieties to show that some the integral models constructed in [24] continue to have geometrically irreducible special fibers at certain places of bad reduction. As mentioned above, this is done by comparing the ordinary loci of such integral models with those with good reduction using p-adic correspondences. We combine this with the integral period map constructed in [25] to prove Theorem 1.

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Conventions

- We will fix a prime number p for the entirety of this paper.
- Given a set X and any Grothendieck site, we will write $\underline{X}$ for the locally constant sheaf over the site associated with the constant presheaf sending every object to X.

- Given a topos, a smooth $\mathbb{Z}_p$-group scheme D and an object S in the topos, an $\underline{D(\mathbb{Z}_p)}$-torsor $\pi : \mathcal{P} \to S$ is an inverse system
$$\{\pi_n : \mathcal{P}_n \to S\}_{n\in\mathbb{Z}_{\geq 1}},$$
where π_n is a torsor under $\underline{D(\mathbb{Z}/p^n\mathbb{Z})}$.
- For any finite set of primes T, we will set
$$\mathbb{A}_f^T = \prod_{\ell\notin T}{}' \mathbb{Q}_\ell,$$
the restricted product over all completions of $\mathbb{Q}$ at finite places not in T. If $T = \{\ell\}$ is a singleton, we will write $\mathbb{A}_f^\ell$ instead.
- For any local or global field F in characteristic 0, we will write $\mathscr{O}_F$ for its ring of integers.
- Suppose that H is an algebraic group over a local or global field F with a model $H_{\mathscr{O}_F}$ over $\mathscr{O}_F$. If R is an $\mathscr{O}_F$-algebra, we will abuse notation and write $H(R)$ instead of $H_{\mathscr{O}_F}(R)$ whenever the integral model is clear from context.
- We will on occasion use the geometric notation for change of scalars. If $f : R \to S$ is a map of rings and M is an R-module, then we will denote the induced S-module $S \otimes_{f,R} M$ by f^*M. If the map f is clear from context, then we will also write M_S for the same S-module.
- If $\varphi : R \to R$ is an endomorphism of R, then a **φ-module** over R is an R-module M equipped with a map $\varphi^*M \to M$ of R-modules.
- Suppose that R is a ring and suppose that $\mathbf{C}$ is an R-linear tensor category that is a faithful tensor sub-category of Mod_R, the category of R-modules. Suppose in addition that $\mathbf{C}$ is closed under taking duals, symmetric and exterior powers in Mod_R. Then, for any object $D \in \mathrm{Obj}(\mathbf{C})$, we will denote by $D^\otimes$ the direct sum of the tensor, symmetric and exterior powers of D and its dual.
- In this paper, 'abelian scheme' will be used exclusively as short-hand for 'abelian scheme up to prime-to-p isogeny' as defined in, for instance, [24, §3.7].

1. Group theoretic preliminaries

The purpose of this short section is to abstract some ideas due to Chai and Hida on a 'pure thought' study of the monodromy of Igusa towers. All the key ideas can already be found in [11] and [4].

The reader can return to consult it as necessary.

Proposition 1.1. *Let H be a connected reductive group over $\mathbb{Q}$ such that $H_{\mathbb{Q}_p}$ contains a maximal torus that splits over a cyclic extension of $\mathbb{Q}_p$ (this hypothesis holds in particular when H is unramified at p). Then H satisfies weak approximation with respect to $\{p\}$; that is, $H(\mathbb{Q})$ is dense in $H(\mathbb{Q}_p)$.*

Proof. This is essentially contained in [30]. If H is semi-simple and simply connected, the result follows directly from Theorem 7.8 of *loc. cit.* In general, let $\widetilde{H}$ be the simply connected cover of the derived group of H. Then we find from Proposition 2.11 of *loc. cit.* that there are quasi-trivial[1] tori T_1 and T_2 over $\mathbb{Q}$, and an integer $m \geq 1$ such that there is a central isogeny: $\widetilde{H}^m \times T_1 \to H^m \times T_2$. In fact, the proof of this result shows that we can choose T_1 and T_2 to have the same splitting field as the maximal central torus of H. It is easy to see H satisfies weak approximation with respect to $\{p\}$ if and only if $H^m \times T_2$ does, so we can replace H by the latter group and assume that it admits a central cover $H_1 \to H$ where H_1 is a product of a semi-simple, simply connected group with a quasi-trivial torus.

[1]This means that the Galois representation attached to the character group is a permutation representation.

Let F be the kernel of $H_1 \to H$: It is a central sub-group of H_1, and so, by our hypothesis, splits over a cyclic extension of $\mathbb{Q}_p$. The result now follows from Proposition 7.10 and Corollary 2 in Ch. 7 of *loc. cit.*. $\square$

Corollary 1.2. *With the hypotheses as above, for any integer $n \geq 1$, the map*

$$H(\mathbb{Z}_{(p)}) \to H(\mathbb{Z}/p^n\mathbb{Z})$$

is surjective.

Proof. Let $P_n = \ker(H(\mathbb{Z}_p) \to H(\mathbb{Z}/p^n\mathbb{Z}))$. By (1.1), $H(\mathbb{Q})P_n = H(\mathbb{Q}_p)$. We now have:

$$H(\mathbb{Z}_p) = H(\mathbb{Q}_p) \cap H(\mathbb{Z}_p) = H(\mathbb{Q})P_n \cap H(\mathbb{Z}_p) = H(\mathbb{Z}_{(p)})P_n.$$

The corollary now follows, since the map $H(\mathbb{Z}_p) \to H(\mathbb{Z}/p^n\mathbb{Z})$ is surjective by the smoothness of $H_{\mathbb{Z}_{(p)}}$. $\square$

1.3. Keep the hypotheses of (1.1). Suppose that we have a finite set of primes T containing p, a reductive group G over $\mathbb{Q}$, and an $H(\mathbb{Z}_p)$-torsor $\mathcal{P}$ over a scheme S, such that the morphism $\mathcal{P} \to S$ is equivariant for an action of $G(\mathbb{A}_f^T)$ that commutes with the $H(\mathbb{Z}_p)$-action on $\mathcal{P}$. In particular, for every $n \geq 1$, the $H(\mathbb{Z}/p^n\mathbb{Z})$-torsor

$$\mathcal{P}_n \to S$$

is $H(\mathbb{Z}/p^n\mathbb{Z}) \times G(\mathbb{A}_f^T)$-equivariant, with the first factor acting trivially on S.

Let $Q \subset H_{\mathbb{Z}_p}$ be a closed $\mathbb{Z}_p$-subgroup scheme such that the quotient $\mathsf{X} = H_{\mathbb{Z}_p}/Q$ is represented by a scheme over $\mathbb{Z}_p$.

We will need, for every $n \geq 1$, the contraction product

$$\mathcal{P}_{\mathsf{X},n} \coloneqq \mathcal{P}_n \times^{H(\mathbb{Z}/p^n\mathbb{Z})} \mathsf{X}(\mathbb{Z}/p^n\mathbb{Z}).$$

We will be interested in the morphism of the sets of connected components

$$\pi_0(\mathcal{P}_{\mathsf{X},n}) \to \pi_0(S). \tag{1.3.1}$$

1.4. With the notation as above, suppose now that we have anothe reductive group M over $\mathbb{Q}$ with the following properties:

- There exists an embedding
$$\psi : M_{\mathbb{A}_f^T} \hookrightarrow G_{\mathbb{A}_f^T}.$$
- There exists a smooth model $M_{\mathbb{Z}_{(p)}}$ for M over $\mathbb{Z}_{(p)}$, and an isomorphism
$$\varphi : M_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \xrightarrow{\simeq} H_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p.$$

In particular, we have an embedding

$$\Phi : M(\mathbb{Z}_{(p)}) \xrightarrow{m \mapsto (\varphi(m), \psi(m))} H(\mathbb{Z}_p) \times G(\mathbb{A}_f^T),$$

inducing for every $n \geq 1$ a map

$$\Phi_n : M(\mathbb{Z}_{(p)}) \xrightarrow{m \mapsto (\varphi_n(m), \psi(m))} H(\mathbb{Z}/p^n\mathbb{Z}) \times G(\mathbb{A}_f^T).$$

For any reductive group D over a field, let $\rho_D : \widetilde{D} \to D$ be the simply connected cover of the derived subgroup of D. Let $\widetilde{M}_{\mathbb{Z}_{(p)}}$ (resp. $\widetilde{H}_{\mathbb{Z}_p}$) be the normalization of $M_{\mathbb{Z}_{(p)}}$ in $\widetilde{M}$ (resp. $\widetilde{H}$), and let $\widetilde{Q}$ be the pre-image of Q in $\widetilde{H}_{\mathbb{Z}_p}$.

Let $Z_{H,\mathbb{Z}_p} \subset H_{\mathbb{Z}_p}$ be the Zariski closure of the center $Z_H \subset H$.

Proposition 1.5. *Suppose that the following conditions hold:*

(1) $\rho_G(\widetilde{G}(\mathbb{Q}_\ell))$ acts trivially on $\pi_0(S)$;
(2) For all $\ell \notin T$, $G_{\mathbb{Q}_\ell}$ is isotropic;

(3) $G(\mathbb{A}_f^T)$ *acts transitively on* $\pi_0(S)$*;*
(4) $\Phi_n(M(\mathbb{Z}_{(p)}))$ *fixes a point* $\varpi \in \pi_0(\mathcal{P}_{\mathsf{X},n})$*;*
(5) Q *contains* $Z_{H,\mathbb{Z}_p}$*;*
(6) *The* $\mathbb{Z}_{(p)}$*-group* $\widetilde{M}_{\mathbb{Z}_{(p)}}$ *and the* $\mathbb{Z}_p$*-group* $\widetilde{Q}$ *are smooth with connected special fiber.*

Then (1.3.1) *is a bijection.*

Proof. We will need the following consequence of the Kneser-Tits conjecture (see [30, Theorem 7.6]): For any simply connected isotropic group D over $\mathbb{Q}_\ell$, $D(\mathbb{Q}_\ell)$ does not admit any finite index sub-groups.

This, combined with hypotheses (1) and (2), implies that $\rho_G(\widetilde{G}(\mathbb{Q}_\ell))$ acts trivially on $\pi_0(\mathcal{P}_{\mathsf{X},n})$ as well. Let ϖ be as in hypothesis (4), and let $F_\varpi \subset \pi_0(\mathcal{P}_{\mathsf{X},n})$ be the fiber over the image of ϖ in $\pi_0(S)$. By hypothesis (3), it is enough to show that F_ϖ is a singleton: Any other fiber is a translate of this by an element of $G(\mathbb{A}_f^T)$.

Hypothesis (4) implies that the subgroup

$$\tilde{H}_n := \{\phi_n(m) :\ m \in M(\mathbb{Z}_{(p)}),\, \psi(m) \in \rho_G(\widetilde{G}(\mathbb{A}_f^T))\} \subset H(\mathbb{Z}/p^n\mathbb{Z})$$

fixes ϖ.

It is now enough to show that $\tilde{H}_n$ acts transitively on the fiber $F_\varpi \subset \pi_0(\mathcal{P}_{\mathsf{X},n})$. For this, it is enough to know that it surjects onto $\mathsf{X}(\mathbb{Z}/p^n\mathbb{Z})$ via the map induced by φ_n. Note, however that $\tilde{H}_n$ contains $\rho_M(\widetilde{M}(\mathbb{Z}_{(p)}))$. Therefore, it is enough to show that the latter surjects onto $\mathsf{X}(\mathbb{Z}/p^n\mathbb{Z})$.

First, note that the natural map

$$\widetilde{H}_{\mathbb{Z}_p}/\widetilde{Q} \to H_{\mathbb{Z}_p}/Q$$

is an isomorphism of fppf sheaves over $\mathbb{Z}_p$. Indeed, it is a monomorphism by definition, and hypothesis (5) implies that it is also surjective.

Now, hypothesis (6) ensures that $\widetilde{H}_{\mathbb{Z}_p}$ and $\widetilde{Q}$ are smooth over $\mathbb{Z}_p$ with connected fibers. Therefore, by Lang's theorem, we have

$$\mathsf{X}(\mathbb{Z}/p^n\mathbb{Z}) = \widetilde{H}(\mathbb{Z}/p^n\mathbb{Z})/\widetilde{Q}(\mathbb{Z}/p^n\mathbb{Z}).$$

It now follows from (1.2) that $\rho_M(\widetilde{M}(\mathbb{Z}_{(p)}))$ maps surjectively onto $\mathsf{X}(\mathbb{Z}/p^n\mathbb{Z})$ via φ_n. □

2. The μ-ordinary locus

Let (G, X) be a Shimura datum with reflex field E, and suppose that G is unramified[2] at p. This is equivalent to requiring that it admit a reductive model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, which we now fix for the remainder of this section. Set $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$: this is a hyperspecial compact open sub-group in $G(\mathbb{Q}_p)$. We will also assume that (G, X) is of Hodge type, so that it is equipped with an embedding

$$(G, X) \hookrightarrow (\mathrm{GSp}(H), \mathrm{S}^{\pm}(H))$$

into a Siegel Shimura datum.

Further, we will fix a $\mathbb{Z}_{(p)}$-lattice $H_{(p)} \subset H$ such that the embedding $G \hookrightarrow \mathrm{GL}(H)$ arises from an embedding of $\mathbb{Z}_{(p)}$-group schemes $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(H_{(p)})$.[3] There now exists a collection of tensors $\{s_\alpha\} \subset H_{(p)}^{\otimes}$ such that $G_{\mathbb{Z}_{(p)}}$ is their point-wise stabilizer in $\mathrm{GL}(H_{(p)})$; cf. [18, p. 1.3.2].

[2] Quasi-split and split over an unramified extension.
[3] This is always possible; cf. [16].

2.1. Since $G_{\mathbb{Q}_p}$ is unramified by hypothesis, and therefore quasi-split, we can find a Borel subgroup of $G_{\mathbb{Q}_p}$, arising as the generic fiber a Borel sub-group scheme

$$B \subset G_{\mathbb{Z}_p}.$$

Fix a maximal torus $T \subset B$ defined over $\mathbb{Z}_p$, and let $X_*(T)$ be the (unramified) $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-module of cocharacters of T.

Let $X_*(T)^+ \subset X_*(T)$ be the subset of **dominant** cocharacters: Here, a cocharacter $\lambda : \mathbb{G}_m \to T$ is dominant if the eigenvalues of $\lambda(p)$ acting on $(\mathrm{Lie}\, B)_{\overline{\mathbb{Q}}_p}$ have non-negative p-adic valuations.

2.2. Fix a place $v|p$ of E, a choice of algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding $E_v \hookrightarrow \overline{\mathbb{Q}}_p$.

Let $K \subset G(\mathbb{A}_f)$ be a neat[4] compact open sub-group of the form $K_pK^p = K_p \times K^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$. Attached to this is the Shimura variety $\mathrm{Sh}_K \coloneqq \mathrm{Sh}_K(G,X)$ over E along with its smooth integral canonical model $\mathcal{S}_K = \mathcal{S}_K(G,X)_{(v)}$ over $\mathscr{O}_{E,(v)}$. Using the representation $H_{(p)}$, we obtain over $\mathcal{S}_K$ an abelian scheme $\mathcal{A}$, defined up to prime-to-p isogeny, and equipped with a canonical class of quasi-polarizaitions $[\lambda]$.

2.3. The representation H has several motivic incarnations: First, over Sh_K, we have the dual $\boldsymbol{H}_p$ of the p-adic Tate module $T_p(\mathcal{A})$: This is a lisse p-adic étale sheaf over Sh_K. Secondly, we have the first relative de Rham cohomology $\boldsymbol{H}_{\mathrm{dR}}$ of $\mathcal{A}$ over $\mathcal{S}_K$: This is a vector bundle over $\mathcal{S}_K$ equipped with a Hodge filtration $\mathrm{Fil}^1 \boldsymbol{H}_{\mathrm{dR}} \subset \boldsymbol{H}_{\mathrm{dR}}$, and the integrable Gauss-Manin connection.

Next, over the special fiber $\mathcal{S}_{K,k(v)}$, we have the contra-variant Dieudonné F-crystal $\boldsymbol{H}_{\mathrm{cris}}$ attached to $\mathcal{A}_{\mathcal{S}_{K,k(v)}}$: It is equipped with a Frobenius operator

$$F : \mathrm{Fr}^*_{\mathcal{S}_{K,k(v)}}\boldsymbol{H}_{\mathrm{cris}} \to \boldsymbol{H}_{\mathrm{cris}},$$

where $\mathrm{Fr}_{\mathcal{S}_{K,k(v)}}$ is the absolute Frobenius endomorphism of $\mathcal{S}_{K,k(v)}$. The evaluation of this F-crystal over the formal completion of $\mathcal{S}_K$ along $\mathcal{S}_{K,k(v)}$ is a vector bundle with topologically nilpotent connection, which can be canonically identified with that obtained from $\boldsymbol{H}_{\mathrm{dR}}$.

Further, over the analytic space $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$, we have the relative first Betti (or singular) cohomology $\boldsymbol{H}_B$ with coefficients in $\mathbb{Z}_{(p)}$: this can be viewed as a variation of Hodge structures over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$.

2.4. On the Betti side, we have a collection of Hodge tensors $\{s_{\alpha,B}\} \subset H^0\bigl(\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}, \boldsymbol{H}_B^{\otimes}\bigr)$ such that the sheaf I_B of isomorphisms $\underline{H}_{(p)} \xrightarrow{\simeq} \boldsymbol{H}_B$ carrying s_α to $s_{\alpha,B}$ is a torsor under the locally constant sheaf $\underline{G_{\mathbb{Z}_{(p)}}}(\mathbb{Z}_p)$. This is because $\mathrm{Sh}_K(\mathbb{C})$ admits a uniformization:

$$\mathrm{Sh}_K(\mathbb{C}) = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})\backslash\bigl(X \times G(\mathbb{A}_f^p)\bigr)/K^p. \tag{2.4.1}$$

Under this uniformization, the local system $\boldsymbol{H}_B$ is the one attached to the representation $H_{(p)}$ of $G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$.

This slightly modified uniformization can be deduced from the tautological uniformization:

$$\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q})\backslash\bigl(X \times G(\mathbb{A}_f)\bigr)/K,$$

combined with weak approximation for connected reductive groups over $\mathbb{Z}_{(p)}$ (cf. for example [18, p. 2.2.6]), which shows that $G(\mathbb{Q})K_p = G(\mathbb{Q}_p)$.

[4]cf. [29, p. 0.6]

2.5. There exists a collection of tensors $\{s_{\alpha,\mathbb{A}_f^p}\} \subset H^0\big(\mathcal{S}_K, \boldsymbol{H}^{\otimes}_{\mathbb{A}_f^p}\big)$ with the following property: Let I^p be the étale sheaf (over $\mathcal{S}_K$) of trivializations $\underline{\mathbb{A}}_f^p \otimes H_{(p)} \xrightarrow{\simeq} \boldsymbol{H}_{\mathbb{A}_f^p}$ carrying, for each α, $1 \otimes s_\alpha$ to $s_{\alpha,\mathbb{A}_f^p}$. Then I^p is a right torsor under the constant sheaf of groups $\underline{G(\mathbb{A}_f^p)}$. More precisely, we have a canonical K^p-torsor over $\mathcal{S}_K$ represented by the inverse limit

$$\mathcal{S}_{K_p} = \varprojlim_{K_1^p \subset K^p} \mathcal{S}_{K_p K_1^p},$$

where $K_1^p \subset K^p$ ranges over the finite-index subgroups of K^p, and I^p is obtained from this torsor as a contraction product

$$I^p = \mathcal{S}_{K_p} \times^{K^p} G(\mathbb{A}_f^p).$$

Therefore, I^p/K^p admits a canonical section $[\eta] \in H^0(\mathcal{S}_K, I^p/K^p)$, called a **$K^p$-level structure**.

In terms of the uniformization in (2.5), given a pair $(\boldsymbol{h}, g^p) \in X \times G(\mathbb{A}_f^p)$ mapping to a point $x \in \mathrm{Sh}_K(\mathbb{C})$, the associated abelian scheme $\mathcal{A}_x$ with its K^p-level structure can be described as follows: The map $\boldsymbol{h} : \mathbb{S} \to G_{\mathbb{R}}$ endows $H_{(p)}$ with the structure of a polarizable $\mathbb{Z}_{(p)}$-Hodge structure of weight 1. This is exactly the cohomology of the abelian scheme $\mathcal{A}_x$, which is now equipped with an identification $H_{(p)} \xrightarrow{\simeq} \boldsymbol{H}_{B,x}$. The level structure is now the K^p-coset of the isomorphism

$$H_{\mathbb{A}_f^p} \xrightarrow[\simeq]{g} H_{\mathbb{A}_f^p} \xrightarrow{\simeq} \mathbb{A}_f^p \otimes \boldsymbol{H}_{B,x} = \boldsymbol{H}_{\mathbb{A}_f^p,x}.$$

2.6. There also exists a collection of tensors

$$\{s_{\alpha,p}\} \subset H^0\big(\mathrm{Sh}_K, \boldsymbol{H}_p^{\otimes}\big)$$

such that the sheaf I_p of isomorphisms $\underline{\mathbb{Z}}_p \otimes H_{(p)} \xrightarrow{\simeq} \boldsymbol{H}_p$ carrying $1 \otimes s_\alpha$ to $s_{\alpha,p}$[5] is a torsor under the locally constant sheaf $\underline{G(\mathbb{Z}_p)}$.

This torsor is actually independent of the choice of symplectic representation H, and can be canonically described as the torsor represented by the tower

$$\mathrm{Sh}_{K^p} = \{\mathrm{Sh}_{K_n}\}_n,$$

where $\mathrm{Sh}_{K_n} \to \mathrm{Sh}_K$ is the finite cover associated with the sub-group

$$K_n = \ker(K_p \to G(\mathbb{Z}/p^n\mathbb{Z})) \times K^p \subset K_p \times K^p = K.$$

Under the canonical comparison isomorphism between $\mathbb{Z}_p \otimes \boldsymbol{H}_B$ and $\boldsymbol{H}_p^{\mathrm{an}}$, $1 \otimes s_{\alpha,B}$ is carried to $s_{\alpha,p}$, for each α.

2.7. There is also a collection of parallel tensors

$$\{s_{\alpha,\mathrm{dR}}\} \subset H^0\big(\mathcal{S}_K, \mathrm{Fil}^0\, \boldsymbol{H}_{\mathrm{dR}}^{\otimes}\big).$$

For each α, the de Rham comparision isomorphism

$$\mathscr{O}_{\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}} \otimes \boldsymbol{H}_B \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{dR}}|_{\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}},$$

carries $1 \otimes s_{\alpha,B}$ to $s_{\alpha,\mathrm{dR}}$.

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[5]We will term such isomorphisms **G-structure preserving**. We leave to the reader the task of formalizing this notion, although we hope that its meaning will be clear from context in the remainder of the paper.

2.8. We also have a collection of tensors $\{s_{\alpha,\mathrm{cris}}\} \subset H^0\big((\mathcal{S}_{K,k(v)}/W(k(v)))_{\mathrm{cris}}, \boldsymbol{H}^{\otimes}_{\mathrm{cris}}\big)$, whose realizations along the formal completion of $\mathcal{S}_K$ along $\mathcal{S}_{K,k(v)}$ agree with $\{s_{\alpha,\mathrm{dR}}\}$ as sections of $\boldsymbol{H}_{\mathrm{dR}}$.

These tensors also have the following property: For every point $x_0 : \operatorname{Spec} k \to \mathcal{S}_{K,k(v)}$ valued in a perfect field k, the evaluation of $\boldsymbol{H}_{\mathrm{cris}}$ on the formal PD thickening $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} W(k)$ gives rise to an F-crystal $\boldsymbol{H}_{\mathrm{cris},x_0}$ over $W(k)$. More precisely, if $\sigma : W(k) \to W(k)$ is the lift of the Frobenius automorphsim of k, we obtain an operator

$$F : \sigma^*\boldsymbol{H}_{\mathrm{cris},x_0} \to \boldsymbol{H}_{\mathrm{cris},x_0}.$$

This in turn induces an isomorphism

$$F : \sigma^*\boldsymbol{H}_{\mathrm{cris},x_0}[p^{-1}]^{\otimes} \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},x_0}[p^{-1}].$$

Now, for every α, the evaluation $s_{\alpha,\mathrm{cris},x_0} \in \boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}$ is F-invariant, when viewed as an element of $\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}[p^{-1}]$:

$$F(\sigma^* s_{\alpha,\mathrm{cris},x_0}) = s_{\alpha,\mathrm{cris},x_0}.$$

Further, there exists an isomorphism

$$\tau : W(k) \otimes H_{(p)} \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},x_0}$$

carrying, for each α, $1 \otimes s_\alpha$ to $s_{\alpha,\mathrm{cris},x_0}$.

2.9. In particular, the $\mathrm{Fr}(W(k))$-module $\boldsymbol{H}_{\mathrm{cris},x_0}[p^{-1}]$ is an F-isocrystal with G-structure in the language of [21]. Given a choice of isomorphism τ as above, there exists a unique $b \in G(\mathrm{Fr}(W(k)))$ such that the following diagram commutes:

$$\begin{tikzcd}
W(k) \otimes H_{(p)} \arrow[r, "\sigma^*\tau"] \arrow[d, "b_{x_0,\tau}"'] & \sigma^*\boldsymbol{H}_{\mathrm{cris},x_0} \arrow[d, "F"] \\
W(k) \otimes H_{(p)} \arrow[r, "\tau"'] & \boldsymbol{H}_{\mathrm{cris},x_0}.
\end{tikzcd}$$

Let $\mathbb{D}$ be the diagonalizable group over $\mathbb{Q}$ with character group $\mathbb{Q}$. Associated with this data is the *Newton cocharacter* $\nu_{x_0,\tau} : \mathbb{D} \to G$, which splits the pull-back to $W(k) \otimes H_{(p)}$ of the slope filtration on $\boldsymbol{H}_{\mathrm{cris},x_0}$.

If τ is replaced by $\tau \circ g$ for $g \in G(W(k))$, then $b_{x_0,\tau}$ is in turn replaced by its σ-conjugate $g^{-1}b_{x_0,\tau}\sigma(g)$, and $\nu_{x_0,\tau}$ is replaced by $g^{-1}\nu_{x_0,\tau}g$.

2.10. The final point in this tensorial history is to do with the p-adic comparison isomorphism. Suppose that we are given a point $x_0 : \operatorname{Spec} k \to \mathcal{S}_{K,k(v)}$ as above and a lift $x : \operatorname{Spec} \mathscr{O}_L \to \mathcal{S}_K$, where $L/\mathrm{Fr}(W(k))$ is a finite extension. Choose a geometric point $\overline{x} : \operatorname{Spec} \overline{L} \to \mathrm{Sh}_K$ lying above the restriction of x to $\operatorname{Spec} L$. Then there exists a canonical crystalline comparison isomorphism

$$B_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \boldsymbol{H}_{p,\overline{x}} \xrightarrow{\simeq} B_{\mathrm{cris}} \otimes_{W(k)} \boldsymbol{H}_{\mathrm{cris},x_0}.$$

This isomorphism preserves G-structure, carrying, for each α, $1 \otimes s_{\alpha,p,\overline{x}}$ to $1 \otimes s_{\alpha,\mathrm{cris},x_0}$.

2.11. Consider the conjugacy class $\{\mu_X\}$ of Shimura cocharacters of $G_{\mathbb{C}}$ associated with the datum X. By the very definition of the reflex field, this conjugacy class is defined over E. Since $G_{\mathbb{Q}_p}$ is quasi-split, Lemma 1.1.3 of [22] implies that this conjugacy class has a representative over E_v. Furthermore, this representative can be chosen in such a way as to extend to a dominant cocharacter

$$\mu_X : \mathbb{G}_{m,\mathscr{O}_{E_v}} \to T_{\mathscr{O}_{E_v}}.$$

Set $\mu_p = \mu_X^{-1}$. Viewing this cocharacter as an element of the Galois module $X_*(T)$ fixed by the subgroup

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_v) \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p),$$

we form its 'norm'

$$N\mu_p = \sum_{\sigma\in\mathrm{Gal}(E_v/\mathbb{Q}_p)} \sigma\mu_p \in X_*(T)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)},$$

which is a cocharacter of T defined over $\mathbb{Z}_p$.

We will also need the 'average'

$$\nu_p = \frac{1}{[E_v : \mathbb{Q}_p]} N\mu_p \in X_*(T)_{\mathbb{Q}}^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}.$$

Let $M_{N\mu} \subset G_{\mathbb{Z}_p}$ be the centralizer of $N\mu_p$. Let $\mathrm{Lie}\, U_{N\mu}$ (resp. $\mathrm{Lie}\, U_{\mu}^-$) be the direct sum of the eigenspaces of $\mathrm{Lie}\, G_{\mathbb{Z}_p}$ on which $N\mu_p(p)$ acts via eigenvalues of positive (resp. negative) p-adic valuation. These subspaces are the Lie algebras of unipotent subgroups $U_{N\mu}$ and $U_{N\mu}^-$ of $G_{\mathbb{Z}_p}$, which are the radicals of *opposite* parabolic subgroups $P_{N\mu}$ and $P_{N\mu}^-$ with common Levi subgroup $M_{N\mu}$.

2.12. Consider the *reflex norm*

$$r_\mu : T_0 = \mathrm{Res}_{\mathscr{O}_{E_v}/\mathbb{Z}_p} \mathbb{G}_m \xrightarrow{\mathrm{Res}(\mu_p)} \mathrm{Res}_{\mathscr{O}_{E_v}/\mathbb{Z}_p} T_{\mathscr{O}_{E_v}} \xrightarrow{\mathrm{Nm}} T$$

associated with μ_p.

Here, the map on the far right is the usual norm map on tori: over $\overline{\mathbb{Z}}_p$ it can be identified with the homomorphism

$$\prod_{\sigma:E_v\to\overline{\mathbb{Q}}_p} T_{\overline{\mathbb{Z}}_p} \to T_{\overline{\mathbb{Z}}_p}$$
$$(x_\sigma) \mapsto \prod_\sigma x_\sigma.$$

Lemma 2.13. *The homomorphism r_μ has central image in $M_{N\mu}$.*

Proof. Working over $\overline{\mathbb{Q}}_p$, this amounts to the following claim: If α is a root for T (that is, a character appearing in the eigenspace decomposition of $\mathrm{Lie}\, G_{\overline{\mathbb{Q}}_p}$) such that $\langle\alpha, \mu_p\rangle > 0$, then $\langle\alpha, N\mu_p\rangle > 0$.

To see this, note that we have

$$\langle\alpha, N\mu_p\rangle = \sum_{\sigma\in\mathrm{Gal}(E_v/\mathbb{Q}_p)} \langle\sigma^{-1}\alpha, \mu_p\rangle.$$

The condition $\langle\alpha, \mu_p\rangle > 0$ ensures that α is a positive root, appearing in $\mathrm{Lie}\, B_{\overline{\mathbb{Q}}_p}$. Since B is defined over $\mathbb{Z}_p$ by hypothesis, the set of positive roots is Galois stable, and since μ_p is dominant, we must have $\langle\sigma^{-1}\alpha, \mu_p\rangle \geq 0$, for all $\sigma \in \mathrm{Gal}(E_v/\mathbb{Q}_p)$. This finishes the proof. $\square$

2.14. To be able to work with p-divisible groups over $\mathscr{O}_{E_v}$ uniformly, without making exceptions for the case $p = 2$, it will be useful to employ the theory of Breuil-Kisin modules from [17], extended to the case $p = 2$ by, among others, W. Kim [15]. This can be summarized as follows:

Suppose that k is a perfect field of characteristic p. Consider $\mathfrak{S}_k = W(k)[|u|]$, the power series ring in over variable over $W(k)$, and equip it with the Frobenius lift $\varphi : \mathfrak{S}_k \to \mathfrak{S}_k$ restricting to the canonical Frobenius automorphism σ on $W(k)$ and satisfying $\varphi(u) = u^p$. Set $\mathcal{E}(u) = u + p$, and consider the category $\mathrm{BT}_{/\mathfrak{S}_k}$ of pairs $(\mathfrak{M}, F_{\mathfrak{M}})$ where:

- $\mathfrak{M}$ is a finite free $\mathfrak{S}_k$-module;
- $F_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is an injective homomorphism whose cokernel is killed by $\mathcal{E}(u)$.

Such pairs are called **Breuil-Kisin modules**. Usually, $F_{\mathfrak{M}}$ will be omitted from the notation, and we will refer to the entire Breuil-Kisin module by its first entry $\mathfrak{M}$.

Then there is a contravariant equivalence of categories between p-divisible groups over $W(k)$ and $\mathrm{BT}_{/\mathfrak{S}_k}$ associating with a p-divisible group $\mathcal{H}$ a Breuil-Kisin module $\mathfrak{M}(\mathcal{H})$. Moreover, this equivalence has the following properties:

- If $\mathcal{H}_0$ is the reduction of $\mathcal{H}$ over k, then the (contravariant) Dieudonné F-crystal $M(\mathcal{H}_0)$ over $W(k)$ can be recovered by base-change along the map $\mathfrak{S}_k \xrightarrow{u \mapsto 0} W(k)$ of the pair $(\varphi^*\mathfrak{M}(\mathcal{H}), \varphi^*F_{\mathcal{H}})$.
- The de Rham realization $M_{\mathrm{dR}}(\mathcal{H})$ of $\mathcal{H}$ over $W(k)$, along with its Hodge filtration $\mathrm{Fil}^1 M_{\mathrm{dR}}(\mathcal{H})$, can be recovered as follows: We have
$$M_{\mathrm{dR}}(\mathcal{H}) = \mathfrak{S}_k/\mathcal{E}(u)\mathfrak{S}_k \otimes_{\mathfrak{S}_k} \varphi^*\mathfrak{M}(\mathcal{H}),$$
and $\mathrm{Fil}^1 M_{\mathrm{dR}}(\mathcal{H})$ is the image in $M_{\mathrm{dR}}(\mathcal{H})$ of the subspace
$$\mathrm{Fil}^1 \varphi^*\mathfrak{M}(\mathcal{H}) = F^{-1}_{\mathfrak{M}(\mathcal{H})}\big(\mathcal{E}(u)\mathfrak{M}(\mathcal{H})\big).$$
- If k'/k is an extension of perfect fields, and $\mathcal{H}' = W(k') \otimes_{W(k)} \mathcal{H}$, then there is a canonical isomorphism of Breuil-Kisin modules:
$$\mathfrak{S}_{k'} \otimes_{\mathfrak{S}_k} \mathfrak{M}(\mathcal{H}) \xrightarrow{\simeq} \mathfrak{M}(\mathcal{H}').$$

2.15. For future use, we note that $\mathrm{BT}_{/\mathfrak{S}_k}$ is embedded in the larger tensor category $\mathrm{BrK}_{/\mathfrak{S}_k}$ of pairs $(\mathfrak{M}, F_{\mathfrak{M}})$ where $\mathfrak{M}$ is a finite free $\mathfrak{S}_k$-module and
$$F_{\mathfrak{M}} : \varphi^*\mathfrak{M}[\mathcal{E}(u)^{-1}] \xrightarrow{\simeq} \mathfrak{M}[\mathcal{E}(u)^{-1}]$$
is an isomorphism of $\mathfrak{S}_k$-modules.

In fact, it is shown in [18] that there is a *covariant* fully faithful tensor functor
$$\mathfrak{M} : \mathrm{Rep}_{\mathrm{cris}}(\Gamma_k) \to \mathrm{BrK}_{/\mathfrak{S}_k}$$
such that, for every p-divisible group $\mathcal{H}$ as above, we have a canonical isomorphism of Breuil-Kisin modules:
$$\mathfrak{M}(\mathcal{H}) \xrightarrow{\simeq} \mathfrak{M}((T_p\mathcal{H})^\vee),$$
Here, $\Gamma_k = \mathrm{Gal}(\overline{\mathrm{Fr}(W(k))}/\mathrm{Fr}(W(k)))$ is the absolute Galois group of $\mathrm{Fr}(W(k))$, and $\mathrm{Rep}_{\mathrm{cris}}(\Gamma_k)$ is the category of Γ_k-stable lattices in crystalline $\mathbb{Q}_p$-representations of Γ_k.

This functor is not exact. However, if $\mathscr{O}_{\mathscr{E}}$ is the p-adic completion of the localization $\big(\mathfrak{S}_{k(v)}\big)_{(p)}$, and $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ is the category of finite free modules $\mathscr{O}_{\mathscr{E}}$-modules $\mathscr{M}$ equipped with an isomorphism
$$F : \varphi^*\mathscr{M} \xrightarrow{\simeq} \mathscr{M},$$
then the functor
$$\mathrm{BrK}_{/\mathfrak{S}_k} \xrightarrow{\mathfrak{M} \mapsto \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}_k} \mathfrak{M}} \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$$

is a fully faithful tensor functor, and the composition

$$\mathscr{M} : \mathrm{Rep}_{\mathrm{cris}}(\Gamma_k) \xrightarrow{D \mapsto \mathscr{M}(\mathfrak{M}(D))} \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$$

is exact.

2.16. Set $H_p = \mathbb{Z}_p \otimes H_{(p)}$, and $\boldsymbol{H}_0 = \mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} H_p$. Equip $\boldsymbol{H}_0$ with the σ-linear map

$$F : \sigma^* \boldsymbol{H}_0 = \mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} H_p \xrightarrow{\sigma(\mu_p(p))} \mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} H_p = \boldsymbol{H}_0.$$

Since $\sigma(\mu_p)$ commutes with $N\mu_p$, there is an F-stable grading

$$\boldsymbol{H}_0 = \bigoplus_{i=0}^{d} \boldsymbol{H}_0^i \tag{2.16.1}$$

given by the eigenspaces for $N\mu_p$. Here, $d = [E_v : \mathbb{Q}_p]$.

We also have an ascending slope filtration $\{S^r \boldsymbol{H}_0\}_{r \in \mathbb{Q}}$ by setting, for any $r \in \mathbb{Q}$,

$$S_r \boldsymbol{H}_0 = \bigoplus_{i \leq rd} \boldsymbol{H}_0^i.$$

Now, $\boldsymbol{H}_0$ is the Dieudonné module associated with a canonical p-divisible group $\mathcal{G}_{0,k(v)}$ over $k(v)$, endowed with a grading arising from (2.16.1). It is also equipped with a G**-structure** in the sense that, for each index α, the tensor

$$\boldsymbol{s}_{\alpha,0} = 1 \otimes s_\alpha \in \mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} H_p^{\otimes} = \boldsymbol{H}_0^{\otimes}$$

is F-invariant.

2.17. We will now define a canonical lift of $\mathcal{G}_{0,k(v)}$ to a p-divisible group $\mathcal{G}_0$ over $\mathscr{O}_{E_v}$, equipped with a grading

$$\mathcal{G}_0 = \bigoplus_{i=0}^{d} \mathcal{G}_0^i,$$

lifting the one on $\mathcal{G}_{0,k(v)}$ arising from (2.16.1).

For this, we will use (2.14), and define the corresponding object $\mathfrak{H}_0$ in $\mathrm{BT}_{/\mathfrak{S}_{k(v)}}$ as follows: We set $\mathfrak{H}_0 = \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} H_p$, and the map $F_{\mathfrak{H}_0}$ will be given by the composition:

$$F_{\mathfrak{H}_0} : \varphi^* \mathfrak{H}_0 = \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} H_p \xrightarrow{\mu_p(\mathcal{E}(u))} \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} H_p = \mathfrak{H}_0.$$

This map clearly commutes with the grading

$$\mathfrak{H}_0 = \bigoplus_{i=0}^{d} \mathfrak{H}_0^i$$

given by the eigenspaces for $N\mu_p$ acting on $\mathfrak{H}_0 = \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} H_p$.

We now take $\mathcal{G}_0$ to be the p-divisible group such that $\mathfrak{M}(\mathcal{G}_0) = \mathfrak{H}_0$. Since we can identify

$$\varphi^* \mathfrak{H}_0 / u \varphi^* \mathfrak{H}_0 = \boldsymbol{H}_0$$

as F-crystals over $W(k)$, $\mathcal{G}_0$ has all the desired properties. It is also equipped with a descending **slope filtration** $\{S^r \mathcal{G}_0\}_{r \in \mathbb{Q}}$ given by setting

$$S^r \mathcal{G}_0 = \bigoplus_{i \geq rd} \mathcal{G}_0^i.$$

Note also that, for each index α, the tensor

$$\boldsymbol{s}_{\alpha,\mathfrak{H}_0} = 1 \otimes s_\alpha \in \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} H_p^{\otimes} = \mathfrak{H}_0^{\otimes},$$

is an F-invariant element. Here, $\mathfrak{H}_0^{\otimes}$ is viewed as an ind-object over the tensor category $\mathrm{BrK}_{/\mathfrak{S}_{k(v)}}$.

Also, let $\mathrm{Fil}^1(\mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} H_p)$ be the eigenspace on which $\mu_p(p)$ acts via multiplication by p; then under the identification

$$M_{\mathrm{dR}}(\mathcal{G}_0) = \varphi^*\mathfrak{H}_0/(u-p)\varphi^*\mathfrak{H}_0 = \mathfrak{S}_{k(v)}/(u-p)\mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} H_p = \mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} H_p$$

the Hodge filtration on $M_{\mathrm{dR}}(\mathcal{G}_0)$ is carried to $\mathrm{Fil}^1(\mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} H_p)$.

Using this, we can now check that the p-divisible group $\mathcal{G}_0$ is a **G-adapted lift** of $\mathcal{G}_{0,k(v)}$ in the terminology of [16, Defn. 3.3].

2.18. Let $u \in \mathscr{O}_{E_v}^\times$ be such that

$$\mathscr{O}_{E_v} = \mathbb{Z}_p[u].$$

Let T_0 be the torus from (2.12), so that

$$u \in T_0(\mathbb{Z}_p) = \mathscr{O}_{E_v}^\times.$$

Then the sub-group generated by u is Zariski dense in T_0.

Now, set $\gamma = r_\mu(u) \in T(\mathbb{Z}_p) \subset M_{N\mu}(\mathbb{Z}_p)$: this is a central element by (2.13). Since it commutes with μ_p, we find that it induces an endomorphism (indeed, an automorphism) of the Breuil-Kisin module $\mathfrak{H}_0$, and hence of the p-divisible group $\mathcal{G}_0$. We will denote this endomorphism by $\boldsymbol{\gamma}$.

For any $\mathscr{O}_{E_v}$-algebra A, set

$$\mathrm{End}_\gamma(A \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0) = \{f \in \mathrm{End}(A \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0) : \boldsymbol{\gamma} f = f\boldsymbol{\gamma}\}.$$

Lemma 2.19. *For any perfect extension $k/k(v)$, the natural map*

$$\mathrm{End}_\gamma(W(k) \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0) \to \mathrm{End}_\gamma(k \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0)$$

is a bijection.

Proof. Using Dieudonné and Breuil-Kisin theory, it is enough to show the following linear algebraic assertion: Suppose that we have an endomorphism f of $W(k) \otimes_{\mathbb{Z}_p} H_p$ that commutes with γ, and which satisfies

$$\mu_p(p) \circ \sigma^* f = f \circ \mu_p(p). \tag{2.19.1}$$

Then the induced endomorphism $1 \otimes f$ of

$$W(k)[|u|] \otimes_{\mathbb{Z}_p} H_p = W(k)[|u|] \otimes_{W(k)} W(k) \otimes_{\mathbb{Z}_p} H_p$$

satisfies

$$\mu_p(\mathcal{E}(u))\varphi^*(1 \otimes f) = (1 \otimes f) \circ \mu_p(\mathcal{E}(u)). \tag{2.19.2}$$

For this, note that the sub-group $u^{\mathbb{Z}} \subset T_0(\mathbb{Z}_p)$ is Zariski dense in T_0. Therefore, for any $\mathbb{Z}_p$-algebra R, any endomorphism of $R \otimes_{\mathbb{Z}_p} H_p$ that commutes with γ will also commute with $r_\mu(T_0(R))$, and so we must have

$$\mu_p(p) f \mu_p(p)^{-1} = f \ ; \ \mu_p(\mathcal{E}(u))(1 \otimes f)\mu_p(\mathcal{E}(u))^{-1} = 1 \otimes f.$$

Combining the first of these identities with (2.19.1) shows that $\sigma^* f = f$, implying that f is defined over $\mathbb{Z}_p$. This in turns shows that $\varphi^*(1 \otimes f) = 1 \otimes f$, and so the second of the identities now implies the desired equality in (2.19.2). $\square$

2.20. Let $\underline{\mathrm{Aut}}_F(\boldsymbol{H}_0)$ be the group scheme over $\mathbb{Z}_p$ obtained as the group of units in the algebra $\mathrm{End}_F(\boldsymbol{H}_0)$. Here, $\mathrm{End}_F(\boldsymbol{H}_0)$ is the algebra of endomorphisms of the F-crystal $\boldsymbol{H}_0$. Write $J_0 \subset \underline{\mathrm{Aut}}_F(\boldsymbol{H}_0)$ for the largest closed sub-group that fixes the tensors $\{\boldsymbol{s}_{\alpha,0}\} \subset \boldsymbol{H}_0^{\otimes}$.

From this description, one finds easily that $J_0(\mathbb{Z}_p)$ (resp. $J_0(\mathbb{Q}_p)$) is the group of automorphisms (resp. self-quasi-isogenies) of the p-divisible group $\mathcal{G}_{0,k(v)}$ whose crystalline realizations are G-structure preserving automorphisms of $\boldsymbol{H}_0$ (resp. $\boldsymbol{H}_0[p^{-1}]$).

Proposition 2.21. *Then, for any $\mathbb{Z}_p$-algebra A, the map*

$$\begin{aligned}\mathrm{GL}\big(A \otimes_{\mathbb{Z}_p} H_p\big) &\to \mathrm{GL}\big(A \otimes_{\mathbb{Z}_p} \boldsymbol{H}_0\big) = \mathrm{GL}\big(W \otimes_{\mathbb{Z}_p} (A \otimes_{\mathbb{Z}_p} H_p)\big)\\ m &\mapsto 1 \otimes m\end{aligned}$$

carries $M_{N\mu}(A)$ into $J_0(A)$, and induces an isomorphism of $\mathbb{Z}_p$-groups $M_{N\mu} \xrightarrow{\simeq} J_0$.

In particular, every element of $J_0(\mathbb{Z}_p)$ preserves the grading and slope filtration on $\mathcal{G}_0$.

Proof. By Lemma 2.7 of [32] (see also (2.13)), μ_p, and hence $\sigma(\mu_p)$, is central in $M_{N\mu}$. From this, it is easy to see that the map defined does indeed carry $M_{N\mu}(A)$ into $J_0(A)$.

Any $h \in J_0(A)$ satisfies

$$h^{-1}\sigma(\mu_p)(p)\sigma(h) = \sigma(\mu_p)(p).$$

Set $d = [E_v : \mathbb{Q}_p]$, and consider the algebra

$$\mathcal{E} = \{f \in W \otimes_{\mathbb{Z}_p} \mathrm{End}(H_p) :\ N\mu_p(p)\sigma^d(f)N\mu_p(p)^{-1} = f\}.$$

Choose $m \in H_p^i$, $f \in \mathcal{E}$, and write $f(1 \otimes m) = \sum_j a_j \otimes m_j$ with $m_j \in H_p^j$ and $a_j \in W$. Then we find:

$$\sum_j a_j \otimes m_j = f(m) = N\mu_p(p)\sigma^d(f)N\mu_p(p)^{-1}(m) = \sum_j p^{j-i}\sigma^d(a_j) \otimes m_j.$$

Since $\sigma^d(a_j)$ and a_j have the same p-adic valuations, we must have $a_j = 0$ for $i \neq j$ and $a_i = \sigma^r(a_i)$. This implies that $\mathcal{E}$ is simply the commutant of $N\mu_p(p)$ in $\mathscr{O}_{E_v} \otimes_{\mathbb{Z}_p} \mathrm{End}(H_p)$.

It is easy to check that $h \in \mathcal{E}$ and that it therefore arises from an element of $M_{N\mu}(A \otimes_{\mathbb{Z}_p} \mathscr{O}_{E_v})$. On the other hand, using the centrality of $\sigma(\mu_p)$ in $M_{N\mu}$, we have:

$$\sigma(h) = \sigma(\mu_p)(p)^{-1}h\sigma(\mu_p)(p) = h.$$

Therefore, we in fact have $h \in M_{N\mu}(A)$.

The last assertion follows because any element of $M_{N\mu}(\mathbb{Z}_p)$ clearly preserves the grading and hence the slope filtration on $\mathcal{G}_0$. □

Corollary 2.22. *Suppose that $k/k(v)$ is a perfect extension, and that f is a self-isogeny of $k \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0$, whose crystalline realization, viewed as an automorphism of $W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0[p^{-1}]$, fixes the tensors $\{1 \otimes \boldsymbol{s}_{\alpha,0}\}$. Then f lifts to a self-isogeny of $W(k) \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0$.*

Proof. By (2.19), it is enough to show that any such isogeny must commute with γ. However, by (2.21), we know that any such isogeny must arise from an element of $M_{N\mu}(\mathbb{Q}_p)$, and so by (2.13) must commute with γ. □

2.23. Let $U^-_{\mu_p} \subset G_{\mathscr{O}_{E_v}}$ be the opposite unipotent associated with μ_p, so that $\mathrm{Lie}\, U^-_{\mu_p} \subset \mathrm{Lie}\, G_{\mathscr{O}_{E_v}}$ is the subspace on which $\mu_p(p)$ acts as p^{-1}. Let $\widehat{U}_G$ be the formal scheme over $\mathscr{O}_{E_v}$ obtained from the completion of $U^-_{\mu_p}$ along its identity section. Let $\boldsymbol{H}_0^{(i)}$ be the eigenspace on which $\mu_p(p)$ acts via p^i, so that we have a direct sum decomposition

$$\boldsymbol{H}_0 = \boldsymbol{H}_0^{(0)} \oplus \boldsymbol{H}_0^{(1)}.$$

We have the unipotent subgroup $U_0 \subset \mathrm{GL}(\boldsymbol{H}_0)$ with Lie algebra $\mathrm{Hom}(\boldsymbol{H}_0^{(1)}, \boldsymbol{H}_0^{(0)})$. Let $\widehat{U}_0$ be its completion along the identity section. Then we have a closed embedding of formallly smooth formal $\mathscr{O}_{E_v}$-schemes

$$\widehat{U}_G \hookrightarrow \widehat{U}_0.$$

Following Faltings [9], we can identify $\widehat{U}_0$ with the deformation space of the p-divisible group $\mathcal{G}_{0,k(v)}$ in such a way that the identity section in $\widehat{U}_0(\mathscr{O}_{E_v})$ corresponds to the canonical lift $\mathcal{G}_0$. Furthermore, for any algebraic extension $k/k(v)$, we can characterize the set $\widehat{U}_G(W(k))$ as consisting of the G-adapted lifts of $\mathcal{G}_{0,k}$; see [16, Prop. 3.4]. We will not need to know precisely what this means, except to note, as we did in (2.17), that $\mathcal{G}_0$ is such a lift.

2.24. Fix an algebraic closure k of $k(v)$. Suppose that we have a point $x_0 \in \mathcal{S}_{K,k(v)}(k)$. We will say that x_0 is **μ-ordinary** if, in the notation of (2.9), with x_0 viewed as a k^p-poin, the trivialization τ can be chosen such that $\nu_{x_0,\tau} = \nu_p$.

By Proposition 7.2 of [34], being μ-ordinary is also equivalent to: There exists a choice of trivialization τ such that $b_{x_0,\tau} = \sigma(\mu_p)(p)$.

It now follows from the main result of [34] that there is an open subscheme $\mathcal{S}^{\mathrm{ord}}_{K,k(v)} \subset \mathcal{S}_{K,k(v)}$, whose closed points are precisely the μ-ordinary points of $\mathcal{S}_{K,k(v)}$. We will call it the **μ-ordinary locus**, and will denote by $\widehat{\mathcal{S}}^{\mathrm{ord}}_{K,k(v)}$ the formal completion of $\mathcal{S}_K$ along its μ-ordinary locus.

Let $x_0 \in \mathcal{S}_{K,k(v)}(k)$ be a μ-ordinary point. Let $\underline{\mathrm{Aut}}_F(\boldsymbol{H}_{\mathrm{cris},x_0})$ be the group scheme over $\mathbb{Z}_p$ obtained as the group of units in the algebra $\mathrm{End}_F(\boldsymbol{H}_{\mathrm{cris},x_0})$ of F-equivariant endomorphisms of $\boldsymbol{H}_{\mathrm{cris},x_0}$, and let

$$J_{x_0} \subset \underline{\mathrm{Aut}}_F(\boldsymbol{H}_{\mathrm{cris},x_0})$$

be the stabilizer of the tensors $\{s_{\alpha,\mathrm{cris},x_0}\}$.

Proposition 2.25. *Let x_0 be a μ-ordinary point in $\mathcal{S}_K(k)$. Then there is a* ***canonical lift*** *of x_0 to a point $x \in \mathcal{S}_K(W(k))$ characterized by the following property: There exists an isomorphism of p-divisible groups*

$$W(k) \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{A}_x[p^\infty]$$

such that, for every α, the associated isomorphism of crystalline realizations

$$W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0 \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},x_0} = \boldsymbol{H}_{\mathrm{dR},x}$$

carries $1 \otimes \boldsymbol{s}_{\alpha,0}$ to $s_{\alpha,\mathrm{cris},x_0}$.

In particular, the p-divisible group $\mathcal{G}_{x_0} = \mathcal{A}_{x_0}[p^\infty]$ is equipped with a canonical grading

$$\mathcal{G}_{x_0} = \bigoplus_{i=0}^{r} \mathcal{G}^i_{x_0},$$

and a slope filtration

$$S^r\mathcal{G}_{x_0} = \bigoplus_{i \geq rd} \mathcal{G}^i_{x_0}.$$

Moreover, there is an isomorphism of $\mathbb{Z}_p$-group schemes $M_{N\mu} \xrightarrow{\simeq} J_{x_0}$.

Proof. When $p > 2$, this is shown in [32]. The same proof more or less works when $p = 2$ as well. We recall the details: Choose a trivialization τ such that $b_{x_0,\tau} = \sigma(\mu_p)(p)$. This immediately implies that there is an isomorphism of p-divisible groups

$$k \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_{x_0}$$

whose crystalline realization carries $1 \otimes \boldsymbol{s}_{\alpha,0}$ to $s_{\alpha,\mathrm{cris},x_0}$, for every α.

Using this, we can identify the deformation space for the p-divisible group $\mathcal{G}_{x_0}$ with $W(k) \otimes_{\mathscr{O}_{E_v}} \widehat{U}_0$, where $\widehat{U}_0$ is as in (2.23).

Moreover, by Prop. 3.4 and the proof of Prop. 4.6 of [16], $\widehat{U}_{x_0}$ is a closed formal subscheme of $\widehat{U}_0$ that is identified with $\widehat{U}_G$. Now, as observed in (2.17), the p-divisible group $W(k)\otimes_{\mathscr{O}_{E_v}}\mathcal{G}_0$ is a G-adapted lift of $\mathcal{G}_{x_0}$. Therefore, by the characterizing property of $\widehat{U}_G(W(k))$, explained in (2.23), we find that this p-divisible group corresponds to a unique point $x\in\mathcal{S}_K(W(k))$ lifting x_0.

The rest of the proposition is now immediate. □

Lemma 2.26. *Suppose that $E_v=\mathbb{Q}_p$; or equivalently, that μ_p is defined over $\mathbb{Q}_p$. Then a point $x_0\in\mathcal{S}_K(k)$ is ordinary if and only if the abelian variety $\mathcal{A}_{x_0}$ is ordinary in the classical sense; that is, if and only if $\mathcal{A}_{x_0}[p^\infty]$ is the extension of an étale p-divisible group by a multiplicative one.*

Proof. This result is certainly well-known, but since we could not find a convenient reference, we provide a proof.

First, observe that in this situation $\mathcal{G}_0$ is an ordinary p-divisible group, isomorphic to

$$\underline{\mathrm{Hom}}(H^0_p,\mathbb{Q}_p/\mathbb{Z}_p)\oplus\underline{\mathrm{Hom}}(H^1_p,\mu_{p^\infty}).$$

Indeed, this is immediate from the fact that $\mu_p=\sigma(\mu_p)=N\mu_p$.

Therefore, from (2.25) we see that, for any μ-ordinary point x_0, $\mathcal{A}_{x_0}$ is an ordinary abelian variety.

Conversely, suppose that $\mathcal{A}_{x_0}$ is an ordinary abelian variety. We want to show that x_0 is μ-ordinary. For this, we can assume that k is algebraically closed. The ordinariness of $\mathcal{A}_{x_0}$ implies that we have a canonical F-stable grading

$$\boldsymbol{H}_{\mathrm{cris},x_0}=\boldsymbol{H}_{\mathrm{cris},x_0}(0)\oplus\boldsymbol{H}_{\mathrm{cris},x_0}(1)$$

into its étale and multiplicative parts. On $\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}$, this gives us an induced grading

$$\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}=\bigoplus_{i\in\mathbb{Z}}\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}(i).$$

The graded pieces can be described as follows: View F as a semi-linear endomorphism of $\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}$, and set

$$\boldsymbol{H}_i^{(n)}=(p^{-i}F)^n(\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0})\cap\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}.$$

Then we have

$$\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}(i)=\bigcap_{n\in\mathbb{Z}_{\geq 1}}\boldsymbol{H}_i^{(n)}$$

In particular, since the tensors $\{s_{\alpha,\mathrm{cris},x_0}\}$ are F-invariant, we find that we must have

$$\{s_{\alpha,\mathrm{cris},x_0}\}\subset\boldsymbol{H}^{\otimes}_{\mathrm{cris},x_0}(0).$$

Let $\mu_0:\mathbb{G}_m\to\mathrm{GL}(\boldsymbol{H}_{\mathrm{cris},x_0})$ be the cocharacter corresponding to this decomposition, so that $\mu_0(p)$ acts trivially on $\boldsymbol{H}_{\mathrm{cris},x_0}(0)$ and via multiplication-by-p on $\boldsymbol{H}_{\mathrm{cris},x_0}(1)$. Then we see that it fixes $s_{\alpha,\mathrm{cris},x_0}$, for each α.[6] Therefore, if we choose a trivialization τ as in (2.9), then the induced grading on $W(k)\otimes_{\mathbb{Z}_p}H_p$ is split by a cocharacter of $G_{W(k)}$, which we denote by $\mu_{0,\tau}$.

Let $\mathcal{A}^{\mathrm{can}}$ be the canonical lift of the ordinary abelian variety $\mathcal{A}_{x_0}$: The corresponding Hodge filtration on $\boldsymbol{H}_{\mathrm{cris},x_0}$ is given by the subspace $\boldsymbol{H}_{\mathrm{cris},x_0}(1)$. We claim that $\mathcal{A}^{\mathrm{can}}$ arises from a lift $x\in\mathcal{S}_K(W(k))$ of x_0. Indeed, it corresponds to the origin for the canonical Serre-Tate formal group structure on the deformation scheme $\widehat{U}_0$ for the abelian variety $\mathcal{A}_{x_0}$. Moreover, by the main result of [28], the completion $\widehat{U}_{x_0}$ of $\mathcal{S}_K$ at x_0 is the translation by a torsion-point of a formal sub-torus of $\widehat{U}_0$. Since there are no such non-trivial torsion points defined over $W(k)$, and since $\mathcal{S}_K$ is smooth over $\mathbb{Z}_p$, it follows that the origin must lie within $\widehat{U}_{x_0}$.

[6] This could also have been seen a bit more elegantly using a little Tannakian theory.

Since μ_0 splits the Hodge filtration on $\boldsymbol{H}_{\mathrm{dR},x}$, it follows that $\mu_{0,\tau}$ must be G-conjugate to μ_p. Therefore, by changing τ by an element of $G(W)$ if necessary, we can assume that it induces identifications $W(k)\otimes_{\mathbb{Z}_p} H^i_p \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},x_0}(i)$, for $i=0,1$. From this, one finds that $m = \mu_p(p)^{-1}b_{x_0,\tau} \in M_{N\mu}(W(k))$. Write $m = g\sigma(g)^{-1}$, for $g \in M_{N\mu}(W(k))$. Then $b_{x_0,\tau\circ g} = \mu_p(p)$, showing that x_0 is indeed μ-ordinary. □

2.27. Let $\underline{\mathrm{Aut}}(\mathcal{A}_{x_0})$ be the group scheme over $\mathbb{Z}_{(p)}$ obtained as the group of units in the algebra $\mathrm{End}(\mathcal{A}_{x_0})$. Note that there is a natural map of $\mathbb{Z}_p$-groups:

$$\mathbb{Z}_p\otimes\underline{\mathrm{Aut}}(\mathcal{A}_{x_0})\to\underline{\mathrm{Aut}}_F(\boldsymbol{H}_{\mathrm{cris},x_0}).$$

Suppose that x_0 lies over a point $x_0'\in\mathcal{S}_K(\mathbb{F}_q)$ for a finite extension $\mathbb{F}_q/k(v)$ contained in k. Then we obtain a $\mathbb{Z}_q$-structure on $\boldsymbol{H}_{\mathrm{cris},x_0}$:

$$\boldsymbol{H}_{\mathrm{cris},x_0}=W(k)\otimes_{\mathbb{Z}_q}\boldsymbol{H}_{\mathrm{cris},x_0'}.$$

For each $r\geq 1$, we now obtain a $\mathbb{Z}_p$-sub-group

$$\underline{\mathrm{Aut}}_F(\mathbb{Z}_{q^r}\otimes_{\mathbb{Z}_q}\boldsymbol{H}_{\mathrm{cris},x_0'})\subset\underline{\mathrm{Aut}}_F(\boldsymbol{H}_{\mathrm{cris},x_0}).$$

Let $I_{\mathrm{cris},x_0',q^r}$ be the intersection of J_{x_0} with $\underline{\mathrm{Aut}}_F(\mathbb{Z}_{q^r}\otimes_{\mathbb{Z}_q}\boldsymbol{H}_{\mathrm{cris},x_0'})$. One can check that $I_{\mathrm{cris},x_0',q^r}=I_{\mathrm{cris},x_0',q^s}$ for all r,s sufficiently divisible. Write I_{cris,x_0} for this common sub-group of J_{x_0}.

Let $x\in\mathcal{S}_K(W(k))$ be the canonical lift of x_0. Associated with this is the $\mathbb{Z}_{(p)}$-group scheme $\underline{\mathrm{Aut}}(\mathcal{A}_x)$, defined just as above. Let $I_{x_0}\subset\underline{\mathrm{Aut}}(\mathcal{A}_{x_0})$ (resp. $I_x\subset\underline{\mathrm{Aut}}(\mathcal{A}_x)$) be the largest closed sub-group that maps into J_{x_0}. Since every automorphism of $\mathcal{A}_{x_0}$ is defined over some finite extension of $\mathbb{F}_q$, we find that I_{x_0} maps into I_{cris,x_0} via the crystalline realization map, and we obtain canonical homomorphisms of groups

$$I_x\to I_{x_0}\ ;\ \mathbb{Z}_p\otimes_{\mathbb{Z}_{(p)}}I_{x_0}\to I_{\mathrm{cris},x_0}\tag{2.27.1}$$

over $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$, respectively.

Proposition 2.28. *The natural homomorphisms in* (2.27.1) *are all isomorphisms of* $\mathbb{Z}_{(p)}$*-group schemes.*

Proof. Choose an isomorphism $W(k)\otimes_{\mathscr{O}_{E_v}}\mathcal{G}_0\xrightarrow{\simeq}\mathcal{A}_x[p^\infty]$ as in (2.25). Via such an isomorphism, the automorphism γ of $\mathcal{G}_0$, which is central in $J_0(\mathbb{Z}_p)$, transfers to a canonical endomorphism γ_x of $\mathcal{A}_x[p^\infty]$.

Suppose that x_0 lies over a point $x_0'\in\mathcal{S}_K(\mathbb{F}_{q^r})$ and that $r\geq 1$ is large enough that all endomorphisms of $\mathcal{A}_{x_0}$ are defined over $\mathbb{F}_{q^r}$. Let

$$\underline{\mathrm{Aut}}_\gamma(\mathcal{A}_x)\subset\underline{\mathrm{Aut}}(\mathcal{A}_x)\ ;\ \underline{\mathrm{Aut}}_\gamma(\mathcal{A}_{x_0})\subset\underline{\mathrm{Aut}}(\mathcal{A}_{x_0})\ ;\ \underline{\mathrm{Aut}}_{\gamma,F}(\boldsymbol{H}_{\mathrm{cris},x_0'})\subset\underline{\mathrm{Aut}}_F(\boldsymbol{H}_{\mathrm{cris},x_0'})$$

be the subgroups of automorphisms that commute with γ_x.

Then, by (2.19), the natural map

$$\underline{\mathrm{Aut}}_\gamma(\mathcal{A}_x)\to\underline{\mathrm{Aut}}_\gamma(\mathcal{A}_{x_0})\tag{2.28.1}$$

is an isomorphism.

Moreover, Tate's theorem provides us an isomorphism of $\mathbb{Z}_p$-groups:

$$\mathbb{Z}_p\otimes_{\mathbb{Z}_{(p)}}\underline{\mathrm{Aut}}_\gamma(\mathcal{A}_{x_0})\xrightarrow{\simeq}\underline{\mathrm{Aut}}_{\gamma,F}(\boldsymbol{H}_{\mathrm{cris},x_0'}).\tag{2.28.2}$$

By (2.22), we find that every element of $J_{x_0}(\mathbb{Z}_p)$ lifts to an automorphism of $\mathcal{A}_x[p^\infty]$. This, combined with Serre-Tate deformation theory immediately implies that the canonical map $I_x\to I_{x_0}$ is an isomorphism.

To show that the second map in (2.27.1) is an isomorphism, we will need a bit more work. Fix a geometric point $\overline{x}:\operatorname{Spec}\overline{\mathrm{Fr}(W(k))}\to\mathrm{Sh}_K$ lying above x, so that we obtain tensors $\{s_{\alpha,p,\overline{x}}\}\subset\boldsymbol{H}^{\otimes}_{p,\overline{x}}$ invariant under the absolute Galois group $\Gamma=\mathrm{Gal}\big(\overline{\mathrm{Fr}(W(k))}/\mathrm{Fr}(W(k))\big)$.

Choose an embedding $\iota : \overline{\mathrm{Fr}(W(k))} \hookrightarrow \mathbb{C}$; then we obtain a natural G-structure preserving isomorphism $\mathbb{Z}_p \otimes \boldsymbol{H}_{B,\iota(\overline{x})} \xrightarrow{\simeq} \boldsymbol{H}_{p,\overline{x}}$. Let $\underline{\mathrm{Aut}}_{\gamma,\mathbf{Hdg}}(\boldsymbol{H}_{B,\iota(\overline{x})})$ be the group scheme of automorphisms of the $\mathbb{Z}_{(p)}$-Hodge structure $\boldsymbol{H}_{B,\iota(\overline{x})}$ that commute with γ_{x_0}, and let $I_{B,\iota(\overline{x})}$ be its largest $\mathbb{Z}_{(p)}$-sub-group that stabilizes the tensors $\{s_{\alpha,B,\iota(\overline{x})}\}$.

There now exist canonical isomorphisms of $\mathbb{Z}_{(p)}$-groups:

$$\underline{\mathrm{Aut}}_{\gamma,\mathbf{Hdg}}(\boldsymbol{H}_{B,\iota(\overline{x})}) \xrightarrow{\simeq} \underline{\mathrm{Aut}}_{\gamma}(\mathcal{A}_x) \xrightarrow{\simeq} \underline{\mathrm{Aut}}_{\gamma}(\mathcal{A}_{x_0}). \tag{2.28.3}$$

Combining with (2.28.2) now provides an isomorphism of $\mathbb{Z}_p$-groups:

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} \underline{\mathrm{Aut}}_{\gamma,\mathbf{Hdg}}(\boldsymbol{H}_{B,\iota(\overline{x})}) \xrightarrow{\simeq} \underline{\mathrm{Aut}}_{\gamma,F}(\boldsymbol{H}_{\mathrm{cris},x_0'}). \tag{2.28.4}$$

Since the various comparison isomorphisms are all G-structure preserving, one finds that (2.28.3) induces isomorphisms of $\mathbb{Z}_{(p)}$-groups

$$I_{B,\iota(\overline{x})} \xrightarrow{\simeq} I_x \xrightarrow{\simeq} I_{x_0}$$

and that (2.28.4) induces an isomorphism of $\mathbb{Z}_p$-groups

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} I_{B,\iota(\overline{x})} \xrightarrow{\simeq} I_{\mathrm{cris},x_0}.$$

Combining these now finishes the proof of the proposition. □

Remark 2.29. Kisin has shown that (2.28) holds in general, without the ordinariness hypothesis,[7] if one works instead with the associated $\mathbb{Q}_p$-groups; see [19]. In [20], it is shown that the result continues to hold without the unramifiedness hypothesis on $G_{\mathbb{Q}_p}$.

We could have of course appealed to these more general results, but found the direct proof above appealing enough to present here.

Corollary 2.30. *Fix a prime $\ell \neq p$, and let $\boldsymbol{H}_{\ell,x_0}$ be the ℓ-adic realization of $\mathcal{A}_{x_0}$: it is a $\mathbb{Q}_\ell$-vector space equipped with a $G(\mathbb{Q}_\ell)$-orbit of isomorphisms $\mathbb{Q}_\ell \otimes H \xrightarrow{\simeq} \boldsymbol{H}_{\ell,x_0}$, and thus a canonical collection of tensors*

$$\{\boldsymbol{s}_{\alpha,\ell,x_0}\} \subset \boldsymbol{H}^{\otimes}_{\ell,x_0}.$$

Then, for any $\mathbb{Z}_{(p)}$-algebra R, $I_{x_0}(R)$ consists of the elements $f \in \big(R \otimes_{\mathbb{Z}_{(p)}} \mathrm{End}(\mathcal{A}_{x_0})\big)^\times$, whose ℓ-adic realization $\boldsymbol{f}_\ell \in (R \otimes_{\mathbb{Z}_{(p)}} \mathrm{End}(\boldsymbol{H}_{\ell,x_0}))^\times$ fixes $\{\boldsymbol{s}_{\alpha,\ell,x_0}\}$ pointwise.

Proof. This is obtained by using the Betti realization of the canonical lift, which shows that the condition on f is independent of the prime ℓ, and can in fact be checked for the p-adic realization of the generic fiber of the canonical lift, where one can use the p-adic comparison isomorphism to conclude. □

Corollary 2.31. *With the notation as above, the following are equivalent:*

(1) $I_{\mathrm{cris},x_0} = J_{x_0}$.
(2) $\mathbb{Z}_p \otimes I_{x_0} = J_{x_0}$.
(3) For some integer $r \geq 1$, there exists an isomorphism $\iota' : \mathbb{F}_{q^r} \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{A}_{x_0'}[p^\infty]$ whose base-change over k satisfies the conditions of (2.25).

□

Definition 2.32. We will say that x_0 is **hypersymmetric** if it satisfies any of the equivalent conditions in (2.31).[8]

[7]Note that all relevant definitions can still be made in the absence of this hypothesis.
[8]The definiton is originally due to Chai [4] in the case where $G = \mathrm{GSp}(H)$.

Remark 2.33. (1) Suppose that $\mathcal{A}_{x_0}$ is hypersymmetric as an abelian variety; that is, suppose that the natural map

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} \mathrm{End}(\mathcal{A}_{x_0}) \to \mathrm{End}_F(\boldsymbol{H}_{\mathrm{cris},x_0}) = \mathrm{End}(\mathcal{A}_{x_0}[p^\infty]).$$

is an isomorphism. Then it is immediate that x_0 is hypersymmetric as a point of $\mathcal{S}_K$. In fact, it is enough to assume that $\mathcal{A}_{x_0}$ is isogenous to a hypersymmetric abelian variety: any abelian variety isogenous to a hypersymmetric one is itself hypersymmetric.

(2) Any ordinary elliptic curve $\mathcal{E}$ is hypersymmetric in the above sense: The right hand side is $\mathbb{Z}_p \times \mathbb{Z}_p$, and so it is enough to know that $\mathrm{End}(\mathcal{E})$ has rank at least 2 as a $\mathbb{Z}_{(p)}$-module (since the image of the map in question is saturated), which is clear, since Frobenius does not act as a scalar.

(3) Suppose that Sh_K admits a modular curve as a sub-Shimura variety via a map of Shimura data[9]

$$(\mathrm{GL}_2, \mathbb{H}) \to (G, X).$$

Then the ordinary locus $\mathcal{S}^{\mathrm{ord}}_{K,\mathbb{F}_p}$ contains many hypersymmetric points: Indeed, the symplectic representation H, viewed as a representation of GL_2, must be isomorphic to a direct sum of the tautological representation, for weight reasons. Therefore, if x_0 is a point of $\mathcal{S}_K$ that is the reduction of a point on the modular curve, then $\mathcal{A}_{x_0}$ is isogenous to a power of the elliptic curve associated with that point. In particular, if we choose the point so that the associated elliptic curve is ordinary, then by the preceding remarks, and by (2.26), we find that x_0 is a hypersymmetric μ-ordinary point.

(4) We do not know a general criterion for when a hypersymmetric μ-ordinary point should exist. See however [36] for the PEL case.

3. The canonical torsor over the μ-ordinary locus

The goal of this section is to define a canonical $M_{N\mu}(\mathbb{Z}_p)$-torsor over the μ-ordinary locus—usually referred to as the Igusa tower—and to relate this to the $G(\mathbb{Z}_p)$-torsor I_p over the generic fiber Sh_K.

As before, we fix an algebraic closure k for $k(v)$.

Proposition 3.1. *Let $\mathcal{G} = \mathcal{A}[p^\infty]|_{\widehat{\mathcal{S}}^{\mathrm{ord}}_K}$. Then there exists a descending filtration $\{S^r\mathcal{G}\}_{r\in\mathbb{Q}}$ of $\mathcal{G}$ by p-divisible subgroups that specializes at every μ-ordinary geometric point x_0 to the filtration $\{S^r\mathcal{G}_{x_0}\}_{r\in\mathbb{Q}}$.*

Proof. Let x_0 be a μ-ordinary point in $\mathcal{S}_K(k)$, and let R_{x_0} be the complete local ring of $\mathcal{S}_K$ at x_0. Set

$$\mathcal{G}_{R_{x_0}} = \mathcal{A}[p^\infty]|_{\mathrm{Spec}\, R_{x_0}}.$$

Then, by Propositions 4.1 and 5.1 of [32], it follows that the slope filtration on $\mathcal{G}_{x_0}$ lifts to a filtration $\{S^r\mathcal{G}_{R_{x_0}}\}_{r\in\mathbb{Q}}$ of $\mathcal{G}_{R_{x_0}}$. The statements there assume $p > 2$, but the proofs go through even for $p = 2$: All they need is an explicit, group-theoretic description of the complete local ring R_{x_0}, and this is available for $p = 2$ by the results of [16].

The content of the proposition is that these filtrations can be glued together as x_0 varies. For this, it is enough to show that, for every morphism

$$\mathrm{Spf}\, A \to \widehat{\mathcal{S}}^{\mathrm{ord}}_K$$

with A a p-adic, formally smooth, formally of finite type $\mathscr{O}_{E_v}$-algebra, the p-divisible group $\mathcal{G}|_{\mathrm{Spf}\, A}$ has a slope filtration inducing the canonical ones from the previous paragraph over each complete local ring. Over $\mathbb{F}_p \otimes A$, this follows from [14, Thm. 2.4.2], which shows the

[9] This is only possible if $E_v = \mathbb{Q}_p$.

statement on the level of Dieudonné F-crystals, and Main Theorem 1 of [6], which shows that the Dieudonné functor is an equivalence of categories over $\mathbb{F}_p \otimes A$.

We need to now show that the slope filtration lifts over A. First, note that the maximal étale quotient

$$\mathrm{gr}^0_S \left(\mathcal{G}|_{\mathrm{Spec}(\mathbb{F}_p\otimes A)}\right)$$

lifts canonically to the maximal étale quotient

$$\mathcal{G}|_{\mathrm{Spec}\,A} \to \mathrm{gr}^0_S \left(\mathcal{G}|_{\mathrm{Spec}\,A}\right).$$

Let $S^{>0}\mathcal{G}|_{\mathrm{Spec}\,A}$ be the kernel of this map. It is enough to show that the induced filtration on $S^{>0}\mathcal{G}|_{\mathrm{Spec}(\mathbb{F}_p\otimes A)}$ lifts. Now, we are dealing with a formal p-divisible group. Let $\mathcal{H}_0$ be a formal p-divisible group over $\mathbb{F}_p \otimes A$, and let $D(\mathcal{H}_0)$ be the evaluation of the covariant Dieudonné F-crystal of $\mathcal{H}_0$ on the formal divided power thickening

$$\mathrm{Spec}(\mathbb{F}_p \otimes A) \hookrightarrow \mathrm{Spf}\,A.$$

It is a finite free A-module, and the $\mathbb{F}_p\otimes A$-module $\mathbb{F}_p \otimes D(\mathcal{H}_0)$ has a canonical Hodge filtration $\mathrm{Fil}^1(\mathbb{F}_p \otimes D(\mathcal{H}_0))$. By [35, Corollary 97], there is an equivalence of categories between liftings over A of $\mathcal{H}_0$, and liftings of the Hodge filtration to a direct summand of $D(\mathcal{H}_0)$. Therefore, given a lift $\mathcal{H}$ of $\mathcal{H}_0$, corresponding to a lift

$$\mathrm{Fil}^1 D(\mathcal{H}_0) \subset D(\mathcal{H}_0),$$

a sub p-divisible group $\mathcal{J}_0 \subset \mathcal{H}_0$ lifts to a sub p-divisible group of $\mathcal{H}$ if and only if

$$\mathrm{Fil}^1 D(\mathcal{H}_0) \cap D(\mathcal{J}_0)$$

is a direct summand lifting $\mathrm{Fil}^1(\mathbb{F}_p \otimes D(\mathcal{J}_0))$. This is a condition that can be checked by verifying at it over the completions of A at every maximal ideal.

Now, apply this to the slope filtration on $S^{>0}\mathcal{G}|_{\mathrm{Spec}(\mathbb{F}_p\otimes A)}$, and use the observation from the first paragraph of the proof to conclude. □

3.2. Set

$$\mathrm{gr}^r\,\mathcal{G} = S^r\mathcal{G}/\cup_{s>r} S^s\mathcal{G}\ ;\ \mathrm{gr}\,\mathcal{G} = \bigoplus_{r\in\mathbb{Q}} \mathrm{gr}^r\,\mathcal{G}.$$

Suppose that we have a formally of finite type p-adic $\mathscr{O}_{E_v}$-algebra A[10], and a morphism of formal schemes

$$y : \mathrm{Spf}\,A \to \widehat{\mathcal{S}}^{\mathrm{ord}}_K.$$

For any integer $n \geq 0$, an isomorphism of finite flat group schemes over A

$$\eta : A \otimes_{\mathscr{O}_{E_v}} \mathrm{gr}\,\mathcal{G}_0[p^n] \xrightarrow{\simeq} \mathrm{gr}\,\mathcal{G}_y[p^n]$$

is said to **preserve $M_{N\mu}$-structure** if, for every point x_0 of $\mathrm{Spf}\,A$ valued in k, there exists an isomorphism

$$\iota : k \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0[p^n] \xrightarrow{\simeq} k \otimes_A \mathcal{G}_y[p^n]$$

such that:

- For every index α, the induced isomorphism of crystalline realizations

$$\eta^{-1}_{\mathrm{cris}} : \big(W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0\big)/p^n \xrightarrow{\simeq} \big(W(k) \otimes_A \boldsymbol{H}_{\mathrm{cris},x_0}\big)/p^n$$

  carries $1 \otimes s_{\alpha,0}$ to $1 \otimes s_{\alpha,\mathrm{cris},x_0}$.
- We have

$$\mathrm{gr}\,\iota = \eta_{x_0} : k \otimes_{\mathscr{O}_{E_v}} \mathrm{gr}\,\mathcal{G}_0[p^n] \xrightarrow{\simeq} k \otimes_A \mathrm{gr}\,\mathcal{G}_y[p^n].$$

[10] By this, we mean that A is p-adicaly complete and that $\mathbb{F}_p \otimes A$ is a finite type $k(v)$-algebra. We will topologize such algebras using the p-adic topology.

An isomorphism of p-divisible groups

$$A \otimes_{\mathscr{O}_{E_v}} \operatorname{gr} \mathcal{G}_0 \xrightarrow{\simeq} \operatorname{gr} \mathcal{G}_y$$

preserves $M_{N\mu}$-structure if the induced isomorphism on p^n-torsion preserves $M_{N\mu}$-structure for every $n \geq 1$.

Given $n \geq 1$, let $\mathcal{I}^M_{p,n}$ be the functor on morphisms

$$y : \operatorname{Spf} A \to \widehat{\mathcal{S}}^{\mathrm{ord}}_K$$

as above that associates with y the set of $M_{N\mu}$-structure preserving isomorphisms

$$A \otimes_{\mathscr{O}_{E_v}} \operatorname{gr} \mathcal{G}_0[p^n] \xrightarrow{\simeq} \operatorname{gr} \mathcal{G}_y[p^n].$$

Similarly, let $\mathcal{I}^M_p$ be the functor that associates with y the set of $M_{N\mu}$-structure preserving isomorphisms

$$A \otimes_{\mathscr{O}_{E_v}} \operatorname{gr} \mathcal{G}_0 \xrightarrow{\simeq} \operatorname{gr} \mathcal{G}_y.$$

Proposition 3.3. *The functor $\mathcal{I}^M_{p,n}$ is represented by an $\underline{M_{N\mu}(\mathbb{Z}/p^n\mathbb{Z})}$-torsor over $\widehat{\mathcal{S}}^{\mathrm{ord}}_K$. Moreover, we have*

$$\mathcal{I}^M_p = \varprojlim_n \mathcal{I}^M_{p,n},$$

and so $\mathcal{I}^M_p$ is an $\underline{M_{N\mu}(\mathbb{Z}_p)}$-torsor over $\widehat{\mathcal{S}}^{\mathrm{ord}}_K$.

Proof. The second assertion is clear once the first has been shown.

For the first, simply observe that there is an obvious $\underline{M_{N\mu}(\mathbb{Z}/p^n\mathbb{Z})}$-action on $\mathcal{I}^M_{p,n}$ via precomposition, and that, by (2.25), the fibers of $\mathcal{I}^M_{p,n}$ over any k-point of $\mathcal{S}^{\mathrm{ord}}_K$ are indeed torsors for this action. □

3.4. Let $\boldsymbol{H}_{p,N\mu}$ be the dual of the Tate module

$$T_p\mathcal{G}_0 = \varprojlim_n \mathcal{G}_0[p^n](\overline{\mathbb{Q}}_p)$$

associated with the p-divisible group $\mathcal{G}_0$. This is a $\Gamma_v = \operatorname{Gal}(\overline{\mathbb{Q}}_p/E_v)$-module, and, by (2.21), we have a canonical embedding

$$M_{N\mu}(\mathbb{Z}_p) \hookrightarrow \operatorname{Aut}(\mathcal{G}_0) \to \operatorname{Aut}_{\Gamma_v}(\boldsymbol{H}_{p,N\mu}).$$

Note that the grading $\mathcal{G}_0 = \oplus_i \mathcal{G}^i_0$ also endows $\boldsymbol{H}_{p,N\mu}$ with a Γ_v-invariant grading

$$\boldsymbol{H}_{p,N\mu} = \bigoplus_{i=0}^{d} \boldsymbol{H}^i_{p,N\mu}.$$

Let $\{\boldsymbol{s}_{\alpha,\mathfrak{H}_0}\} \subset \mathfrak{H}^{\otimes}_0$ be the F-invariant tensors from (2.17). By the full faithfulness of the functor described in (2.15), they correspond to Γ_v-invariant tensors

$$\{\boldsymbol{s}_{\alpha,p,N\mu}\} \subset \boldsymbol{H}^{\otimes}_{p,N\mu}.$$

We can view $\boldsymbol{H}_{p,N\mu}$ as a pro-étale sheaf over $\operatorname{Spec} E_v$. Now, consider the pro-étale sheaf $I_{N\mu}$ over $\operatorname{Spec} E_v$ that associates with every E_v-scheme T, the set of $M_\nu(\mathbb{Z}_p)$-equivariant isomorphisms of sheaves

$$\boldsymbol{H}_{p,N\mu}|_T \xrightarrow{\simeq} \underline{H}_p$$

that respect the grading on either side, and are also G-structure preserving, in the sense that they carry $\{\boldsymbol{s}_{\alpha,p,N\mu}\}$ to $\{s_\alpha\}$.

The key result of this section is the following:

Proposition 3.5. *$I_{N\mu}$ is non-empty, and thus a left $Z(M_{N\mu})(\mathbb{Z}_p)$-torsor over E_v. Here, $Z(M_{N\mu}) \subset M_{N\mu}$ is the center.*

Proof. It is clear that any two sections of $I_{N\mu}$ differ by a unique automorphism of $\underline{H}_p$ that preserves G-structure and grading, and that commutes with $M_{N\mu}(\mathbb{Z}_p)$. Therefore, $I_{N\mu}$, if non-empty, is a $Z(M_{N\mu})(\mathbb{Z}_p)$-torsor.

It remains to show the non-emptiness. For this, let $\mathcal{I}_{\mathrm{LT}}$ be the $T_0(\mathbb{Z}_p) = \mathscr{O}_{E_v}^\times$-torsor arising from the Lubin-Tate tower over E_v associated with the uniformizer p. More explicitly, let $\mathcal{G}_Q$ be the Lubin-Tate formal $\mathscr{O}_{E_v}$-module associated with the polynomial $x^{q_v} + p$; then

$$\mathcal{I}_{\mathrm{LT},n} = \mathcal{G}_Q[p^n] \backslash \mathcal{G}_Q[p^{n-1}]$$

with the transition maps given by multiplication-by-p.

We will now show that $\mathcal{I}_{N\mu}$ is the push-forward of this torsor along the map

$$T_0(\mathbb{Z}_p) \xrightarrow{t \mapsto r_\mu(t)^{-1}} Z(M_{N\mu})(\mathbb{Z}_p),$$

where r_μ is as in (2.13).

Explicitly, this means the following: Let $\mathrm{rec}_{\mathrm{LT}} : \Gamma_v \to T_0(\mathbb{Z}_p)$ be the Lubin-Tate character associated with $\mathcal{I}_{\mathrm{LT}}$, which describes the action of Γ_v on the Tate module of $\mathcal{G}_Q$. Then we have to show that the Galois representation $\boldsymbol{H}_{p,\mathrm{LT}}$ obtained from the composition

$$\Gamma_v \xrightarrow{\mathrm{rec}_{\mathrm{LT}}^{-1}} T_0(\mathbb{Z}_p) \xrightarrow{r_\mu} Z(M_{N\mu})(\mathbb{Z}_p) \hookrightarrow \mathrm{GL}(H_p)$$

is isomorphic to $\boldsymbol{H}_{p,N\mu}$.

Let $\mathrm{Rep}_{\mathbb{Z}_p}(T_0)$ be the category of algebraic representations of T_0 on finite free $\mathbb{Z}_p$-modules. Given $\rho : T \to \mathrm{GL}(D)$ in this category, we obtain the associated Galois representation $\boldsymbol{D}_{\mathrm{LT}}$ with underlying $\mathbb{Z}_p$-module D. As shown in [1, Prop. 3.5.2] (following an argument due to Rapoport-Zink [31]), $\boldsymbol{D}_{\mathrm{LT}}$ is a $\mathbb{Z}_p$-lattice in a crystalline representation of Γ_v, and therefore, as explained in (2.15), we can associate with it a Breuil-Kisin module $\mathfrak{M}_{\mathrm{LT}}(D)$.

By the full faithfulness of the functor in (2.15), our proof will be completed by checking that there is an isomorphism of Breuil-Kisin modules

$$\mathfrak{M}_{\mathrm{LT}}(H_p) \xrightarrow{\simeq} \mathfrak{H}_0.$$

This is best done by considering the entire category $\mathrm{Rep}_{\mathbb{Z}_p}(T_0)$. Given $\rho : T \to \mathrm{GL}(D)$ here, we can associate with a Breuil-Kisin module $\mathfrak{M}_{\mathrm{naive}}(D)$ in a more direct way: Note that we have a distinguished cocharacter $\mu_0 \in X_*(T_0)$ determined by the choice of embedding $E_v \hookrightarrow \overline{\mathbb{Q}}_p$, so that $r_\mu \circ \mu_0 = \mu_p$. This cocharacter is actually defined over $\mathscr{O}_{E_v}$. Set

$$\mathfrak{M}_{\mathrm{naive}}(D) = \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} D,$$

with the Breuil-Kisin module structure given by

$$F : \varphi^*\mathfrak{M}_{\mathrm{naive}}(D) = \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} D \xrightarrow{\rho(\mu_0(\mathcal{E}(u)))} \mathfrak{S}_{k(v)} \otimes_{\mathbb{Z}_p} D = \mathfrak{M}_{\mathrm{naive}}(D).$$

Observe that, by construction, we have an identification of Breuil-Kisin modules $\mathfrak{H}_0 = \mathfrak{M}_{\mathrm{naive}}(D)$. Therefore, we will be done if we can show that, for every D as above, we have an isomorphism of Breuil-Kisin modules

$$\mathfrak{M}_{\mathrm{LT}}(D) \xrightarrow{\simeq} \mathfrak{M}_{\mathrm{naive}}(D).$$

If D is the tautological representation of T_0 on E_v, then $\boldsymbol{D}_{\mathrm{LT}}$ is the dual of the Tate module of $\mathcal{G}_Q$, and the desired isomorphism is given in [1, Prop. 2.2.1]. Since D generates the tensor category $\mathrm{Rep}_{\mathbb{Z}_p}(T_0)$, if we knew that $\mathfrak{M}_{\mathrm{LT}}$ is an exact functor (as $\mathfrak{M}_{\mathrm{naive}}$ clearly is), then we would be done. In general, however the functor from (2.15) is not exact. However, this is still okay, since, as follows from the discussion in (2.15), the composite of both functors with the fully faithful functor $\mathscr{M}$ *is* exact, and this is sufficient for us to conclude. ∎

3.6. Let $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$ be the rigid analytic space over E_v associated with the formal scheme $\widehat{\mathcal{S}}_K^{\mathrm{ord}}$ in the sense of Berthelot; see the appendix to [6]. It admits an immersion

$$\widehat{\mathcal{S}}_K^{\mathrm{ord,an}} \to \mathrm{Sh}_K^{\mathrm{an}}$$

to the rigid analytic space associated with the E_v-scheme Sh_K. In particular, we can restrict the $\underline{G(\mathbb{Z}_p)}$-torsor I_p over $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$.

Lemma 3.7. *The restriction of I_p over $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$ admits a canonical reduction of structure group to a $\underline{P^-_{N\mu}(\mathbb{Z}_p)}$-torsor*

$$I_p^P \hookrightarrow I_p|_{\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}}.$$

Proof. Note that (3.1) gives us a filtration $\{S^r\mathcal{G}\}_{r\in\mathbb{Q}}$, which translates over the rigid analytic space $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$ to an ascending filtration of the p-adic sheaf $\boldsymbol{H}_p|_{\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}}$:

$$\{S_r\boldsymbol{H}_p|_{\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}}\}_{r\in\mathbb{Q}}.$$

On the other hand, we have an ascending filtration of the $\mathbb{Z}_p$-module H_p given by

$$S_rH_p = \bigoplus_{i<rd} H_p^i,$$

whose stabilizer in $G_{\mathbb{Z}_p}$ is precisely $P^-_{N\mu}$. The desired reduction of structure group is given by the sheaf of G-admissible trivializations

$$\underline{H}_p \xrightarrow{\simeq} \boldsymbol{H}_p|_{\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}}$$

that carry $\{\underline{S_rH_p}\}$ onto $\{S_r\boldsymbol{H}_p|_{\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}}\}$.

To check that this is indeed a torsor under $\underline{P^-_{N\mu}(\mathbb{Z}_p)}$, it is enough to do so over any classical point $x\in\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$ arising from the canonical lift of a point $x_0\in\mathcal{S}^{\mathrm{ord}}_{K,k(v)}(k)$. Here, it is immediate from (2.25) and (3.5). ∎

Proposition 3.8. *Let*

$$I_p^M = I_p^P \times^{P^-_{N\mu}(\mathbb{Z}_p)} M_{N\mu}(\mathbb{Z}_p)$$

be the induced $M_{N\mu}(\mathbb{Z}_p)$-torsor over $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$. By slight abuse of notation, write $I_{N\mu}$ for the $\underline{Z(M_{N\mu})(\mathbb{Z}_p)}$-torsor over $\widehat{\mathcal{S}}_K^{\mathrm{ord},an}$ obtained from the torsor over $\operatorname{Spec} E_v$ *in* (3.4).

Then there is a canonical isomorphism of $M_{N\mu}(\mathbb{Z}_p)$-torsors:

$$I_p^M \times^{Z(M_{N\mu})(\mathbb{Z}_p)} I_{N\mu} \xrightarrow{\simeq} \mathcal{I}_p^M|_{\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}}.$$

Proof. First, observe that the contraction product on the left is indeed an $M_{N\mu}(\mathbb{Z}_p)$-torsor in a natural way. Indeed, since $Z(M_{N\mu})$ is central in $M_{N\mu}$, the action of $m\in M_{N\mu}(\mathbb{Z}_p)$ given by

$$I_p^M\times I_{N\mu} \xrightarrow{(\eta,\beta)\mapsto(\eta\circ m,\beta)} I_p^M\times I_{N\mu}$$

descends to the quotient $I_p^M\times^{Z(M_{N\mu})(\mathbb{Z}_p)} I_{N\mu}$, and gives it the structure of an I_p^M-torsor.

To finish, it suffices to construct a $Z(M_{N\mu})(\mathbb{Z}_p)\times M_{N\mu}(\mathbb{Z}_p)$-equivariant map

$$\varpi : I_p^M\times I_{N\mu} \to \mathcal{I}_p^M|_{\widehat{\mathcal{S}}_K^{\mathrm{ord,an}}}.$$

The point is that the right-hand side can be identified with the $\underline{M_{N\mu}(\mathbb{Z}_p)}$-torsor of isomorphisms

$$\theta : \operatorname{gr}\boldsymbol{H}_{p,N\mu} \xrightarrow{\simeq} \operatorname{gr}\boldsymbol{H}_p,$$

such that, for each $n\geq 1$, the induced isomorphism

$$\theta_n : \operatorname{gr}\boldsymbol{H}_{p,N\mu}/p^n \xrightarrow{\simeq} \operatorname{gr}\boldsymbol{H}_p/p^n$$

can be lifted, étale locally, to a G-structure and slope filtration preserving isomorphism

$$\tilde{\theta}_n : \boldsymbol{H}_{p,N\mu}/p^n \xrightarrow{\simeq} \boldsymbol{H}_p/p^n.$$

A section α of I_p^M can be interpreted as an isomorphism $\alpha : \mathrm{gr}\, \underline{H}_p \xrightarrow{\simeq} \mathrm{gr}\, \boldsymbol{H}_p$, and a section β of $I_{p,N\mu}$ can be seen as an $M_{N\mu}(\mathbb{Z}_p)$-equivariant isomorphism $\beta : \mathrm{gr}\, \boldsymbol{H}_{p,N\mu} \xrightarrow{\simeq} \mathrm{gr}\, \underline{H}_p$.

We now set

$$\theta((\alpha, \beta)) = \alpha \circ \beta : \mathrm{gr}\, \boldsymbol{H}_{p,N\mu} \xrightarrow{\simeq} \mathrm{gr}\, \boldsymbol{H}_p.$$

It can be checked that $\theta((\alpha, \beta))$ is a sction of $\mathcal{I}^M_{p,N\mu}$ and that θ has all the desired properties. □

Remark 3.9. Though we will not need this, it is not hard to see that both the μ-ordinary locus and the canonical torsor over it defined above are independent of the choice of the symplectic representation H.

4. p-isogenies and p-Hecke correspondences

4.1. Suppose that D is a smooth affine group scheme over $\mathbb{Z}_p$, and that $\mathcal{P}$ is an $\underline{D(\mathbb{Z}_p)}$-torsor over an $\mathscr{O}_{E_v}$-scheme S. The contraction product

$$\mathcal{P} \times^{D(\mathbb{Z}_p)} D(\mathbb{Q}_p) = \left(\mathcal{P} \times \underline{D(\mathbb{Q}_p)}\right) / \underline{D(\mathbb{Z}_p)}$$

gives us an $\underline{D(\mathbb{Q}_p)}$-equivariant pro-étale sheaf over S that we will refer to as the $D(\mathbb{Q}_p)$-*torsor associated with* $\mathcal{P}$. We obtain an $\underline{D(\mathbb{Z}_p)}$-equivariant inclusion of sheaves

$$\mathcal{P} \subset \mathcal{P} \times^{D(\mathbb{Z}_p)} D(\mathbb{Q}_p)$$

by taking the image of $\mathcal{P} \times \{1\}$ in the right hand side.

4.2. Suppose that $\mathcal{P}_1$, $\mathcal{P}_2$ are two $\underline{D(\mathbb{Z}_p)}$-torsors over S. An *isogeny* $\alpha : \mathcal{P}_1 \dashrightarrow \mathcal{P}_2$ is a $\underline{D(\mathbb{Q}_p)}$-equivariant isomorphism of sheaves

$$\alpha : \mathcal{P}_1 \times^{D(\mathbb{Z}_p)} D(\mathbb{Q}_p) \xrightarrow{\simeq} \mathcal{P}_2 \times^{D(\mathbb{Z}_p)} D(\mathbb{Q}_p).$$

Given such an isogeny, pro-étale locally on S, there exists $h \in D(\mathbb{Q}_p)$ such that

$$\alpha(\mathcal{P}_1)h = \mathcal{P}_2 \subset \mathcal{P}_2 \times^{D(\mathbb{Z}_p)} D(\mathbb{Q}_p).$$

The class of h in $D(\mathbb{Z}_p)\backslash D(\mathbb{Q}_p)/D(\mathbb{Z}_p)$ is well-defined, and gives rise to a section

$$\mathsf{typ}(\alpha) \in H^0(S, \underline{D(\mathbb{Z}_p)\backslash D(\mathbb{Q}_p)/D(\mathbb{Z}_p)}),$$

which we will call the *type* of the isogeny α.

When D is reductive, $T \subset D$ is a maximal torus with a choice of Borel subgroup $B \supset T$ of D, we can use the Cartan decomposition to obtain an identification

$$D(\mathbb{Z}_p)\backslash D(\mathbb{Q}_p)/D(\mathbb{Z}_p) = \mathsf{C}_D^+,$$

where C_D^+ is the set of coroots of T that are dominant with respect to B and defined over $\mathbb{Z}_p$. In this situation, we will also use $\mathsf{typ}(\alpha)$ to refer to the corresponding section of $\underline{\mathsf{C}_D^+}$. In particular, when $\mathsf{typ}(\alpha)$ is *constant* on S and equals $\lambda \in \mathsf{C}_D^+$, we will say that α is of type λ.

4.3. Suppose still that D is reductive, and let $Q \subset D$ be a parabolic subgroup with Levi quotient L. Suppose that we have two $\underline{Q(\mathbb{Z}_p)}$-torsors $\mathcal{P}_{1,Q}$ and $\mathcal{P}_{2,Q}$ over S, and an isogeny $\alpha : \mathcal{P}_{1,Q} \dashrightarrow \mathcal{P}_{2,Q}$. Then via change of structure group along the morphisms $Q(\mathbb{Z}_p) \to D(\mathbb{Z}_p)$ and $Q(\mathbb{Z}_p) \to L(\mathbb{Z}_p)$, respectively, we obtain an isogeny of $\underline{D(\mathbb{Z}_p)}$-torsors

$$\alpha_D : \mathcal{P}_{1,D} \dashrightarrow \mathcal{P}_{2,D},$$

as well as an isogeny of $\underline{L(\mathbb{Z}_p)}$-torsors

$$\alpha_L : \mathcal{P}_{1,L} \dashrightarrow \mathcal{P}_{2,L}.$$

The dominant chamber in $X_*(T)$ for D determines one for L as well, and we can consider the corresponding subspace C_L^+ as above. The next lemma is immediate from the definitions.

Lemma 4.4. *Suppose that α_D has type $\lambda \in \mathsf{C}_D^+$. Then α_L has type ν, where $\nu \in \mathsf{C}_L^+$ is such that the image of $D(\mathbb{Z}_p)\lambda(p)D(\mathbb{Z}_p) \cap Q(\mathbb{Q}_p)$ in $L(\mathbb{Q}_p)$ intersects $L(\mathbb{Z}_p)\nu(p)L(\mathbb{Z}_p)$ non-trivially.*

□

4.5. Suppose that $\lambda : \mathbb{G}_m \to D$ is a cocharacter defined over $\mathbb{Z}_p$. Let $M_\lambda \subset D$ be the Levi subgroup centralizing λ, and let $U_\lambda^+, U_\lambda^- \subset D$ be the unipotent subgroups whose Lie algebras are the sum of the positive (resp. negative) valuation eigenspaces for $\lambda(p)$. Let

$$P_\lambda^\pm = M_\lambda U_\lambda^\pm$$

be the parabolic subgroups associated with λ. We then have the projective $\mathbb{Z}_p$-scheme Par_λ associating with every $\mathbb{Z}_p$-algebra R the set of parabolic subgroups $Q \subset D_R$ that are fppf locally on $\operatorname{Spec} R$ conjugate to P_λ^-. Note that conjugating P_λ^- by D induces an isomorphism of homogeneous spaces

$$D/P_\lambda^- \xrightarrow{\simeq} \mathrm{Par}_\lambda$$

over $\mathscr{O}_{E_v}$.

For any $n \geq 1$, set

$$\mathcal{P}_{\mathrm{Par}_\lambda,n} = \mathcal{P} \times^{D(\mathbb{Z}_p)} \mathrm{Par}_\lambda(\mathbb{Z}/p^n\mathbb{Z}).$$

Also, set

$$\mathcal{P}_\lambda = \mathcal{P}/\big(D(\mathbb{Z}_p) \cap \lambda(p)D(\mathbb{Z}_p)\lambda(p)^{-1}\big).$$

These are both finite étale covers of S.

Lemma 4.6. *For n sufficiently large, we have a $D(\mathbb{Z}_p)$-equivariant surjection:*

$$\mathrm{Par}_\lambda(\mathbb{Z}/p^n\mathbb{Z}) \to D(\mathbb{Z}_p)\lambda(p)D(\mathbb{Z}_p)/D(\mathbb{Z}_p).$$

Here, $D(\mathbb{Z}_p)$ acts everywhere on the left. If λ is in addition miniscule *then we have a $D(\mathbb{Z}_p)$-equivariant* isomorphism*:*

$$\mathrm{Par}_\lambda(\mathbb{F}_p) \xrightarrow{\simeq} D(\mathbb{Z}_p)\lambda(p)D(\mathbb{Z}_p)/D(\mathbb{Z}_p).$$

In particular, we have, for n sufficiently large, a surjective map of finite étale S-schemes $\mathcal{P}_{\mathrm{Par}_\lambda,n} \to \mathcal{P}_\lambda$; and, if λ is minuscule, we have an isomorphism $\mathcal{P}_{\mathrm{Par}_\lambda,1} \xrightarrow{\simeq} \mathcal{P}_\lambda$.

Proof. Observe that the map $m \mapsto m\lambda(p)D(\mathbb{Z}_p)$ sets up an isomorphism:

$$D(\mathbb{Z}_p)/\big(D(\mathbb{Z}_p) \cap \lambda(p)D(\mathbb{Z}_p)\lambda(p)^{-1}\big) \xrightarrow{\simeq} D(\mathbb{Z}_p)\lambda(p)D(\mathbb{Z}_p)/D(\mathbb{Z}_p).$$

For $n \geq 1$, set

$$U_D(n) = \ker(D(\mathbb{Z}_p) \to D(\mathbb{Z}/p^n\mathbb{Z})).$$

Then there are bijections

$$D(\mathbb{Z}_p)/U_D(n)P_\lambda^-(\mathbb{Z}_p) \xrightarrow{\simeq} D(\mathbb{Z}/p^n\mathbb{Z})/P_\lambda^-(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{g \mapsto gP_\lambda^- g^{-1}} \mathrm{Par}_\lambda(\mathbb{Z}/p^n\mathbb{Z}).$$

Indeed, the first is because of the smoothness of D, and for the second, this follows from Lang's theorem, and the fact that a torsor under P_λ over $\mathbb{Z}/p^n\mathbb{Z}$ is trivial if and only if its base-change to $\mathbb{F}_p$ is trivial.

It is now enough to show that that $D(\mathbb{Z}_p) \cap \lambda(p)^{-1} D(\mathbb{Z}_p)\lambda(p)$ contains $U_D(n)P_\lambda(\mathbb{Z}_p)$ for n large enough. Then we have:

$$U_D(n) = (U_\lambda^+(\mathbb{Z}_p) \cap U_D(n)) \cdot (M_\lambda(\mathbb{Z}_p) \cap U_D(n)) \cdot (U_\lambda^-(\mathbb{Z}_p) \cap U_D(n)),$$

so that

$$U_D(n)P_\lambda^-(\mathbb{Z}_p) = (U_\lambda^+(\mathbb{Z}_p) \cap U_D(n))P_\lambda^-(\mathbb{Z}_p).$$

Moreover, we have

$$P_\lambda^-(\mathbb{Z}_p) \subset \lambda(p)P_\lambda^-(\mathbb{Z}_p)\lambda(p)^{-1} \ ; \ \lambda(p)U_\lambda^+(\mathbb{Z}_p)\lambda(p)^{-1} \subset U_\lambda^+(\mathbb{Z}_p).$$

Therefore, everything comes down to the easily checked fact that, for n sufficiently large, we have

$$U_D(n) \cap U_\lambda^+(\mathbb{Z}_p) \subset \lambda(p)U_\lambda^+(\mathbb{Z}_p)\lambda(p)^{-1}.$$

Now, suppose that λ is *minuscule*. This means that $\operatorname{Lie} U_\lambda^+ \subset \operatorname{Lie} D$ is the eigenspace on which $\lambda(p)$ acts via p under conjugation. Moreover, U_λ^+ is a commutative unipotent group scheme over $\mathbb{Z}_p$, and is thus isomorphic to its Lie algebra. We have to show, under this assumption, that

$$U_D(1) \cap U_\lambda^+(\mathbb{Z}_p) = \lambda(p)U_\lambda^+(\mathbb{Z}_p)\lambda(p)^{-1}.$$

But it is easily seen that both sides are equal to $U_\lambda^+(\mathbb{Z}_p)^p \subset U_\lambda^+(\mathbb{Z}_p)$, the subgroup generated by p^{th}-powers. □

4.7. We now return to the notation of §3. Given two morphisms $s_1, s_2 : S \to \mathrm{Sh}_K$, a quasi-isogeny of S-abelian schemes up to prime-to-p isogeny

$$\xi : \mathcal{A}_{s_1} \dashrightarrow \mathcal{A}_{s_2}$$

is G**-admissible** if the associated map of $\mathbb{Q}_p$-sheaves

$$\xi^* : \boldsymbol{H}_{p,s_2}[p^{-1}] \xrightarrow{\simeq} \boldsymbol{H}_{p,s_1}[p^{-1}]$$

carries, for each α, s_{α,p,s_2} to s_{α,p,s_1}.

To any G-admissible quasi-isogeny ξ, we can attach a canonical isogeny

$$\alpha(\xi) : I_{p,s_1} \dashrightarrow I_{p,s_2}$$

of $G(\mathbb{Z}_p)$-torsors over S. Indeed, given a G-structure preserving trivialization η_0 of $\boldsymbol{H}_{p,s_1}$, $(\xi^*)^{-1} \circ \eta_0$ is a G-structure preserving trivialization of $\boldsymbol{H}_{p,s_2}[p^{-1}]$. In fact, it is a section of $I_{p,s_2} \times^{G(\mathbb{Z}_p)} G(\mathbb{Q}_p)$, and we set $\mathsf{typ}(\xi) = \mathsf{typ}(\alpha(\xi))$. If this is constant and is represented by $\lambda(p)$, for some $\lambda \in \mathsf{C}_G^+$, we will say that ξ **has type** λ.

Via comparison with the Betti realization, it follows that any G-admissible quasi-isogeny ξ as above induces a G-structure preserving isomorphism of prime-to-p étale realizations

$$\xi^*_{\mathbb{A}_f} : \boldsymbol{H}_{\mathbb{A}_f^p,s_2} \xrightarrow{\simeq} \boldsymbol{H}_{\mathbb{A}_f^p,s_1}.$$

Moreover, in the notation of (2.5), this gives us an isomorphism of $G(\mathbb{A}_f^p)$-torsors $s_1^* I^p \xrightarrow{\simeq} s_2^* I^p$ over S. The canonical K^p-level structure $[\eta]$ over $\mathcal{S}_K$ pulls back to sections

$$[\eta_{s_i}] \in H^0(S, s_i^* I^p / K^p),$$

for $i = 1, 2$. We will say that ξ *preserves level structures* if $\xi^*_{\mathbb{A}_f^p}$ carries $[\eta_{s_2}]$ to $[\eta_{s_1}]$

The next lemma follows easily from the uniformization in (2.4), and its relation to the moduli description of the points of Sh_K as described in (2.5).

Lemma 4.8. *If F is a field in characteristic* 0*, and $x, y \in \mathrm{Sh}_K(F)$. If there exists a G-admissible, level structure preserving* isomorphism $\mathcal{A}_x \xrightarrow{\simeq} \mathcal{A}_y$*, then $x = y$.*

□

Definition 4.9. Fix $\lambda \in \mathsf{C}_G^+$, and let Isog_λ be the functor on E_v-schemes associating with T the set of triples (s_1, s_2, ξ), where $s_1, s_2 \in \mathrm{Sh}_K(T)$, and $\xi : \mathcal{A}_{s_1} \dashrightarrow \mathcal{A}_{s_2}$ is an admissible quasi-isogeny of type λ that preserves level structures.

Lemma 4.10. Isog_λ *is represented by a scheme of finite type over E. Moreover, let*

$$s_\lambda : \mathrm{Isog}_\lambda \xrightarrow{(s_1, s_2, \xi) \mapsto s_1} \mathrm{Sh}_K \; ; \; t_\lambda : \mathrm{Isog}_\lambda \xrightarrow{(s_1, s_2, \xi) \mapsto s_2} \mathrm{Sh}_K$$

be the source and target morphisms, respectively. Then s_λ (resp. t_λ) is isomorphic to $\pi_\lambda : I_{p,\lambda} \to \mathrm{Sh}_K$ (resp. $\pi_{-\lambda} : I_{p,-\lambda} \to \mathrm{Sh}_K$), the finite étale covers obtained from I_p by the construction in (4.5)*.*

Proof. We will prove the assertion for s_λ. The one for t_λ is shown in completely analogous fashion.

It is enough to construct a $G(\mathbb{Z}_p)$-equivariant isomorphism of étale sheaves

$$\mathrm{Isog}_\lambda \times_{s_\lambda, \mathrm{Sh}_K} I_p \xrightarrow{\simeq} I_p \times_{\mathrm{Sh}_K} \underline{G(\mathbb{Z}_p)\lambda(p)G(\mathbb{Z}_p)/G(\mathbb{Z}_p)},$$

where the action on the right is the diagonal one whose quotient gives $I_{p,\lambda}$, and the action on the left is via that on I_p.

Suppose that we have a section $((s_1, s_2, \xi), \iota)$ on the left hand side over some scheme S. This gives us the $\mathbb{Z}_p$-lattice

$$\iota^{-1}(\xi^* \boldsymbol{H}_{p,s_2}) \subset \underline{H}_p[p^{-1}].$$

This gives a section $\underline{gG(\mathbb{Z}_p)}$ of $\underline{G(\mathbb{Z}_p)\lambda(p)G(\mathbb{Z}_p)/G(\mathbb{Z}_p)}$ such that

$$g\underline{H}_p = \iota^{-1}(\xi^* \boldsymbol{H}_{p,s_2}).$$

It can now be checked that

$$((s_1, s_2, \xi), \iota) \mapsto (\iota, \underline{gG(\mathbb{Z}_p)})$$

is the desired isomorphism. □

4.11. As before, we fix an algebraic closure $k/k(v)$. Suppose that we have two morphisms

$$x_0, y_0 : \operatorname{Spec} k \to \mathcal{S}^{\mathrm{ord}}_{K,k(v)}.$$

A quasi-isogeny

$$\xi : \mathcal{A}_{x_0} \dashrightarrow \mathcal{A}_{y_0}$$

is G**-admissible** if, for every α, the associated map on crystalline realizations

$$\xi^* : \boldsymbol{H}_{\mathrm{cris},x_0}[p^{-1}] \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},y_0}[p^{-1}]$$

carries $s_{\alpha,\mathrm{cris},x_9}$ to $s_{\alpha,\mathrm{cris},y_0}$.

If S is any p-adic, formally of finite type, formal $\mathscr{O}_{E_v}$-scheme[11] with two morphisms $s_1, s_2 : S \to \widehat{\mathcal{S}}^{\mathrm{ord}}_K$, a quasi-isogeny

$$\xi : \mathcal{A}_{s_1} \dashrightarrow \mathcal{A}_{s_2}$$

is G**-admissible** if it is so over every point of $S(k)$.

It follows from (4.12) below that any such quasi-isogeny induces a G-structure preserving map of $\mathbb{A}_f^p$-sheaves:

$$\xi^*_{\mathbb{A}_f^p} : s_1^* \boldsymbol{H}_{\mathbb{A}_f} \xrightarrow{\simeq} s_2^* \boldsymbol{H}_{\mathbb{A}_f^p}.$$

[11]That is, a formal $\mathscr{O}_{E_v}$-scheme that can be covered by affine formal schemes of the form $\operatorname{Spf} A$, with A a p-adic formally of finite type $\mathscr{O}_{E_v}$-algebra.

Therefore, just as in (4.7), it makes sense to require ξ to a be a **level structure preserving** quasi-isogeny.

Lemma 4.12. *If $x_0, y_0 \in \mathcal{S}_K(k)$ are two μ-ordinary points , then any G-admissible isogeny*[12]

$$\xi_0 : \mathcal{A}_{x_0} \to \mathcal{A}_{y_0}$$

lifts to an isogeny

$$\xi : \mathcal{A}_x \to \mathcal{A}_y,$$

where $x, y \in \mathcal{S}_K(W(k))$ are the canonical lifts of x_0, y_0. Moreover, the induced isogeny of abelian varieties over $\mathrm{Fr}(W(k))$ is G-admissible in the sense of (4.7).

Proof. Immediate from (2.22) and (2.25). The last assertion follows from the fact that the p-adic comparison isomorphism preserves G-structure; see (2.10). $\square$

4.13. Suppose that we have a morphism $s : S \to \widehat{\mathcal{S}}_K^{\mathrm{ord}}$. Associated with this is the $\underline{M_{N\mu}(\mathbb{Z}_p)}$-torsor $\mathcal{I}_p^M$ parameterizing $M_{N\mu}$-structure preserving isomorphisms

$$\mathrm{gr}\,\mathcal{G}_{0,S} \xrightarrow{\simeq} \mathrm{gr}\,\mathcal{G}_s.$$

The associated $\underline{M_{N\mu}(\mathbb{Q}_p)}$-torsor $\mathcal{I}_{p,s}^M \times^{M_{N\mu}(\mathbb{Z}_p)} M_{N\mu}(\mathbb{Q}_p)$ can be described as the sheaf parameterizing $M_{N\mu}$-structure preserving *quasi-isogenies*

$$\eta : \mathrm{gr}\,\mathcal{G}_{0,S} \dashrightarrow \mathrm{gr}\,\mathcal{G}_s,$$

where the quasi-isogeny η is said to **preserve $M_{N\mu}$-structure** if, for every k-point x_0 factoring through S, the crystalline realization of η_{x_0}

$$\eta^{-1}_{\mathrm{cris},x_0} : W(k) \otimes_{\mathscr{O}_{E_v}} \mathrm{gr}\,\boldsymbol{H}_0[p^{-1}] \xrightarrow{\simeq} \mathrm{gr}\,\boldsymbol{H}_{\mathrm{cris},x_0}[p^{-1}]$$

is equal to $\mathrm{gr}\,\tilde{\eta}_{x_0}$, for some G-structure preserving isomorphism

$$\tilde{\eta}_{x_0} : W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0[p^{-1}] \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},x_0}[p^{-1}].$$

4.14. As is easily seen from this description, any G-admissible quasi-isogeny

$$\xi : \mathcal{A}_{x_1} \dashrightarrow \mathcal{A}_{x_2}$$

for $x_1, x_2 : S \to \widehat{\mathcal{S}}_K^{\mathrm{ord}}$ gives rise to a canonical isogeny of $\underline{M_{N\mu}(\mathbb{Z}_p)}$-torsors

$$\beta(\xi) : x_1^*\mathcal{I}_p^M \dashrightarrow x_2^*\mathcal{I}_p^M$$

carrying a quasi-isogeny $\eta : \mathrm{gr}\,\mathcal{G}_{0,S} \dashrightarrow \mathrm{gr}\,\mathcal{G}_{x_1}$ to $\mathrm{gr}\,\xi[p^\infty] \circ \eta$.

We set $\mathsf{typ}(\xi) = \mathsf{typ}(\beta(\xi))$. If this section is constant and is represented by $\lambda(p)$ for some $\lambda \in \mathsf{C}^+_{M_{N\mu}}$, then we say that ξ **has type** λ.

Remark 4.15. The type of an isogeny ξ as above can be computed as follows: Let us assume that the type is constant. Fix a point $x_0 \in S(k)$. We then have two lattices

$$\boldsymbol{H}_{\mathrm{cris},x_1}, \xi^*\boldsymbol{H}_{\mathrm{cris},x_2} \subset \boldsymbol{H}_{\mathrm{cris},x_1}[p^{-1}].$$

Fix a trivialization

$$\iota : W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0 \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},x_1}$$

as in (2.25). Then the type of ξ is the unique $\lambda \in \mathsf{C}^+_{M_{N\mu}}$ such that

$$\iota^{-1}(\xi^*\boldsymbol{H}_{\mathrm{cris},x_2}) = m\lambda(p)(W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0),$$

for some $m \in M_{N\mu}(\mathbb{Z}_p)$.

[12] That is, an isogeny that is admissible when viewed as a quasi-isogeny.

In particular, suppose that we have a map $s : S \to \mathcal{S}^{\mathrm{ord}}_{K,k(v)}$, and that $\mathrm{Fr} : S \to S$ is the absolute Frobenius. Then we have the canonical Frobenius isogeny:

$$F_s : \mathcal{A}_s \to \mathcal{A}_{\mathrm{Fr}(s)}$$

Then the explicit description of $\boldsymbol{H}_0$ in (2.17) shows that F_s has type $\sigma(\mu_p)$. In fact, since $\sigma(\mu_p)$ is centralized by $M_{N\mu}$, we find that any other G-admissible isogeny of type $\sigma(\mu_p)$ must differ from F_s by a G-admissible automorphism of $\mathcal{A}_s$.

4.16. Fix a μ-ordinary point $x_0 \in \mathcal{S}_K(k)$. Let $\mathrm{Isog}^{\mathrm{ord}}_{x_0}(k)$ be the set of pairs (y_0, ξ), where $y_0 \in \mathcal{S}_{K,k(v)}(k)$, and where $\xi : \mathcal{A}_{x_0} \dashrightarrow \mathcal{A}_{y_0}$ is a G-admissible, level structure preserving quasi-isogeny. Let $x \in \mathcal{S}_K(W(k))$ be the canonical lift of x_0. By (2.25), we can a G-structure preserving isomorphism $\eta : W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{G}_0 \xrightarrow{\simeq} \mathcal{G}_x$. There now exists a unique coset $m_\eta(\xi)M_{N\mu}(\mathbb{Z}_p) \in M_{N\mu}(\mathbb{Q}_p)/M_{N\mu}(\mathbb{Z}_p)$ such that

$$\eta^*_{\mathrm{cris}}(\xi^* \boldsymbol{H}_{\mathrm{cris},y_0}) = m_\eta(\xi)(W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0).$$

This gives us a map:

$$m_\eta : \mathrm{Isog}^{\mathrm{ord}}_{x_0}(k) \to M_{N\mu}(\mathbb{Q}_p)/M_{N\mu}(\mathbb{Z}_p). \tag{4.16.1}$$

For any cocharacter $\lambda \in \mathsf{C}^+_M$, let

$$\mathrm{Isog}^{\mathrm{ord}}_{\lambda,x_0}(k) \subset \mathrm{Isog}^{\mathrm{ord}}_{x_0}(k)$$

be the subset of pairs (y_0, ξ), where ξ has type λ.

Proposition 4.17. *The map* (4.16.1) *is a bijection. Moreover, it maps the subset* $\mathrm{Isog}^{\mathrm{ord}}_{\lambda,x_0}(k)$ *onto* $M_{N\mu}(\mathbb{Z}_p)\lambda(p)M_{N\mu}(\mathbb{Z}_p)/M_{N\mu}(\mathbb{Z}_p)$.

Proof. This can be deduced from the very general description of isogeny classes found in [19, §1], but we give a direct proof here for the convenience of the reader.

Once the first assertion is known, the second is immediate from the definitions and (4.15).

To prove the bijectivity of (4.16.1), we will use the canonical lift.

First, suppose that we have two G-admissible quasi-isogenies

$$\xi_1 : \mathcal{A}_{x_0} \dashrightarrow \mathcal{A}_{y_0} \ ; \ \xi_2 : \mathcal{A}_{x_0} \dashrightarrow \mathcal{A}_{z_0}$$

with $m_\eta(\xi_1)M_{N\mu}(\mathbb{Z}_p) = m_\eta(\xi_2)M_{N\mu}(\mathbb{Z}_p)$. Then one finds that the quasi-isogeny

$$\xi_1 \circ \xi_2^{-1} : \mathcal{A}_{z_0} \dashrightarrow \mathcal{A}_{y_0}$$

is in fact an honest isomorphism $\mathcal{A}_{z_0} \xrightarrow{\simeq} \mathcal{A}_{y_0}$. By (4.12), it lifts to an isomorphism

$$\mathcal{A}_z \xrightarrow{\simeq} \mathcal{A}_y$$

of the canonical lifts that is a G-admissible, level structure preserving isomorphism over $\mathrm{Fr}(W(k))$. But then $y = z$, by (4.8), and hence $y_0 = z_0$. Therefore, to finish the proof of injectivity, it is enough to know that $\mathcal{A}_{x_0}$ has no non-trivial G-admissible, level structure preserving automorphisms, but this is a consequence of the fact that $\mathcal{S}_K$ is a scheme, because of the neatness of K.[13]

[13] In more detail: The $\mathbb{Z}_{(p)}$-group I_{x_0} is compact-mod-scalars over $\mathbb{R}$, and so the embedding

$$I_{x_0}(\mathbb{Z}_{(p)}) \hookrightarrow I_{x_0}(\mathbb{A}^p_f) \hookrightarrow G(\mathbb{A}^p_f)$$

has discrete image. Here, the second embedding is obtained from the étale realization for I_{x_0} acting on $x_0^* \boldsymbol{H}_{\mathbb{A}^p_f}$, which can be identified with $\underline{H}_{\mathbb{A}^p_f}$ in a G-structure preserving way. This embedding can be chosen so that the level structure preserving elements of $I_{x_0}(\mathbb{Z}_{(p)})$ map into K^p, and thus form a finite subgroup of K^p. Since K^p is neat by hypothesis, this finite subgroup has to be trivial.

To prove surjectivity, we will first define a map

$$\varpi : G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to \mathrm{Isog}^{\mathrm{ord}}_{x_0}(k)$$

as follows: Fix an embedding $\iota : W(k) \hookrightarrow \mathbb{C}$, and suppose that the canonical lift x, viewed as a point $\iota(x) \in \mathrm{Sh}_K(\mathbb{C})$ is the image of a pair (h, g^p) under the uniformization (2.4.1). This means that we have chosen a G-structure preserving trivialization $\alpha : H_{(p)} \xrightarrow{\simeq} \boldsymbol{H}_{B,\iota(x)}$ such that the induced Hodge structure on $H_{(p)}$ corresponds to $\boldsymbol{h} \in X$.

As observed in (2.4), given a coset $gG(\mathbb{Z}_p)$, there exists $\gamma \in G(\mathbb{Q}) \cap gG(\mathbb{Z}_p)$. Now, the point $\tilde{y} = [(\gamma^{-1}\boldsymbol{h}, \gamma^{-1}g)] \in \mathrm{Sh}_K(\mathbb{C})$ depends only on $gG(\mathbb{Z}_p)$, as we find from the observation that, if $\gamma' \in G(\mathbb{Q})$ is another choice satisfying $\gamma' \in gG(\mathbb{Z}_p)$, then $\gamma^{-1}\gamma' \in G(\mathbb{Z}_{(p)})$.

Notice now that the action of γ on H gives a natural map $\boldsymbol{H}_{B,\tilde{y}} \to \boldsymbol{H}_{B,\iota(x)}$, which is the Betti realization of a G-admissible, level structure preserving isogeny

$$\tilde{\varpi}(g) : \mathcal{A}_{\iota(x)} \to \mathcal{A}_{\tilde{y}}.$$

This implies that the point $\tilde{y}$ arises from a point $\overline{y}$ defined over the algebraic closure $\overline{\mathrm{Fr}(W(k))}$ of $\mathrm{Fr}(W(k))$ in $\mathbb{C}$. In particular, if $y_0 \in \mathcal{S}_K(k)$ is the reduction of $\tilde{y}$, then $\tilde{\varpi}(g)$ reduces to a G-admissible, level structure preserving isogeny

$$\varpi(g) : \mathcal{A}_{x_0} \to \mathcal{A}_{y_0}.$$

Indeed, this is immediate from the fact that the various comparison isomorphisms preserve G-structure.

We claim that the composition $m_\eta \circ \varpi(g)$ can be described as follows: Write $\overline{x}$ for the points x viewed as a $\overline{\mathrm{Fr}(W(k))}$-valued point, and observe that in the notation of the proof of (3.4), the trivialization η induces a G-structure preserving isomorphism

$$\eta_p^* : \boldsymbol{H}_{p,\overline{x}} \xrightarrow{\simeq} \boldsymbol{H}_{p,N\mu}.$$

Choose $g_0 \in G(\mathbb{Z}_p)$ such that the composition

$$H_p \xrightarrow[\simeq]{g_0} H_p \xrightarrow[\simeq]{\alpha_p} \boldsymbol{H}_{p,\overline{x}} \xrightarrow[\simeq]{\eta_p^*} \boldsymbol{H}_{p,N\mu}$$

is a section of the sheaf $I_{N\mu}$ from (3.4).

Using the Iwasawa decomposition

$$G(\mathbb{Q}_p) = U^-_{N\mu}(\mathbb{Q}_p) M_{N\mu}(\mathbb{Q}_p) G(\mathbb{Z}_p),$$

we can write $g \in G(\mathbb{Q}_p)$ in the form $n^-(g)m(g)k(g)$, where the coset $m(g)M_{N\mu}(\mathbb{Z}_p)$ is canonically determined.

It can now be checked from the definitions that, for all $g \in G(\mathbb{Q}_p)$, we have

$$m_\eta(\varpi(g))M_{N\mu}(\mathbb{Z}_p) = m(g_0^{-1}g)M_{N\mu}(\mathbb{Z}_p).$$

This finishes the proof of surjectivity and thus of the proposition.

∎

4.18. We will say that $\lambda \in \mathsf{C}^+_{M_{N\mu}}$ is *effective* if $\lambda(p)$ acts on H_p via eigenvalues of non-negative p-adic valuation. In this case, any admissible quasi-isogeny of type λ is actually an honest isogeny at all geometric points. This can be seen for instance from the description of the type in (4.15) above.

Let $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_\lambda$ be the functor on p-adically complete formally of finite type $\mathscr{O}_{E_v}$-schemes S, associating with S the set of triples (x_1, x_2, ξ), where

$$x_1, x_2 : S \to \widehat{\mathcal{S}}^{\mathrm{ord}}_K$$

and $\xi : \mathcal{A}_{x_1} \to \mathcal{A}_{x_2}$ is a G-admissible, level structure preserving *isogeny* of type λ.

We will write $\mathrm{Isog}_\lambda^{\mathrm{ord}}$ for the reduction of $\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}$ mod p.

We have the source and target maps

$$s_\lambda^{\mathrm{ord}}:\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}\xrightarrow{(x_1,x_2,\xi)\mapsto x_1}\widehat{\mathcal{S}}_K^{\mathrm{ord}}\ ;\ t_\lambda^{\mathrm{ord}}:\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}\xrightarrow{(x_1,x_2,\xi)\mapsto x_2}\widehat{\mathcal{S}}_K^{\mathrm{ord}}.$$

4.19. Fix a G-structure preserving isogeny $\xi_\lambda:\mathcal{G}_0\to\mathcal{G}_0$ of type λ. For instance, such an isogeny arises from the map

$$\boldsymbol{H}_0\xrightarrow{h\mapsto\lambda(p)h}\boldsymbol{H}_0.$$

In the notation of (2.23), let

$$\widehat{U}_{G,\lambda}\subset\widehat{U}_G\times\widehat{U}_G$$

be the closed formal subscheme consisting of lifts (x,y) where ξ_λ lifts to an isogeny $\xi:\mathcal{G}_x\to\mathcal{G}_y$.

Suppose that we have a point $(x_0,y_0,\xi_0)\in\mathrm{Isog}_\lambda^{\mathrm{ord}}(k)$. Let $\widehat{U}_{x_0}$ (resp. $\widehat{U}_{y_0}$) be the completion of $\mathcal{S}_K$ at x_0 (resp. y_0). Then the completion $\widehat{U}_{(x_0,y_0,\xi_0)}$ is a closed formal subscheme of $\widehat{U}_{x_0}\times\widehat{U}_{y_0}$.

As seen in the proof of (2.25), given a choice of G-structure preserving isomorphism

$$\eta_{x_0}:W(k)\otimes_{\mathscr{O}_{E_v}}\mathcal{G}_0\xrightarrow{\simeq}\mathcal{G}_{x_0},$$

we can identify $\widehat{U}_{x_0}$ with $\widehat{U}_{G,W(k)}=W(k)\otimes_{\mathscr{O}_{E_v}}\widehat{U}_G$. Similarly, a choice of trivialization η_{y_0} for $\mathcal{G}_{y_0}$ allows us to identify $\widehat{U}_{y_0}$ with $\widehat{U}_{G,W(k)}$. By the definition of the type of ξ_0, we can choose η_{x_0} and η_{y_0} such that

$$\eta_{y_0}^{-1}\circ\xi_0\circ\eta_{x_0}=1\otimes\alpha_\lambda:W(k)\otimes_{\mathscr{O}_{E_v}}\mathcal{G}_0\to W(k)\otimes_{\mathscr{O}_{E_v}}\mathcal{G}_0.$$

The following lemma is immediate.

Lemma 4.20. *Under the isomorphisms $\widehat{U}_{x_0}\xrightarrow{\simeq}\widehat{U}_{G,W(k)}$ and $\widehat{U}_{y_0}\xrightarrow{\simeq}\widehat{U}_{G,W(k)}$ chosen above, the subspace $\widehat{U}_{(x_0,y_0,\xi_0)}\subset\widehat{U}_{x_0}\times\widehat{U}_{y_0}$ is mapped onto $\widehat{U}_{G,\lambda,W(k)}$.*

□

Proposition 4.21.

(1) *$\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}$ is represented by a formally of finite type, p-adically complete formal scheme over $\mathscr{O}_{E_v}$.*
(2) *The source and target maps factor as*

$$s_\lambda^{\mathrm{ord}}:\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}\xrightarrow{\alpha_\lambda}\mathcal{I}_{p,\lambda}^M\xrightarrow{\pi_\lambda^{\mathrm{ord}}}\widehat{\mathcal{S}}_K^{\mathrm{ord}}\ ;\ t_\lambda^{\mathrm{ord}}:\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}\xrightarrow{\beta_\lambda}\mathcal{I}_{p,-\lambda}^M\xrightarrow{\pi_{-\lambda}^{\mathrm{ord}}}\widehat{\mathcal{S}}_K^{\mathrm{ord}},$$

where the maps α_λ and β_λ are finite flat homeomorphisms, and

$$\pi_\lambda^{\mathrm{ord}}:\mathcal{I}_{p,\lambda}^M\to\widehat{\mathcal{S}}_K^{\mathrm{ord}}\ ;\ \pi_{-\lambda}^{\mathrm{ord}}:\mathcal{I}_{p,-\lambda}^M\to\widehat{\mathcal{S}}_K^{\mathrm{ord}}$$

are the finite étale covers obtained from $\mathcal{I}_p^M$ in (4.5).

Proof. The first assertion is standard: In fact, the morphism

$$(s_\lambda,t_\lambda):\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}\to\widehat{\mathcal{S}}_K^{\mathrm{ord}}\times\widehat{\mathcal{S}}_K^{\mathrm{ord}}$$

is locally of finite type and formally unramified. Indeed, it is formally unramified because homomorphisms between abelian schemes have at most one lift over any nilpotent thickening; see [13, Lemma 1.1.3], and it is locally of finite type over $\widehat{\mathcal{S}}_K^{\mathrm{ord}}\times\widehat{\mathcal{S}}_K^{\mathrm{ord}}$, since that is the case for the homomorphism scheme $\underline{\mathrm{Hom}}(\pi_1^*\mathcal{A},\pi_2^*\mathcal{A})$, where, for $i=1,2$

$$\pi_i:\widehat{\mathcal{S}}_K^{\mathrm{ord}}\times\widehat{\mathcal{S}}_K^{\mathrm{ord}}\to\widehat{\mathcal{S}}_K^{\mathrm{ord}}$$

is the natural projection onto the i^{th}-factor.

Moreover, by [10, Proposition 2.7], (s_λ,t_λ) satisfies the valuative criterion of properness, and therefore $\widehat{\mathrm{Isog}}_\lambda^{\mathrm{ord}}$ is a union of finite, formally unramified formal schemes over $\widehat{\mathcal{S}}_K^{\mathrm{ord}}\times\widehat{\mathcal{S}}_K^{\mathrm{ord}}$.

By (4.17) (and (4.6)), both s_λ and t_λ have finite fibers, from which it follows that s^{ord}_λ and t^{ord}_λ are both finite morphisms.

We now construct the morphisms α_λ and β_λ. This proceeds just as in (4.10). For instance, for α_λ, one needs to construct an $M_{N\mu}(\mathbb{Z}_p)$-equivariant isomorphism

$$\widehat{\mathrm{Isog}}^{\mathrm{ord}}_\lambda \times_{s^{\mathrm{ord}}_\lambda, \widehat{\mathcal{S}}^{\mathrm{ord}}_K} \mathcal{I}^M_p \to \mathcal{I}^M_p \times_{\widehat{\mathcal{S}}^{\mathrm{ord}}_K} \underline{M_{N\mu}(\mathbb{Z}_p)\lambda(p)^{-1}M_{N\mu}(\mathbb{Z}_p)/M_{N\mu(\mathbb{Z}_p)}}.$$

This is done as follows: Given a section $((x_1,x_2,\xi),\eta)$ on the left hand side over a connected p-adic formal scheme S, we send it to $(\eta, mM_{N\mu}(\mathbb{Z}_p))$, where $mM_{N\mu}(\mathbb{Z}_p)$ is the unique coset

$$mM_{N\mu}(\mathbb{Z}_p) \subset M_{N\mu}(\mathbb{Z}_p)\lambda(p)^{-1}M_{N\mu}(\mathbb{Z}_p)$$

such that, for every point $t \in S(k)$ valued in an algebraically closed field k, we have

$$W(k) \otimes_{\mathscr{O}_{E_v}} \mathrm{gr}\,\boldsymbol{H}_0 = m \cdot \eta_{\mathrm{cris},x_1\circ t}(\xi_t^* \boldsymbol{H}_{\mathrm{cris},x_2\circ t}) \subset \boldsymbol{H}_0[p^{-1}].$$

It now follows from (4.20) that the rank of the fibers of the finite morphisms α_λ and β_λ are constant. Since $\mathcal{I}^M_{p,\lambda}$ and $\mathcal{I}^M_{p,-\lambda}$ are formally smooth over $\mathscr{O}_{E_v}$, this immediately implies that α_λ and β_λ are both also flat.

Since α_λ and β_λ are finite flat, to check that they are homeomorphisms it is enough to see that they induce bijections on k-valued points for all algebraically closed fields k. But this is immediate from (4.17). □

Corollary 4.22. *Suppose that $\mathcal{S}_{K,k(v)}$ has a hypersymmetric μ-ordinary point. Then both maps s_λ and t_λ induce isomorphisms on the schemes of connected components.*

Proof. By (4.21), this is equivalent to showing that the maps $\pi^{\mathrm{ord}}_\lambda$ and $\pi^{\mathrm{ord}}_{-\lambda}$ induce isomorphisms on the schemes of connected components. This can be done using (4.6), the existence of the hypersymmetric point, and results from § 1. We do not give the details since we will not need this result, but see the proof of (5.25) below. □

4.23. Let $\mathbb{G}_m \subset T$ be the group of scalars, and let $\chi_0 : \mathbb{G}_m \hookrightarrow T$ be the canonical inclusion. For any $\lambda \in \mathsf{C}^+_M$, there exists $i \in \mathbb{Z}_{>0}$ such that $\lambda(i) := i \cdot \chi_0 + \lambda$ is effective. We can now define $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_\lambda$ to be the moduli of tuples (x,y,ξ), where ξ is a quasi-isogeny from $\mathcal{A}_x$ to $\mathcal{A}_y$ such that the tuple $(s,t,[p^i]\circ\xi)$ is a section of $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda(i)}$. This definition does not depend on the choice of i, as shown by:

Lemma 4.24. *Suppose that λ is effective; then the map*

$$\widehat{\mathrm{Isog}}^{\mathrm{ord}}_\lambda \to \widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda(1)}$$
$$(s,t,\xi) \mapsto (s,t,p\xi)$$

is an isomorphism of functors.

Proof. This amounts to the following assertion, which is easily deduced from the definitions: Suppose that $x_0, y_0 \in \mathcal{S}^{\mathrm{ord}}_{K,k(v)}(k)$ and that $\xi : \mathcal{A}_{x_0} \to \mathcal{A}_{y_0}$ is a G-admissible p-isogeny of type $\lambda(1)$. Then ξ factors uniquely through the multiplication-by-p endomorphism of $\mathcal{A}_{x_0}$. □

4.25. We now look at the relation between the two definitions of admissible quasi-isogenies, one over Sh_K, and the other over $\widehat{\mathcal{S}}^{\mathrm{ord}}_K$. Suppose that we have $\lambda \in \mathsf{C}^+_G$, and consider the E_v-scheme $E_v \otimes_E \mathrm{Isog}_\lambda$: It restricts to a finite morphism

$$\mathrm{Isog}^{\mathrm{an}}_\lambda \to \widehat{\mathcal{S}}^{\mathrm{ord,an}}_K \times \widehat{\mathcal{S}}^{\mathrm{ord,an}}_K.$$

Similarly, if we have $\nu \in \mathsf{C}^+_M$, then we can restrict $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_\nu$ to obtain another finite morphism

$$\widehat{\mathrm{Isog}}^{\mathrm{ord,an}}_\nu \to \widehat{\mathcal{S}}^{\mathrm{ord,an}}_K \times \widehat{\mathcal{S}}^{\mathrm{ord,an}}_K.$$

For every $\lambda \in \mathsf{C}_G^+$, let $\mathcal{S}(\lambda) \subset \mathsf{C}_M^+$ be the subset of cocharacters ν such that $\nu(p)$ can be lifted to an element of $P_{N\mu}^-(\mathbb{Q}_p) \cap G(\mathbb{Z}_p)\lambda(p)G(\mathbb{Z}_p)$.

Proposition 4.26. *There exists a canonical identification*

$$\bigsqcup_{\lambda \in \mathsf{C}_G^+} \mathrm{Isog}_\lambda^{\mathrm{an}} \xrightarrow{\simeq} \bigsqcup_{\nu \in \mathsf{C}_M^+} \widehat{\mathrm{Isog}}_\nu^{\mathrm{ord,an}}$$

of spaces over $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}} \times \widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$. *It maps* $\mathrm{Isog}_\lambda^{\mathrm{an}}$ *into the disjoint union*

$$\bigsqcup_{\nu \in \mathcal{S}(\lambda)} \widehat{\mathrm{Isog}}_\nu^{\mathrm{ord,an}}$$

Proof. After replacing λ by $\lambda(i)$ for some integer $i \in \mathbb{Z}_{\geq 0}$, if necessary, we can and will assume that it is an *effective* cocharacter.

Giving a point of $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}} \times \widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$ valued in a smooth affinoid $\mathbb{Q}_p$-algebra R amounts to specifying an open, bounded, integrally closed subring $R^\circ \subset R$, and a pair of maps $x^\circ, y^\circ : \mathrm{Spf}\, R^\circ \to \widehat{\mathcal{S}}_K$ of formal $\mathscr{O}_{E_v}$-schemes. Two such pieces of data, corresponding to tuples $(R_1^\circ, x_1^\circ, y_1^\circ)$ and $(R_2^\circ, x_2^\circ, y_2^\circ)$, give the same point if there exists a bounded subring $R^\circ \subset R$ containing both R_1°, R_2° such that the maps x_1°, x_2° and y_1°, y_2° agree when restricted to $\mathrm{Spf}\, R^\circ$.

Now, suppose that we have an R-valued point t of $\widehat{\mathcal{S}}_K^{\mathrm{ord,an}} \times \widehat{\mathcal{S}}_K^{\mathrm{ord,an}}$ represented by a tuple $(R^\circ, x^\circ, y^\circ)$.

Giving a lift t to an R-valued point of $\mathrm{Isog}_\lambda^{\mathrm{an}}$ amounts to the following: We have abelian schemes $\mathcal{A}_{x^\circ}$ and $\mathcal{A}_{y^\circ}$ over R°, giving rise to abelian schemes $\mathcal{A}_x, \mathcal{A}_y$ over R. Let X be the affinoid rigid analytic space associated with R; then any classical point of X gives rise to a point of Sh_K. Now, the lift to $\mathrm{Isog}_\lambda^{\mathrm{an}}$ is given by a level structure preserving isogeny $\xi : \mathcal{A}_x \dashrightarrow \mathcal{A}_y$ of abelian schemes over R that is G-admissible of type λ at every classical point of X.

Suppose that we are given such a lift. By [10, Prop. I.2.7], ξ extends to an isogeny $\xi^\circ : \mathcal{A}_{x^\circ} \to \mathcal{A}_{y^\circ}$ of abelian schemes over R°. Moreover, using the compatibility of the cohomological realizations of $\{s_\alpha\}$ with the p-adic comparison isomorphism (2.10), one finds that $\xi^\circ$is G-admissible in the sense of (4.11).

Giving a lift to an R-valued point $\widehat{\mathrm{Isog}}_\nu^{\mathrm{ord,an}}$ on the other hand, amounts to specifying a possibly larger bounded subring R_1° containing R°, and a level structure preserving p-isogeny

$$\xi_1^\circ : \mathcal{A}_{x^\circ}|_{\mathrm{Spec}\, R_1^\circ} \to \mathcal{A}_{y^\circ}|_{\mathrm{Spec}\, R_1^\circ}$$

of abelian schemes over R_1° that is G-admissible of type ν in the sense of (4.11).

If we are given such a lift of t, then the restriction of ξ_1° over $\mathrm{Spec}\, R$ is a G-admissible p-isogeny $\xi : \mathcal{A}_x \to \mathcal{A}_y$ in the sense of (4.7).

The proposition can now be deduced from (3.8) and (4.4). □

4.27. It will be useful to make part of (4.20) more explicit in the situation where $E_v = \mathbb{Q}_p$. In this case, we have $N\mu = \mu_p$, and the eigenspace decomposition of H_p is $H_p = H_p^0 \oplus H_p^1$.

A formal group $\widehat{T}$ over $W(k)$ is **diagonalizable** if there exists a finite $\mathbb{Z}_p$-module M such that, for any Artinian local $W(k)$-algebra B with maximal ideal $\mathfrak{m}_B$, we have

$$\widehat{T}(B) = \mathrm{Hom}(M, 1 + \mathfrak{m}_B).$$

In this situation, we will say that M is the **character group** for $\widehat{T}$. In the situation where M is finite free over $\mathbb{Z}_p$ with dual module $M^\vee$, we also have $\widehat{T}(B) = M^\vee \otimes_{\mathbb{Z}_p} (1 + \mathfrak{m}_B)$. In this case, we will call $\widehat{T}$ a **formal torus**, and $M^\vee$ the **cocharacter group** for $\widehat{T}$.

Let $\widehat{U}_0$ be the formal torus over $W(k)$ with cocharacter group $\mathrm{Hom}(H_p^1, H_p^0)$. Fix $\lambda \in \mathsf{C}_M^+$. Then we have two isogenies of formal tori

$$\psi_\lambda^1, \psi_\lambda^0 : \widehat{U}_0 \to \widehat{U}_0$$

whose induced maps on cocharacter groups are given by

$$\mathrm{Hom}(H^1_p, H^0_p) \xrightarrow{\alpha\mapsto\alpha\circ\lambda^1(p)} \mathrm{Hom}(H^1_p, H^0_p)\ ;\ \mathrm{Hom}(H^1_p, H^0_p) \xrightarrow{\alpha\mapsto\lambda^0(p)\circ\alpha} \mathrm{Hom}(H^1_p, H^0_p),$$

respectively. Here, for $i = 0, 1$, $\lambda^i(p)$ is the restriction of $\lambda(p)$ to H^i_p. Set

$$\widehat{U}_{0,\lambda} = \ker\big(\widehat{U}_0 \times \widehat{U}_0 \xrightarrow{\psi^1_\lambda\times\psi^0_\lambda} \widehat{U}_0 \times \widehat{U}_0 \xrightarrow{(u,v)\mapsto u\cdot v^{-1}} \widehat{U}_0\big).$$

This is a diagonalizable formal subgrup of $\widehat{U}_0$.

As in (2.20), let $U^-_{\mu_p} \subset G_{\mathbb{Z}_p}$ be the opposite unipotent associated with μ_p. Then the action of $\mathrm{Lie}\, U^-_{\mu_p}$ on H_p gives us an embedding of $\mathbb{Z}_p$-modules

$$\mathrm{Lie}\, U^-_{\mu_p} \subset \mathrm{Hom}(H^1_p, H^0_p).$$

Let $\widehat{U}_G \subset \widehat{U}_0$ be the formal sub-torus with cocharacter group $\mathrm{Lie}\, U^-_{\mu_p}$. Set

$$\widehat{U}_{G,\lambda} = (\widehat{U}_G \times \widehat{U}_G) \cap \widehat{U}_{0,\lambda}.$$

This is a formal diagonalizable subgroup of $\widehat{U}_G \times \widehat{U}_G$. The two natural projections give us homomorphisms

$$p^1_\lambda : \widehat{U}_{G,\lambda} \to \widehat{U}_G\ ;\ p^0_\lambda : \widehat{U}_{G,\lambda} \to \widehat{U}_G,$$

whose kernels are isomorphic to $\ker \psi^1_\lambda|_{\widehat{U}_G}$ and $\psi^2_\lambda|_{\widehat{U}_G}$, respectively.

Proposition 4.28. *Suppose that x_0, y_0 are μ-ordinary points in $\mathcal{S}_K(k)$. Let $\widehat{U}_{x_0}$ and $\widehat{U}_{y_0}$ be the formal $W(k)$-schemes obtained by completing $\mathcal{S}_K$ at x_0 and y_0, respectively, and let $\widehat{U}_\xi$ be the complete local ring of $\mathrm{Isog}^{\mathrm{ord}}_\lambda$ at (x_0, y_0, ξ). Then there exist identifications $\widehat{U}_{x_0} \xrightarrow{\simeq} \widehat{U}_G$ and $\widehat{U}_{y_0} \xrightarrow{\simeq} \widehat{U}_G$ such that $\widehat{U}_\xi$ is identified with the formal subscheme $\widehat{U}_{G,\lambda} \subset \widehat{U}_G \times \widehat{U}_G$.*

Proof. The p-divisible groups $\mathcal{G}_{x_0}, \mathcal{G}_{y_0}$ are both extensions of an étale p-divisible group by a multiplicative one. More precisely, as in (2.26), we have

$$\mathcal{G}_0 \xrightarrow{\simeq} \underline{\mathrm{Hom}}(H^1_p, \mu_{p^\infty}) \oplus \underline{\mathrm{Hom}}(H^0_p, \mathbb{Q}_p/\mathbb{Z}_p),$$

and we can choose G-structure preserving isomorphisms

$$\eta_{x_0} : k \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_{x_0}\ ;\ \eta_{y_0} : k \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_{y_0}.$$

For all the p-divisible groups above, we will denote their multiplicative and étale parts with a superscript mult and ét, respectively.

The isogeny ξ_0 induces maps

$$\xi^{\mathrm{mult}}_0 : \mathcal{G}^{\mathrm{mult}}_{x_0} \to \mathcal{G}^{\mathrm{mult}}_{y_0}\ ;\ \xi^{\text{ét}}_0 : \mathcal{G}^{\text{ét}}_{x_0} \to \mathcal{G}^{\text{ét}}_{y_0}$$

By the definition of the type of ξ, we can find trivializations η_{x_0} and η_{y_0} such that ξ^{mult}_0 and $\xi^{\text{ét}}_0$ are identified with

$$\underline{\mathrm{Hom}}(H^1_p, \mu_{p^\infty}) \xrightarrow{\beta\mapsto\beta\circ\lambda^1(p)} \underline{\mathrm{Hom}}(H^1_p, \mu_{p^\infty})\ ;\ \underline{\mathrm{Hom}}(H^0_p, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\gamma\mapsto\gamma\circ\lambda^0(p)} \underline{\mathrm{Hom}}(H^0_p, \mathbb{Q}_p/\mathbb{Z}_p),$$

respectively. Denote these maps by $\beta^{\mathrm{mult}}_\lambda : \mathcal{G}^{\mathrm{mult}}_0 \to \mathcal{G}^{\mathrm{mult}}_0$ and $\beta^{\text{ét}}_\lambda : \mathcal{G}^{\text{ét}}_0 \to \mathcal{G}^{\text{ét}}_0$, respectively. Then we obtain an isogeny

$$\beta_\lambda = \beta^{\mathrm{mult}}_\lambda \oplus \beta^{\text{ét}}_\lambda : \mathcal{G}_0 \to \mathcal{G}_0.$$

Now, for any Artin local $W(k)$-algebra $(B, \mathfrak{m}_B)$ with residue field k, and any pair of p-divisible groups $\mathcal{H}_1, \mathcal{H}_2$ over B, write

$$\widehat{\mathrm{Ext}}^1(\mathcal{H}_1, \mathcal{H}_2)$$

for the set of extensions of p-divisible groups of $\mathcal{H}_1$ by $\mathcal{H}_2$ over B, which reduce to the trivial extension over k.

Then we have:

$$\begin{aligned}\widehat{U}_0(B) = \operatorname{Hom}(H^1_p, H^0_p) \otimes_{\mathbb{Z}_p} (1+\mathfrak{m}_B) &= \operatorname{Hom}(H^1_p, H^0_p) \otimes \widehat{\operatorname{Ext}}^1_B(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) \\ &= \widehat{\operatorname{Ext}}^1_B\big(\underline{\operatorname{Hom}}(H^0_p, \mathbb{Q}_p/\mathbb{Z}_p), \underline{\operatorname{Hom}}(H^1_p, \mu_{p^\infty})\big) \\ &= \widehat{\operatorname{Ext}}^1_B\big(B \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0^{\text{ét}}, B \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0^{\text{mult}}\big).\end{aligned}$$

Therefore, we can identify $\widehat{U}_0$ with the deformation space over $W(k)$ of the p-divisible group $k \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0$. Note that the identity element of $\widehat{U}_0(W(k))$ corresponds to the canonical lift $W(k) \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0$ under this identification.

For each extension in $\widehat{U}_0(B)$, we can obtain two new extensions by pushing out along the homomorphism $\beta^{\text{mult}}_\lambda$, and by pulling back along the homomorphism $\beta^{\text{ét}}_\lambda$. The first of these corresponds to the endomorphism ψ^1_λ and the second to the endomorphism ψ^0_λ of $\widehat{U}_0$.

Moreover, suppose that we have two extensions $\mathcal{G}_{x_1}, \mathcal{G}_{x_2}$ corresponding to points $x_1, x_2 \in \widehat{U}_0(B)$. It is now clear that β_λ lifts to a homomorphism between $\mathcal{G}_{x_1}$ and $\mathcal{G}_{x_2}$ if and only if we have $\psi^1_\lambda(x_1) = \psi^0_\lambda(x_2)$. Putting all this together we find that $\widehat{U}_{0,\lambda} \subset \widehat{U}_0 \times \widehat{U}_0$ is precisely the subspace parameterizing pairs (x_1, x_2) such that β_λ lifts to an isogeny $\mathcal{G}_{x_1} \to \mathcal{G}_{x_2}$.

Using η_{x_0} and η_{y_0} we can now identify the deformation spaces of both $\mathcal{G}_{x_0}$ and $\mathcal{G}_{y_0}$ with $\widehat{U}_0$. The formal schemes $\widehat{U}_{x_0}$ and $\widehat{U}_{y_0}$ can be identified with formal subschemes of the deformation spaces for the p-divisible groups $\mathcal{G}_{x_0}$ and $\mathcal{G}_{y_0}$, respectively, and thus with formal subschemes of $\widehat{U}_0$. To finish, it is now enough to show that, under these identifications, both $\widehat{U}_{x_0}$ and $\widehat{U}_{y_0}$ map onto $\widehat{U}_G$.

We will do this for $\widehat{U}_{x_0}$, which will suffice by the symmetry of the situation. Consider the logarithm map

$$\ell : \widehat{U}_0(W(k)) = \operatorname{Hom}(H^1_p, H^0_p) \otimes (1 + pW(k)) \xrightarrow{1\otimes\log} \operatorname{Hom}(H^1_p, H^0_p) \otimes pW(k).$$

Here, in the first equality, we have identified the formal Ext group with $1 + pW(k)$ using Kummer theory, and the last map is given by the usual p-adic logarithm.

We now claim that we have

$$\ell(\widehat{U}_{x_0}(W(k))) \subset \ell(\widehat{U}_G(W(k))) = \operatorname{Lie} U^-_{\mu_p} \otimes (1 + pW(k)).$$

The second equality is immediate from the definition, so the inclusion of the left hand side in the right is the main thing to check. Assume this has been done. Then we claim that the proposition follows. Indeed, both $\widehat{U}_G$ and $\widehat{U}_{x_0}$ are smooth formal subschemes of $\widehat{U}_0$ of the same dimension, and so, if we have

$$\widehat{U}_{x_0}(W(k)) \subset \widehat{U}_G(W(k)),$$

then it immediately implies $\widehat{U}_{x_0} = \widehat{U}_G$. If $p \neq 2$, then ℓ is injective, and we are done. If $p = 2$, then the kernel of ℓ consists exactly of the 2-torsion points. But even in this case, we know by [28, Theorem 3.7] that $\widehat{U}_{x_0}$ is a translate by a torsion point of a formal sub-torus of $\widehat{U}_0$; in fact, since it contains the canonical lift, $\widehat{U}_{x_0}$ is itself a formal sub-torus of $\widehat{U}_0$. Let X_1 (resp. X_2) be the character group of the formal torus $\widehat{U}_0/\widehat{U}_G$ (resp. $\widehat{U}_0/\widehat{U}_{x_0}$): these are direct summands of $\operatorname{Hom}(H^1_p, H^0_p)$. Via the map ℓ, we find that $2X_1 = 2X_2$, and hence that $X_1 = X_2$. This shows that the result remains valid also when $p = 2$.

It remains now to show the inclusion

$$\ell(\widehat{U}_{x_0}(W(k))) \subset \operatorname{Lie} U^-_{\mu_p} \otimes (1 + pW(k)).$$

For this, we will need the following interpretation of the map ℓ: Given a lift $x \in \widehat{U}_0(W(k))$, we obtain a p-divisible group $\mathcal{G}_x$, whose de Rham realization is canonically identified with

$W(k) \otimes_{\mathbb{Z}_p} H_p$. The Hodge filtration arising from this identification is of the form

$$\mathrm{Fil}^1_x(W(k) \otimes_{\mathbb{Z}_p} H_p) = (1 + pN_x)(W(k) \otimes_{\mathbb{Z}_p} H^1_p),$$

for some $N_x \in W(k) \otimes_{\mathbb{Z}_p} \mathrm{Hom}(H^1_p, H^0_p)$. It follows from a computation of Katz [7, A.3] that, at least up to sign, we have

$$pN_x = \ell(x) \in \mathrm{Hom}(H^1_p, H^0_p) \otimes pW(k).$$

Now, by Prop. 3.4 and the proof of Prop. 4.6 of [16], if $x \in \widehat{U}_0(W(k))$ lie in $\widehat{U}_{x_0}(W(k))$, then the corresponding lift $\mathcal{G}_x$ is G-adapted. Among other things, this means that we have:

$$\{1 \otimes s_\alpha\} \subset \mathrm{Fil}^0 \bullet_x (W(k) \otimes_{\mathbb{Z}_p} H^{\otimes}_p),$$

where $\mathrm{Fil}^{\bullet}_x(W(k) \otimes_{\mathbb{Z}_p} H^{\otimes}_p)$ is the filtration induced from Fil^1_x. This implies that $pN_x = \ell(x) \in \mathrm{Lie}\, U^-_{\mu_p}$, and finishes the proof of the proposition. Indeed, the argument in [18, Lemma 1.5.6] shows that we have

$$\ell(x) \in (\mathrm{Lie}\, G + \mathrm{Lie}\, P_{\mu_p}) \cap \mathrm{Hom}(H^1_p, H^0_p),$$

and this latter intersection is easily seen to be equal to $\mathrm{Lie}\, U^-_{\mu_p}$. □

5. Degree lowering for special endomorphisms

In this section, we apply the above considerations to the special case of GSpin Shimura varieties, and show that the certain irreducible special divisors in their generic fibers continue to have irreducible reduction over $\overline{\mathbb{F}}_p$. Combined with the methods of [25], this yields a quick proof of the irreducibility of the moduli of primitively polarized K3 surfaces of fixed degree in any characteristic.

5.1. We begin by quickly presenting the required paraphernalia for a GSpin Shimura variety, and direct the reader to [24] and [12] for more details.

The starting point is a quadratic space (V, Q) over $\mathbb{Q}$ with signature $(n, 2)$ for some $n \geq 4$. The quadratic form Q gives rise to a symmetric pairing

$$[x, y]_Q = Q(x + y) - Q(x) - Q(y)$$

on V.

Associated with this is the reductive group $G = \mathrm{GSpin}(V, Q)$ over $\mathbb{Q}$, as well as a Hermitian symmetric domain X that parameterizes the space of oriented negative definite planes in $V_{\mathbb{R}}$. The pair (G, X) is a Shimura datum of Hodge type with reflex field $\mathbb{Q}$; a choice of symplectic representation given by the Clifford algebra $H \coloneqq C(V, Q)$, on which G acts via left multiplication.

We will fix a prime p and assume that the quadratic space has been chosen so that it admits a **self-dual lattice** $V_{\mathbb{Z}(p)} \subset V_{\mathbb{Q}_p}$. This is a $\mathbb{Z}_{(p)}$-lattice on which the quadratic form is $\mathbb{Z}_p$-valued, and is such that the associated bilinear form is non-degenerate. Note that when $p = 2$ this forces n to be even.

In this situation, $G_{\mathbb{Q}_p}$ is unramified and admits a reductive model

$$G_{\mathbb{Z}_{(p)}} = \mathrm{GSpin}(V_{\mathbb{Z}_{(p)}}, Q).$$

Therefore, with $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, and for any neat level subgroup of the form $K = K_p \times K^p$, we have the associated Shimura variety $\mathrm{Sh}_K \coloneqq \mathrm{Sh}_K(G, X)$ over $\mathbb{Q}$, and the integral canonical model $\mathcal{S}_K$ over $\mathbb{Z}_{(p)}$.

5.2. The lattice

$$H_{\mathbb{Z}_{(p)}} = C(V_{\mathbb{Z}_{(p)}}, Q) \subset C(V, Q)$$

gives us an abelian scheme $\mathcal{A} \to \mathcal{S}_K$ up to prime-to-p isogeny, as in § 3.

The lattice $V_{\mathbb{Z}_{(p)}} \subset V$ gives rise to canonical sub-sheaves

$$\boldsymbol{V}_? \subset \underline{\mathrm{End}}(\boldsymbol{H}_?)$$

for $? = B, p, \mathrm{dR}, \mathrm{cris}$. For every morphism $T \to \mathcal{S}_K$, we have a canonical $\mathbb{Z}_{(p)}$-submodule

$$V(T) \subset \mathrm{End}(\mathcal{A}_T)$$

whose cohomological realizations are sections of $\boldsymbol{V}_?$ for appropriate values of ?. The space $V(T)$ has a canonical positive definite quadratic form

$$Q : V(T) \to \mathbb{Z}_{(p)}$$

characterized by the identity $Q(f)\mathrm{id}_{\mathcal{A}_T} = f \circ f \in \mathrm{End}(\mathcal{A}_T)$.

5.3. For every $m \in \mathbb{Z}_{(p)}^{>0}$, we now have a morphism

$$\mathcal{Z}(m) \to \mathcal{S}_K$$

parameterizing, for each $T \to \mathcal{S}_K$, special endomorphisms $f \in V(T) \backslash pV(T)$ with $Q(f) = m$.

For the properties of this map, we direct the reader to [12, §7.1]. Here, we simply summarize its properties.

To begin, $\mathcal{Z}(m)$ is locally of finite type and formally unramified over $\mathcal{S}_K$, and also flat over $\mathbb{Z}_{(p)}$. Moreover, it is a local complete intersection.

Next, there is a canonical open subscheme

$$\mathcal{Z}^{\mathrm{pr}}(m) \subset \mathcal{Z}(m),$$

characterized by the property that the de Rham realization of the universal special endomorphism over $\mathcal{Z}^{\mathrm{pr}}(m)$ spans a local direct summand of $\boldsymbol{V}_{\mathrm{dR}}$.

Let $Z(m) \to \mathrm{Sh}_K$ be the generic fiber of $\mathcal{Z}(m)$. Then we have $Z(m) \subset \mathcal{Z}^{\mathrm{pr}}(m)$; cf. [24, Lemma 6.16] and its proof.

Also, let $\widehat{\mathcal{Z}}^{\mathrm{ord}}(m) \to \widehat{\mathcal{S}}_K^{\mathrm{ord}}$ be the restriction of $\mathcal{Z}(m)$ to the completion along the ordinary locus, and let $\mathcal{Z}^{\mathrm{ord}}(m)$ be its special fiber. Then $\mathcal{Z}^{\mathrm{ord}}(m)$ is a smooth, dense open subscheme of $\mathcal{Z}^{\mathrm{pr}}(m)_{\mathbb{F}_p}$.[14] In particular, $\mathcal{Z}^{\mathrm{pr}}(m)$ is regular in codimension 1 and, being a local complete intersection, is also normal, by Serre's criterion.

Warning 5.4. *In the cited references, the condition that f not belong to $pV(T)$ is omitted. This is an open condition, which is even closed over the generic fiber. It helps us pick out exactly the components that will be useful in what follows. Note that, if $p^2 \nmid m$, then f can never belong to $pV(T)$. In this case, the map $\mathcal{Z}(m) \to \mathcal{S}_K$ is a countable union of finite morphisms, and equals $\mathcal{Z}^{\mathrm{pr}}(m)$.*

5.5. Recall from (2.6) that we have a canonical étale $G(\mathbb{Z}_p)$-torsor I_p over Sh_K. Set $V_p = \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} V_{\mathbb{Z}_{(p)}}$. Given $m \in \mathbb{Z}_{(p)}^{>0}$, fix a unimodular element[15] $v_m \in V_{\mathbb{Z}_{(p)}}$ such that $v_m \circ v_m = m$. Let $G_{v_m} \subset G_{\mathbb{Z}_{(p)}}$ be the stabilizer of v_m: It is a smooth group scheme over $\mathbb{Z}_{(p)}$; cf. [24, Prop. 2.9].[16]

Let $V_m \subset V$ be the orthogonal complement of $\langle v_m \rangle$, and let $V_{m,\mathbb{Z}_{(p)}}$ be the $\mathbb{Z}_{(p)}$-lattice in V_m induced from $V_{\mathbb{Z}_{(p)}}$. The generic fiber of G_{v_m} can now be identified with $\mathrm{GSpin}(V_m)$. If

[14] For smoothness, see also (5.19) below.

[15] This means that v_m spans a direct summand of V_p.

[16] The proof there requires $p \neq 2$, but works just as well when $p = 2$ and the bilinear form associated with Q is non-degenerate.

$X_m \subset X$ is the subspace of oriented negative definite planes contained in V_m, then the pair $(G_{v_m,\mathbb{Q}}, X_m)$ is once again a GSpin Shimura datum.

If $\gamma \in G(\mathbb{Z}_{(p)})$, then conjugation by γ gives us another map of Shimura data:

$$\mathrm{int}(\gamma) : (G_{v_m}, X_m) \to (G, X).$$

Given $g \in G(\mathbb{A}_f)$, set $K_{m,g,\gamma} = \mathrm{int}(\gamma)^{-1}(gKg^{-1}) \subset G_{v_m}(\mathbb{A}_f)$. Then we obtain a finite unramified map

$$\iota_{m,\gamma,g} : \mathrm{Sh}_{K_{m,\gamma,g}} \coloneqq \mathrm{Sh}_{K_{m,\gamma,g}}(G_{v_m,\mathbb{Q}}, X_m) \to \mathrm{Sh}_K$$

of GSpin Shimura varieties, which on the level of $\mathbb{C}$-points is obtained from the map

$$X_m \times G_{v_m}(\mathbb{A}_f) \xrightarrow{(x,h)\mapsto(\gamma\cdot x,\mathrm{int}(\gamma)(h)g)} X \times G(\mathbb{A}_f)$$

Taking the normalization of $\mathcal{S}_K$ in $\mathrm{Sh}_{K_{m,\gamma,g}}$ now gives us an integral model $\mathcal{S}_{K_{m,\gamma,g}}$ over $\mathbb{Z}_{(p)}$. Let $\mathcal{S}^{\mathrm{sm}}_{K_{m,\gamma,g}} \subset \mathcal{S}_{K_m}$ be the complement of the singular locus in $\mathcal{S}_{K_{m,\gamma,g},\mathbb{F}_p}$. Then the arguments in [24, Lemmas 6.16,7.1] show that the natural map

$$\mathcal{S}^{\mathrm{sm}}_{K_{m,\gamma,g}} \to \mathcal{S}_K$$

lifts canonically to an open and closed embedding

$$\mathcal{S}^{\mathrm{sm}}_{K_{m,\gamma,g}} \to \mathcal{Z}^{\mathrm{pr}}(m).$$

Following [2, Prop. 2.7.4], one finds that the generic fiber $Z(m)$ is the union of its open and closed subschemes of the form $\mathrm{Sh}_{K_{m,\gamma,g}}$ as γ and g vary. Therefore, since $\mathcal{Z}^{\mathrm{pr}}(m)$ is normal, the Zariski closure

$$\mathcal{S}^{\mathrm{pr}}_{K_{m,\gamma,g}} \subset \mathcal{Z}^{\mathrm{pr}}(m)$$

of $\mathcal{S}^{\mathrm{sm}}_{K_{m,\gamma,g}}$ is an open and closed subscheme; moreover every connected component of $\mathcal{Z}^{\mathrm{pr}}(m)$ is obtained as the connected component of $\mathcal{S}^{\mathrm{pr}}_{K_{m,\gamma,g}}$, for some pair (γ, g).

5.6. Let $f \in V\big(\mathcal{A}|_{\mathcal{Z}(m)}\big)$ be the tautological special endomorphism. Consider the sub-sheaf

$$I_{p,v_m} \subset I_p|_{Z(m)}$$

of G-structure preserving trivializations $\iota : \underline{H}_p \xrightarrow{\simeq} \boldsymbol{H}_p$ such that $\iota(v_m) = \boldsymbol{f}_p$; here, $\boldsymbol{f}_p$ is the p-adic realization of f. Since f spans a direct summand of $V(\mathcal{Z}(m))$, it follows that $\boldsymbol{f}_p$ generates a direct summand of $\boldsymbol{V}_p$. Then, by [24, Lemma 2.8], I_{p,v_m} is a $G_{v_m}(\mathbb{Z}_p)$ torsor over $Z(m)$.[17]

Let $I_{\mathbb{F}_p,v_m}$ be the induced $G_{v_m}(\mathbb{F}_p)$-torsor: it is finite étale over $Z(m)$.

Lemma 5.7. *Given a sub-group $H \subset G_{v_m}(\mathbb{F}_p)$, the following statements are equivalent:*

(1) The finite étale cover $I_{\mathbb{F}_p,v_m}/H$ over $Z(m)$ is relatively geometrically irreducible.
(2) The spinor norm $\nu : G_{v_m}(\mathbb{F}_p) \to \mathbb{F}_p^\times$ is surjective when restricted to H.

Here, we say that a finite étale cover is relatively geometrically irreducible if its restriction over any geometrically irreducible component of the target is once again irreducible.

Proof. As discussed in (5.5), $Z(m)$ is the union of Shimura varieties of the form $\mathrm{Sh}_{K_{m,\gamma,g}}$, and so it suffices to prove the result over each of these varieties.

By construction the p-primary part $K_{m,\gamma,g,p} \subset G_{v_m}(\mathbb{Q}_p)$ is equal to $G_{v_m}(\mathbb{Z}_p)$. One sees that the finite étale cover $I_{\mathbb{F}_p,v_m}/H$ is represented over $\mathrm{Sh}_{K_{m,\gamma,g}}$ by the Shimura variety $\mathrm{Sh}_{K_{p,H}K^p_{m,\gamma,g}}$, where $K^p_{m,\gamma,g} \subset G_{v_m}(\mathbb{A}^p_f)$ is the prime-to-p part of $K_{m,\gamma,g}$, and $K_{p,H} \subset G_{v_m}(\mathbb{Z}_p)$ is the pre-image of H under the map

$$\mathrm{int}(\gamma) : G_{v_m}(\mathbb{Z}_p) \to G(\mathbb{Z}_p) \to G(\mathbb{F}_p)$$

[17] Once again, the proof of this result goes through even when $p = 2$.

On the other hand, the description of the connected components of Shimura varieties in [8, Thm. 2.4] shows that, for any compact open $K' \subset G_{v_m}(\mathbb{A}_f)$, we have

$$\pi_0(\mathrm{Sh}_{K',\overline{\mathbb{Q}}}) = \mathbb{A}_f^\times/\mathbb{Q}_{>0}\nu(K'),$$

where $\nu : G_{v_m} \to \mathbb{G}_m$ is the spinor norm. Therefore, the map

$$\pi_0(\mathrm{Sh}_{K_{p,H}K_m^p,\overline{\mathbb{Q}}}) \to \pi_0(\mathrm{Sh}_{K,\overline{\mathbb{Q}}})$$

is a bijection if and only if $\nu(H) = \nu(G_{v_m}(\mathbb{F}_p)) \subset \mathbb{F}_p^\times$. Since ν is a surjective map of group schemes with connected kernel, the lemma follows. □

5.8. The (inverse) Shimura cocharacter μ_p determines an eigenspace decomposition of V_p:

$$V_p = V_p^1 \oplus V_p^0 \oplus V_p^{-1}.$$

Here, $V_p^{\pm 1}$ are complementary isotropic lines and $V_p^0 \subset V_p$ is the subspace orthogonal to both.

Let $U_{\mu_p}^- \subset G_{\mathbb{Z}_p}$ be the opposite unipotent associated with μ_p. As observed more generally in (4.27), we have a canonical inclusion

$$\mathrm{Lie}\, U_{\mu_p}^- \subset \mathrm{Hom}(H_p^1, H_p^0). \tag{5.8.1}$$

This can be made explicit. Namely, let $G_0 = \mathrm{SO}(V_p)$ be the special orthogonal quotient of $G_{\mathbb{Z}_p}$. Then we can also identify $U_{\mu_p}^-$ with the unipotent subgroup of G_0 associated with the isotropic line V_p^{-1}. That is, we have

$$\mathrm{Lie}\, U_{\mu_p}^- = \{(\varphi,\psi) \in \mathrm{Hom}(V_p^0, V_p^{-1}) \times \mathrm{Hom}(V_p^1, V_p^0) :\ \varphi^\vee + \psi = 0\} \subset \mathrm{End}(V_p).$$

Here, we have used the non-degenerate bilinear form on V_p to identify V_p^0 with its own dual, and V_p^1 with the dual of V_p^{-1}, and hence the dual $\varphi^\vee$ of φ with a map $\varphi^\vee : V_p^1 \to V_p^0$. In what follows, we can and will identify $\mathrm{Lie}\, U_{\mu_p}^-$ with its image in $\mathrm{Hom}(V_p^0, V_p^{-1})$.

Fix generators $v^\pm$ of $V_p^{\pm 1}$. Then, as explained in [24, §1], under the left multiplication action of V_p on H_p, we have

$$H_p^1 = \ker(v^+) = \mathrm{im}(v^+)\ ;\ H_p^0 = \ker(v^-) = \mathrm{im}(v^-).$$

The embedding (5.8.1) can now be described as follows: Suppose that we have a map $\varphi : V_p^0 \to V_p^{-1}$ in $\mathrm{Lie}\, U_{\mu_p}^-$. There exists a unique $v_\varphi^0 \in V_p^0$ such that, for all $v \in V_p^0$, we have

$$[v_\varphi^0, v]_Q \cdot v^- = \psi(v).$$

Now, one can check that, up to sign, under (5.8.1), (φ,ψ) maps to left multiplication by the element $v^- v_\varphi^0$ in the Clifford algebra.

5.9. The scheme of parabolic subgroups Par_{μ_p} over $\mathbb{Z}_p$ can be canonically identified with the space of isotropic lines in V_p. Note that we have

$$\begin{aligned} I_{p,\mu_p} \times_{\mathrm{Sh}_K} Z(m) &= (I_p \times^{G(\mathbb{Z}_p)} (G(\mathbb{Z}_p)\mu_p(p)^{-1}G(\mathbb{Z}_p))/G(\mathbb{Z}_p)) \times_{\mathrm{Sh}_K} Z(m) \\ &= (I_p \times_{\mathrm{Sh}_K} Z(m)) \times^{G(\mathbb{Z}_p)} \mathrm{Par}_{\mu_p}(\mathbb{F}_p) \\ &= I_{p,v_m} \times^{G_{v_m}(\mathbb{Z}_p)} \mathrm{Par}_{\mu_p}(\mathbb{F}_p). \end{aligned}$$

Here, the second identification follows from (4.6), and the fact that μ_p is minuscule.

Let $\mathrm{Par}^\circ_{\mu_p}(v_m)$ (resp. $\mathrm{Par}^\perp_{\mu_p}(v_m)$) be the set of isotropic lines in $V_{\mathbb{F}_p}$ that are not orthogonal to v_m (resp. orthogonal to v_m and linearly independent from v_m). Under the action of $G_{v_m}(\mathbb{F}_p)$, $\mathrm{Par}_{\mu_p}(\mathbb{F}_p)$ has the following orbit decomposition:

$$\mathrm{Par}_{\mu_p}(\mathbb{F}_p) = \begin{cases} \{\langle v_m\rangle\} \sqcup \mathrm{Par}^\circ_{\mu_p}(v_m) \sqcup \mathrm{Par}^\perp_{\mu_p}(v_m), \text{ if } v_m \text{ is isotropic mod } p; \\ \mathrm{Par}^\circ_{\mu_p}(v_m) \sqcup \mathrm{Par}^\perp_{\mu_p}(v_m), \text{ otherwise.} \end{cases}$$

Indeed, each of the purported orbits is clearly preserved by the action of $G_{v_m}(\mathbb{F}_p)$. Therefore, it is enough to show that each is in fact an orbit. But this is an easy consequence of Witt's extension theorem [3, §4, Thm. 1].

Therefore, we obtain a finite étale cover

$$\pi^\circ_{\mu_p} : I^\circ_{p,\mu_p} \coloneqq I_{p,v_m} \times^{G_{v_m}(\mathbb{Z}_p)} \mathrm{Par}^\circ_{\mu_p}(v_m) \subset I_{p,\mu_p}|_{Z(m)} \to Z(m).$$

Lemma 5.10. *The map $\pi^\circ_{\mu_p}$ is relatively geometrically irreducible.*

Proof. By (5.7), it is enough to show that the stabilizer $H \subset G_{v_m}(\mathbb{F}_p)$ of a line $N \in \mathrm{Par}^\circ_{\mu_p}(v_m)$ maps surjectively on $\mathbb{F}_p^\times$ via the spinor norm.

Set

$$V' = (\langle v_m\rangle \oplus N)^\perp \subset V_{\mathbb{F}_p}.$$

Then clearly H contains the $\mathbb{F}_p$-points of the subgroup $G' \subset G_{\mathbb{F}_p}$ that fixes both v_m and N point-wise.

We claim that V' is a non-degenerate quadratic subspace of V, and therefore that $G' = \mathrm{GSpin}(V')$ maps surjectively onto $\mathbb{G}_m$ via the spinor norm. Combined with Lang's theorem and the connectedness of $\mathrm{Spin}(V')$[18], this will complete the proof of the lemma.

We claim that $\langle v_m\rangle \oplus N$ is isometric to a hyperbolic plane over $\mathbb{F}_p$. This will clearly suffice.

If v_m is isotropic mod p, then N is a complementary isotropic line to v_m, and the claim follows.

If v_m is non-isotropic mod p, choose a generator e for N. Set $\alpha = Q(v_m), \beta = [e, v_m]$: these are both in $\mathbb{F}_p^\times$. Set

$$f = -\beta^{-1}\alpha e + v_m.$$

Then we find that $Q(f) = 0$, and that e, f generate complementary isotropic lines, thus giving us the desired hyperbolic plane. □

5.11. Fix $m \in \mathbb{Z}^{>0}_{(p)}$. Define a functor $\mathrm{Isog}_{\mu_p}(m)$ on $\mathbb{Q}$-schemes as follows: For any $\mathbb{Q}$-scheme T, $\mathrm{Isog}_{\mu_p}(m)(T)$ consists of tuples (s, t, ξ, f) such that:

- $(s, t, \xi) \in \mathrm{Isog}_{\mu_p}(T)$ (cf. (4.7));
- $f \in V(s)\backslash pV(s)$ is such that $f \circ f = m$;
- $\xi f \xi^{-1} \in p^{-1}V(t)\backslash V(t)$.

Note that the last condition is well-defined: Given any G-admissible quasi-isogeny $\xi : \mathcal{A}_s \dashrightarrow \mathcal{A}_t$, the induced automorphism

$$\mathrm{End}(\mathcal{A}_s)[p^{-1}] \xrightarrow[\simeq]{f \mapsto \xi f \xi^{-1}} \mathrm{End}(\mathcal{A}_t)[p^{-1}]$$

will carry $V(s)[p^{-1}]$ onto $V(t)[p^{-1}]$. Indeed, this can be checked on the level of cohomological realizations, where it is immediate from the definitions.

We have the 'source' and 'target' maps

$$s_{\mu_p}(m) : \mathrm{Isog}_{\mu_p}(m) \xrightarrow{(s,t,\xi,f)\mapsto(s,f)} Z(m)\ ;\ t_{\mu_p}(m) : \mathrm{Isog}_{\mu_p}(m) \xrightarrow{(s,t,\xi,f)\mapsto(t,p(\xi f\xi^{-1}))} Z(p^2m).$$

Proposition 5.12.

(1) *The map $s_{\mu_p}(m)$ is finite étale and relatively geometrically irreducible.*
(2) *The map $t_{\mu_p}(m)$ is an isomorphism.*

Proof. Consider the map

$$\mathrm{Isog}_{\mu_p}(m) \xrightarrow{(\alpha_{\mu_p},(s,f))} I_{p,\mu_p} \times_{\mathrm{Sh}_K} Z(m),$$

[18] Recall that $n \geq 4$.

where α_{μ_p} is the map from (4.10). To prove (1), given (5.10) and (4.10), it is enough to show that the above map carries $\mathrm{Isog}_{\mu_p}(m)$ into I°_{p,μ_p}.

For this, we first note that the map can be defined by descent from a map

$$\mathrm{Isog}_{\mu_p}(m)\times_{Z(m)} I_{p,v_m}\to I_{p,v_m}\times_{Z(m)}\underline{G(\mathbb{Z}_p)\mu_p(p)G(\mathbb{Z}_p)/G(\mathbb{Z}_p)} = I_{p,v_m}\times_{Z(m)}\mathrm{Par}_{\mu_p}(\mathbb{F}_p).$$

This is given as follows: Let $V_p^1\subset V_p$ be the line on which $\mu_p(z)$ acts via $z\mapsto z$. Given a section $((s,t,\xi,f),\iota)$ over a connected scheme T on the left-hand side, the right-hand side is (η,N), where $N\subset V_{\mathbb{F}_p}$ is an isotropic line with the following property: There exists $g\in G(\mathbb{Z}_p)$ such that:

- $\mathbb{F}_p\otimes_{\mathbb{Z}_p} gV_p^{-1}=N$;
- $v_m\in p^{-1}g\mu_p(p)V_p\backslash g\mu_p(p)V_p$.

To complete the proof of (2), it is enough to see that the conditions together are equivalent to saying that $N\in\mathrm{Par}^{\circ}_{\mu_p}(\mathbb{F}_p)$. But we see that

$$g\mu_p(p)V_p = p^{-1}gV_p^{-1}\oplus gV_p^0\oplus pgV_p^1.$$

Therefore, the second condition is equivalent to saying that the component of v_m in gV_p^1 is non-zero mod p, which in turn is equivalent to N not being orthogonal to v_m.

As for (2), proceeding just as above gives us a map

$$\mathrm{Isog}_{\mu_p}(m)\times_{Z(p^2m)} I_{p,v_{p^2m}}\to I_{p,v_{p^2m}}\times_{Z(p^2m)}\mathrm{Par}_{\mu_p}(\mathbb{F}_p),$$

and we can finish by showing that the image of any tuple $((s,t,\xi),\iota')$ is $(\iota',\langle v_{p^2m}\rangle)$. Indeed, just as above, if (ι',N') is in the image of the map, then there exists $g\in G(\mathbb{Z}_p)$ such that:

- $\mathbb{F}_p\otimes gV_p^1=N'$;
- $v_{p^2m}\in pg\mu_p(p)^{-1}V_p$.

Since

$$pg\mu_p(p)^{-1}V_p = p^2gV_p^{-1}\oplus pgV_p^0\oplus gV_p^1,$$

the second condition can hold if and only if v_{p^2m} generates N' mod p. □

5.13. Assume that V_p^0 is isotropic: This is always the case as soon as $n\geq 3$. Then there exists a unique non-central co-character $\lambda_0\in\mathsf{C}_M^+$ that is conjugate to μ_p under $G(\mathbb{Z}_p)$. Concretely, λ_0 determines an eigenspace decomposition of the form $V_p^0=V_p^0(-1)\oplus V_p^0(0)\oplus V_p^0(1)$, with $V_p^0(-1),V_p^0(1)$ isotropic lines in V_p^0.

Let the notation now be as in (4.27), so that we have an embedding of formal tori $\widehat{U}_G\subset\widehat{U}_0$ over $W(k)$ corresponding to the embedding of their cocharacter groups in (5.8.1).

The action of λ_0 on H_p^i for $i=0,1$ breaks it up into eigenspaces

$$H_p^i=H_p^i(0)\oplus H_p^i(1).$$

It also breaks up $\mathrm{Lie}\,U^-_{\mu_p}$ and $\mathrm{Hom}(H_p^1,H_p^0)$ compatibly into eigenspaces

$$\mathrm{Lie}\,U^-_{\mu_p}=\bigoplus_{i=-1}^{1}\mathrm{Lie}\,U^-_{\mu_p}(i)\ ;\ \mathrm{Hom}(H_p^1,H_p^0)=\bigoplus_{i=-1}^{1}\mathrm{Hom}(H_p^1,H_p^0)(i),$$

where

$$\mathrm{Lie}\,U^-_{\mu_p}(i)=\begin{cases}\mathrm{Hom}(V_p^0(1),V_p^{-1}) & \text{if } i=-1;\\ \mathrm{Hom}(V_p^0(0),V_p^{-1}) & \text{if } i=0;\\ \mathrm{Hom}(V_p^0(-1),V_p^{-1}) & \text{if } i=1.\end{cases},$$

and

$$\mathrm{Hom}(H_p^1, H_p^0)(i) = \begin{cases} \mathrm{Hom}(H_p^1(1), H_p^0(0)) & \text{if } i = -1; \\ \mathrm{Hom}(H_p^1(0), H_p^0(0)) \oplus \mathrm{Hom}(H_p^1(1), H_p^0(1)) & \text{if } i = 0; \\ \mathrm{Hom}(H_p^1(0), H_p^0(1)) & \text{if } i = 1. \end{cases}$$

This gives us decompositions of formal tori:

$$\widehat{U}_G = \widehat{U}_G(-1) \times \widehat{U}_G(0) \times \widehat{U}_G(1) \; ; \; \widehat{U}_0 = \widehat{U}_0(-1) \times \widehat{U}_0(0) \times \widehat{U}_0(1). \tag{5.13.1}$$

5.14. As in (4.27), we have the formal diagonalizable subgroup

$$\widehat{U}_{G,\lambda_0} \subset \widehat{U}_G \times \widehat{U}_G.$$

We will now describe this subgroup more explicitly.

For this, we must recall the definition. We have two morphisms

$$\psi^1_{\lambda_0}, \psi^0_{\lambda_0} : \widehat{U}_G \to \widehat{U}_0,$$

and we have

$$\widehat{U}_{G,\lambda_0} = \{(x,y) \in \widehat{U}_G \times \widehat{U}_G : \; \psi^1_{\lambda_0}(x) = \psi^0_{\lambda_0}(y)\}.$$

On the level of cocharacter groups, the maps $\psi^1_{\lambda_0}$ and $\psi^0_{\lambda_0}$ correspond to the homomorphisms

$$\mathrm{Lie}\, U^-_{\mu_p} \xrightarrow{\varphi \mapsto v^- v^0_\varphi \lambda(p)} \mathrm{Hom}(H_p^1, H_p^0) \; ; \; \mathrm{Lie}\, U^-_{\mu_p} \xrightarrow{\varphi \mapsto \lambda(p) v^- v^0_\varphi} \mathrm{Hom}(H_p^1, H_p^0),$$

respectively. From this, one checks that, in terms of the decomposition (5.13.1), we have

$$\psi^1_{\lambda_0}|_{\widehat{U}_G(-1)} = p \cdot \mathrm{id} \; ; \psi^1_{\lambda_0}|_{\widehat{U}_G(1)} = \mathrm{id} \; ; \psi^0_{\lambda_0}|_{\widehat{U}_G(-1)} = \mathrm{id} \; ; \psi^1_{\lambda_0}|_{\widehat{U}_G(1)} = p \cdot \mathrm{id};$$
$$\psi^1_{\lambda_0}|_{\widehat{U}_G(0)} = \psi^0_{\lambda_0}|_{\widehat{U}_G(0)}.$$

Moreover, $\psi^1_{\lambda_0}$ (equivalently, $\psi^0_{\lambda_0}$) maps $\widehat{U}_G(0)$ isomorphically onto its image in $\widehat{U}_0$. On the level of cocharacter groups, this amounts to the fact that the composition

$$\mathrm{Lie}\, U^-_{\mu_p}(0) = \mathrm{Hom}(V_p^0(0), V_p^{-1}) \xrightarrow{\varphi \mapsto v^- v^0_\varphi} \mathrm{Hom}(H_p^1(0), H_p^0(0)) \oplus \mathrm{Hom}(H_p^1(1), H_p^0(1)) \to$$
$$\xrightarrow{(\mathrm{id}, p \cdot \mathrm{id})} \mathrm{Hom}(H_p^1(0), H_p^0(0)) \oplus \mathrm{Hom}(H_p^1(1), H_p^0(1))$$

maps onto a direct sum of the target.

Therefore, we obtain a decomposition

$$\widehat{U}_{G,\lambda_0} = \widehat{U}_{G,\lambda_0}(-1) \times \widehat{U}_{G,\lambda_0}(0) \times \widehat{U}_{G,\lambda_0}(1),$$

with

$$\widehat{U}_{G,\lambda_0}(i) = \begin{cases} \{(x, x^p) : \; x \in \widehat{U}_G(-1)\} & \text{if } i = -1; \\ \{(x,x) : \; x \in \widehat{U}_G(0)\} & \text{if } i = 0; \\ \{(x^p, x) : \; x \in \widehat{U}_G(1)\} & \text{if } i = 1. \end{cases} \tag{5.14.1}$$

We can also express this in terms of character groups. Indeed, we can identify the character groups of both $\widehat{U}_G$ and $\widehat{U}_{G,\lambda_0}$ with

$$\mathrm{Hom}(V_p^{-1}, V_p^0) \xrightarrow[\simeq]{f \mapsto f(v^-)} V_p^0.$$

Via this identification, the inclusion

$$\widehat{U}_{G,\lambda} \hookrightarrow \widehat{U}_G \times \widehat{U}_G$$

corresponds to the map of character groups

$$V_p^0 \oplus V_p^0 \xrightarrow{(v,w) \mapsto \pi_1(v) + \pi_2(w)} V_p^0,$$

where, in terms of the decomposition

$$V_p^0 = V_p^0(-1) \oplus V_p^0(0) \oplus V_p^0(1),$$

we have

$$\pi_1 = (p,1,1)\ ;\ \pi_2 = (1,1,p).$$

5.15. For every $m \in \mathbb{Z}_p \backslash \{0\}$, choose $v_m^0 \in V_p^0$ such that $Q(v_m^0) = m$, and let $M_{v_m^0} \subset M_{\mu_p}$ be the stabilizer of v_m^0. Let $\mathrm{Par}^\circ_{\lambda_0}(v_m) \subset \mathrm{Par}_{\lambda_0}$ be the open subscheme parameterizing for each $\mathbb{Z}_p$-algebra R, the set of isotropic lines $N \subset R \otimes_{\mathbb{Z}_p} V_p^0$ that are locally spanned by a generator w satisfying

$$[w, v_m^0]_Q \in R^\times.$$

Fix a line $N \in \mathrm{Par}^\circ_{\lambda_0}(v_m)(\mathbb{Z}_p)$, and let $Q_{v_m^0} \subset M_{v_m^0}$ be its stabilizer. We will consider the spinor norm $\nu : M_{v_m^0} \to \mathbb{G}_m$: this is a homomorphism of $\mathbb{Z}_p$-group schemes, and is the restriction of the spinor norm on G. Let

$$\widetilde{M}_{v_m^0} \subset M_{v_m^0}\ ;\ \widetilde{Q}_{v_m^0} \subset Q_{v_m^0}$$

be the kernels of the spinor norm.

Lemma 5.16. *With the notation as above:*

1. *The group schemes $\widetilde{M}_{v_m^0}$ and $\widetilde{Q}_{v_m^0}$ are smooth over $\mathbb{Z}_p$.*
2. *$\widetilde{M}_{v_m^0}$ is the Zariski closure in $M_{v_m^0}$ of the derived subgroup of $M_{v_m^0,\mathbb{Q}_p}$, which is simply connected.*
3. *$M_{v_m^0}$ acts transitively on $\mathrm{Par}^\circ_{\lambda_0}(v_m)$.*

Proof. First, we note that $M_{v_m^0}$ and $Q_{v_m^0}$ are smooth group schemes over $\mathbb{Z}_p$. For this, first observe that the Levi subgroup M_{μ_p} can be identified with $\mathrm{GSpin}(V_p^0)$. Therefore, just as in (5.6), we can use [24, Prop. 2.9] to conclude that $M_{v_m^0}$ is smooth over $\mathbb{Z}_p$.

Now, for $Q_{v_m^0}$, consider the rank 2 direct summand $V' \subset V_p^0$ generated by N and v_m^0. As in the proof of (5.10), one can show that V' is isometric to a hyperbolic plane over $\mathbb{Z}_p$. Let $M_{V'} \subset M_{\mu_p}$ be the pointwise stabilizer of V': This is isomorphic to $\mathrm{GSpin}(U')$, where $U' \subset V_p^0$ is the orthogonal complement to V', and is thus a reductive $\mathbb{Z}_p$-group scheme. We now have an exact sequence

$$1 \to M_{V'} \to Q_{v_m^0} \xrightarrow{\chi} \mathbb{G}_m$$

of $\mathbb{Z}_p$-group schemes, where $\chi : Q_{v_m^0} \to \mathbb{G}_m$ gives the action of $Q_{v_m^0}$ on N. Since $Q_{v_m^0}$ contains the subgroup $\mathrm{GSpin}(V')$, we find that χ is surjective, which shows that $Q_{v_m^0}$ is smooth over $\mathbb{Z}_p$.

To prove the first assertion, it is now enough to observe that the spinor norm restricted to $Q_{v_0^m}$ is a submersion onto $\mathbb{G}_m$ since it clearly is one when restricted to $\mathrm{GSpin}(V')$. The second assertion is also immediate, since $\widetilde{M}_{v_m^0}$ is flat and is isomorphic to the simply connected Spin group associated with the orthogonal complement of $\langle v_m^0 \rangle$ in V_p^0.

Finally, the third assertion can be deduced from Witt's extension theorem [3, §4, Thm. 1]. □

5.17. The restriction of the $M_{N\mu}(\mathbb{Z}_p)$-torsor $\mathcal{I}_p^M$ over $\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)$ admits a canonical reduction of structure group to an $M_{v_m^0}(\mathbb{Z}_p)$-torsor $\mathcal{I}^M_{p,v_m^0}$. Here is how it is obtained: Suppose that we have $s : T \to \widehat{\mathcal{Z}}^{\mathrm{ord}}(m)$, and a section $\eta \in \mathcal{I}_p^M(T)$. For every point $x_0 : \mathrm{Spec}\, k \to T$ valued in an algebraically closed field k, we get an induced isomorphism

$$\eta^{-1}_{\mathrm{cris},x_0} : W(k) \otimes_{\mathscr{O}_{E_v}} \mathrm{gr}\, \boldsymbol{H}_0 \xrightarrow{\simeq} \mathrm{gr}\, \boldsymbol{H}_{\mathrm{cris},s\circ x_0}.$$

This arises from a G-structure and slope filtration preserving isomorphism

$$\tilde{\eta}_{\mathrm{cris},x_0} : W(k) \otimes_{\mathscr{O}_{E_v}} \boldsymbol{H}_0 \xrightarrow{\simeq} \boldsymbol{H}_{\mathrm{cris},s\circ x_0}.$$

This in turn induces an isomorphism

$$W(k)\otimes_{\mathbb{Z}_p} V_p \xrightarrow{\simeq} \boldsymbol{V}_{\mathrm{cris},s\circ x_0}.$$

Now, the slope filtration on $\boldsymbol{H}_{\mathrm{cris},s\circ x_0}$ induces an increasing filtration $S_\bullet \boldsymbol{V}_{\mathrm{cris},s\circ x_0}$, and $\tilde{\eta}_{\mathrm{cris},x_0}$ now induces an isometry

$$W(k)\otimes_{\mathbb{Z}_p} V_p^0 \xrightarrow{\simeq} \mathrm{gr}^S_0 \boldsymbol{V}_{\mathrm{cris},s\circ x_0}, \tag{5.17.1}$$

which depends only on $\eta^{-1}_{\mathrm{cris},x_0}$ and not on the choice of lift $\tilde{\eta}_{\mathrm{cris},x_0}$.

We now define a subsheaf

$$\mathcal{I}^M_{p,v^0_m}(T)\subset \mathcal{I}^M_p(T)$$

consisting of sections η such that, for every point $x_0 : \mathrm{Spec}\, k\to T$ valued in an algebraically closed field k, the induced isomorphism (5.17.1) carries $1\otimes v^0_m \in W(k)\otimes_{\mathbb{Z}_p} V^0_p$ to $\boldsymbol{f}_{\mathrm{cris},s\circ x_0}\in \boldsymbol{V}^0_{\mathrm{cris},s\circ x_0}$. Here, $\boldsymbol{f}_{\mathrm{cris},s\circ x_0}$ is the crystalline realization of f at $s\circ x_0$.

To see that this does give a reduction of structure group, it suffices to check when $T=\mathrm{Spec}\, k$, and here it is immediate from [24, Lemma 2.8].

Set

$$\mathcal{I}^\circ_{p,\lambda_0}(m) = \mathcal{I}^M_p|_{\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)} \times^{M_{v^0_m}(\mathbb{Z}_p)} \mathrm{Par}^\circ_{\lambda_0}(v_m)(\mathbb{F}_p).$$

This is a finite étale scheme over $\mathcal{Z}^{\mathrm{ord}}(m)$.

5.18. Let the notation be as in (5.13). For every $m\in\mathbb{Z}_p\backslash\{0\}$, we obtain two maps:

$$\mathrm{Lie}\, U^-_{\mu_p} \xrightarrow{\varphi\mapsto v^- v^0_\varphi v^0_m} \mathrm{Hom}(H^1_p,H^0_p)\ ;\ \mathrm{Lie}\, U^-_{\mu_p} \xrightarrow{\varphi\mapsto v^0_m v^- v^0_\varphi} \mathrm{Hom}(H^1_p,H^0_p),$$

which induce two morphisms of formal tori

$$s_m, t_m : \widehat{U}_G \to \widehat{U}_0.$$

Set

$$\widehat{U}_{G,v^0_m} = \ker(s_m - t_m : \widehat{U}_G\to\widehat{U}_0).$$

Note that the difference between the corresponding maps of cocharacter groups is just

$$\mathrm{Lie}\, U^-_{\mu_p} \xrightarrow{\varphi\mapsto [v^0_\varphi, v^0_m] v^-} \mathrm{Hom}(H^1_p,H^0_p).$$

From this, one finds that, if we identify the character group of $\widehat{U}_G$ with V^0_p as in (5.13), then we obtain a corresponding identification of the character group of $\widehat{U}_{G,v^0_m}$ with $V^0_p/\langle v^0_m\rangle$. In particular, $\widehat{U}_{G,v^0_m}$ is also a formal torus over $W(k)$.

Lemma 5.19. *Given a point $(x_0,f)\in\mathcal{Z}^{\mathrm{ord}}(m)(k)$ with k algebraically closed, there exists an isomorphism $\widehat{U}_{x_0}\xrightarrow{\simeq}\widehat{U}_G$ as in (4.28) that identifies the completion of $\mathcal{Z}(m)$ at (x_0,f) with $\widehat{U}_{G,v^0_m}$.*

Proof. This is shown just as in (4.28), using Serre-Tate ordinary theory. □

5.20. Let $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}$ be as in (4.18). Fix $m\in\mathbb{Z}^{>0}_{(p)}$. Define a functor $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}(m)$ on p-adically complete formal schemes as follows: For any such formal scheme T, $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}(m)(T)$ consists of tuples (s,t,ξ,f) such that:

- $(s,t,\xi)\in\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}(T)$;
- $f\in V(s)\backslash pV(s)$ is such that $f\circ f = m$;
- $\xi f\xi^{-1}\in p^{-1}V(t)\backslash V(t)$.

Just as in (5.11), it is the G-admissibility of ξ that ensures that $\xi f\xi^{-1}$ is an element of $V(t)[p^{-1}]$, so that the last condition is indeed sensible.

As before, let $\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)$ be the restriction of $\mathcal{Z}(m)$ over $\widehat{\mathcal{S}}_K^{\mathrm{ord}}$. Then we have the 'source' and 'target' morphisms

$$s_{\lambda_0}^{\mathrm{ord}}(m):\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)\xrightarrow{(s,t,\xi,f)\mapsto(s,f)}\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)\;;\;t_{\lambda_0}^{\mathrm{ord}}(m):\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)\xrightarrow{(s,t,\xi,f)\mapsto(t,p(\xi f\xi^{-1}))}\widehat{\mathcal{Z}}^{\mathrm{ord}}(m).$$

Proposition 5.21.

(1) *$\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)$ is representable over $\mathbb{F}_p$ and is of finite type.*

(2) *The map $s_{\lambda_0}^{\mathrm{ord}}(m)$ has a factoring*

$$\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)\xrightarrow{\alpha_{\lambda_0}^{\mathrm{ord}}(m)}\mathcal{I}_{p,\lambda_0}^{\circ}(m)\xrightarrow{\pi_{\lambda_0}^{\circ}}\widehat{\mathcal{Z}}^{\mathrm{ord}}(m),$$

where $\alpha_{\lambda_0}^{\mathrm{ord}}(m)$ is a finite flat homeomorphism.

(3) *The map $t_{\lambda_0}^{\mathrm{ord}}(m)$ is an isomorphism.*

Proof. The map

$$\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)\xrightarrow{(s,t,\xi,f)\mapsto(s,f)\times(s,t,\xi)}\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)\times_{\widehat{\mathcal{S}}_K^{\mathrm{ord}},s_{\lambda_0}^{\mathrm{ord}}}\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}\tag{5.21.1}$$

exhibits $\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)$ as a sub-functor of the right hand side. On the other hand, the latter functor is representable and finite over $\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)$; cf. (4.21). So, to show representability, it is enough to show that $\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)$ is an open sub-functor of the right hand side of (5.21.1).

Suppose that we are given an $\mathbb{F}_p$-scheme T and a map

$$\big((s,f),(s,t,\xi)\big):T\to\mathcal{Z}^{\mathrm{ord}}(m)\times_{\mathcal{S}_{K,\mathbb{F}_p}^{\mathrm{ord}},s_{\lambda_0}^{\mathrm{ord}}}\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}.$$

Now, $\xi f\xi^{-1}$ belongs to $p^{-n}V(t)$, for some $n\geq 1$. Therefore $f'=p^n(\xi f\xi^{-1})$ belongs to $V(t)$. The locus in T where φ factors through $\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}$ is now exactly the locus where f' does not factor through the multiplication-by-p^n map. This is also the locus over which the induced endomorphism $f'[p^n]$ of the p^n-torsion $\mathcal{A}_t[p^n]$ is non-zero, which shows that it must be open, and completes the proof of (1).

We now move on to (2). By (4.21), we have a factoring:

$$s_{\lambda_0}^{\mathrm{ord}}:\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)\xrightarrow{(\alpha_{\lambda_0}^{\mathrm{ord}},(s,f))}\mathcal{I}_{p,\lambda_0}^{M}\times_{\widehat{\mathcal{S}}_K^{\mathrm{ord}}}\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)\to\widehat{\mathcal{Z}}^{\mathrm{ord}}(m).$$

Here, α_{λ_0} is the map described in (4.21). There is was constructed by descent from the base-change over $\mathcal{I}_p^M$, but, since we are working over $\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)$, we can use the reduction of structure group $\mathcal{I}_{p,v_m^0}^M$ instead. Therefore, exactly as in the proof of (5.12), it follows that the first map in the above composition factors through a map

$$\alpha_{\lambda_0}^{\mathrm{ord}}(m):\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)\to\mathcal{I}_{p,\lambda_0}^{\circ}(m)$$

that is a finite homeomorphism onto its image.

Similarly, in the factorization

$$t_{\lambda_0}^{\mathrm{ord}}:\widehat{\mathrm{Isog}}_{\lambda_0}^{\mathrm{ord}}(m)\xrightarrow{(\beta_{\lambda_0}^{\mathrm{ord}},(t,p^{-1}(\xi f\xi^{-1})))}\mathcal{I}_{p,-\lambda_0}^{M}\times_{\widehat{\mathcal{S}}_K^{\mathrm{ord}}}\widehat{\mathcal{Z}}^{\mathrm{ord}}(p^2m)\to\widehat{\mathcal{Z}}^{\mathrm{ord}}(p^2m),$$

the first map factors through the section

$$\widehat{\mathcal{Z}}^{\mathrm{ord}}(p^2m)\to\mathcal{I}_{p,-\lambda_0}^{M}\times_{\widehat{\mathcal{S}}_K^{\mathrm{ord}}}\widehat{\mathcal{Z}}^{\mathrm{ord}}(p^2m),$$

obtained from the $M_{v_m^0}(\mathbb{F}_p)$-equivariant inclusion

$$\{\langle v_m\rangle\}\subset\mathrm{Par}_{\lambda_0}(\mathbb{F}_p)$$

via a map

$$\beta^{\mathrm{ord}}_{\lambda_0}(m) : \widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0} \to \widehat{\mathcal{Z}}^{\mathrm{ord}}(p^2m)$$

that is again a finite homeomorphism onto its image.

To finish, it suffices to check that $\alpha^{\mathrm{ord}}_{\lambda_0}$ is flat and surjective, and that $\beta^{\mathrm{ord}}_{\lambda_0}$ is an isomorphism. Both these facts can be checked on the level of the complete local rings at any geometric point of $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}(m)$.

This can be done using information from (4.28), (5.13) and (5.19). Indeed, these together imply the following: Suppose that (s_0, t_0, ξ_0, f_0) is a point of $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}(m)$ valued in an algebraically closed field k. Then there exists a decomposition

$$V^0_p = V^0_p(-1) \oplus V^0_p(0) \oplus V^0_p(1)$$

with $V^0_p(\pm 1)$ complementary isotropic lines, and $V^0_p(0)$ their mutual orthogonal complement with the following properties:

- In terms of this decomposition, the component $v^0_m(1) \in V^0_p(1)$ of v^0_m, we have

(5.21.2) $$v^0_m(1) \not\equiv 0 \pmod p.$$

- Consider the endomorphisms $\pi_1 = (p, 1, 1)$ and $\pi_2 = (1, 1, p)$ of V^0_p: We then have

$$p \cdot \pi_1(v^0_m) = \pi_2(v^0_{p^2m}).$$

- If $\widehat{U}_G$ is the formal torus over $W(k)$ with character group V^0_p, the completions of $\mathcal{S}_K$ at s_0 and t_0 can be identified with $\widehat{U}_G$, so that the following hold:
  - The completions of $\mathcal{Z}(m)$ and $\mathcal{Z}(p^2m)$ at (s_0, f_0) and $(t_0, p(\xi_0 f_0 \xi_0^{-1}))$, respectively, can be identified with the formal tori with character groups $V^0_p/\langle v^0_m\rangle$ and $V^0_p/\langle v^0_{p^2m}\rangle$, respectively.
  - The completion of $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}$ at (s_0, t_0, ξ_0) can be identified with the formal sub-torus of $\widehat{U}_G \times \widehat{U}_G$ corresponding to the quotient

    $$V^0_p \oplus V^0_p \xrightarrow{\pi_1 \oplus \pi_2} V^0_p$$

    on the level of character groups.

Putting this all together, we find that the completion of $\widehat{\mathrm{Isog}}^{\mathrm{ord}}_{\lambda_0}(m)$ at (s_0, t_0, ξ_0, f_0) is given by the formal torus over $W(k)$ with character group $V^0_p/\langle \pi_1(v^0_m)\rangle$, and that the maps to the completions of $\widehat{\mathcal{Z}}^{\mathrm{ord}}(m)$ at (s_0, f_0) and of $\widehat{\mathcal{Z}}^{\mathrm{ord}}(p^2m)$ at $(t_0, p(\xi_0 f_0 \xi_0^{-1}))$ correspond to isogenies of formal tori, given on the level of character groups by:

$$V^0_p/\langle v^0_m\rangle \xrightarrow{\pi_1} V^0_p/\langle \pi_1(v^0_m)\rangle \ ; \ V^0_p/\langle v^0_{p^2m}\rangle \xrightarrow{\pi_2} V^0_p/\langle \pi_1(v^0_m)\rangle,$$

respectively. It is now immediate from condition (5.21.2) and the definitions of π_1 and π_2 that the first map of character groups maps onto a sub-group of index p and that the second is an isomorphism.

Combining everything now, we find that $\alpha^{\mathrm{ord}}_{\lambda_0}(m)$ is given by a finite flat morphism of degree p, and that $\beta^{\mathrm{ord}}_{\lambda_0}(m)$ is an isomorphism, thus completing the proof of the proposition. □

5.22. Let $\widehat{\mathcal{Z}}^{\mathrm{ord,an}}(m)$ be the associated rigid analytic space over $\mathbb{Q}_p$. From (5.12), we obtain a map

$$s_{\mu_p}(m) \circ t_{\mu_p}(m)^{-1} : \widehat{\mathcal{Z}}^{\mathrm{ord,an}}(p^2m) \to \widehat{\mathcal{Z}}^{\mathrm{ord,an}}(m).$$

On the other hand, (5.21) gives us another map

$$s^{\mathrm{ord}}_{\lambda_0}(m) \circ t^{\mathrm{ord}}_{\lambda_0}(m)^{-1} : \widehat{\mathcal{Z}}^{\mathrm{ord,an}}(p^2m) \to \widehat{\mathcal{Z}}^{\mathrm{ord,an}}(m)$$

Corollary 5.23. *These two maps agree.*

Proof. Suppose that $T = \operatorname{Spf} R$, where R is a normal p-adically complete local ring, and that we have a map of formal schemes $f : T \to \widehat{Z}^{\mathrm{ord}}(p^2 m)$. It is enough to show that, for ever such map, as R varies, we have

$$s_{\mu_p}(m) \circ t_{\mu_p}(m)^{-1} \circ f^{\mathrm{an}} = s^{\mathrm{ord}}_{\lambda_0}(m) \circ t^{\mathrm{ord}}_{\lambda_0}(m)^{-1} \circ f^{\mathrm{an}}.$$

This can be deduced from the following assertion: If $(x, y, \xi, f) \in \mathrm{Isog}_{\mu_p}(m)(R[p^{-1}])$, and if x, y extend to maps

$$\tilde{x}, \tilde{y} : T = \operatorname{Spf} R \to \widehat{\mathcal{S}}^{\mathrm{ord}}_K,$$

Then ξ extends to a G-admissible isogeny over T of type λ_0. Indeed (4.26) shows that ξ extends to a G-admissible isogeny

$$\tilde{\xi} : \mathcal{A}_{\tilde{x}} \to \mathcal{A}_{\tilde{y}}$$

of type either μ_p or λ_0.

Moreover, f also extends to an element $\tilde{f} \in V(\tilde{x})$ such that

$$\tilde{\xi}\tilde{f}\tilde{\xi}^{-1} \in p^{-1}V(\tilde{y})\backslash V(\tilde{y}).$$

However, if $\tilde{\xi}$ were of type μ_p, then, as observed in (4.15), it must be isomorphic to the relative Frobenius map at every point. In fact, since it is level structure preserving, it has to *be* the relative Frobenius at every point: the set $M_{\mu_p}(\mathbb{Z}_p)\mu_p(p)M_{\mu_p}(\mathbb{Z}_p)/M_{\mu_p}(\mathbb{Z}_p)$ from (4.17) is a singleton.

But this is not possible: If x_0 is a μ-ordinary point in $\mathcal{S}_{K,\mathbb{F}_p}$, $f \in V(x_0)$, and $F : \mathcal{A}_{x_0} \to \mathcal{A}^{(p)}_{x_0} = \mathcal{A}_{x_0^{(p)}}$ is the relative Frobenius, then we would have:

$$FfF^{-1} = f^{(p)} \in V(x_0^{(p)}) \subset \operatorname{End}\bigl(\mathcal{A}_{x_0^{(p)}}\bigr).$$

□

5.24. Write

$$\pi : s^{\mathrm{ord}}_{\lambda_0}(m) \circ t^{\mathrm{ord}}_{\lambda_0}(m)^{-1} : \widehat{\mathcal{Z}}^{\mathrm{ord}}(p^2 m) \to \widehat{\mathcal{Z}}^{\mathrm{ord}}(m)$$

For any $m \in \mathbb{Z}^{>0}_{(p)}$, we obtain a diagram:

(5.24.1)
$$\begin{tikzcd}
\pi_0\bigl(\mathcal{Z}^{\mathrm{pr}}(p^2 m)_{\overline{\mathbb{F}}_p}\bigr) \arrow[r, "\simeq"] & \pi_0\bigl(\mathcal{Z}^{\mathrm{ord}}(p^2 m)_{\overline{\mathbb{F}}_p}\bigr) \arrow[r] \arrow[d, "\pi_*"] & \pi_0\bigl(Z(p^2 m)_{\overline{\mathbb{Q}}}\bigr); \arrow[d, "\pi_*", "\simeq"'] \\
\pi_0\bigl(\mathcal{Z}^{\mathrm{pr}}(m)_{\overline{\mathbb{F}}_p}\bigr) \arrow[r, "\simeq"'] & \pi_0\bigl(\mathcal{Z}^{\mathrm{ord}}(m)_{\overline{\mathbb{F}}_p}\bigr) \arrow[r] & \pi_0\bigl(Z(m)_{\overline{\mathbb{Q}}}\bigr).
\end{tikzcd}$$

The horizontal maps are the natural ones. The maps on the left are bijections because the ordinary locus is dense in the special fiber of $\mathcal{Z}^{\mathrm{pr}}(m)$. The vertical map on the right hand side is a bijection by (5.12) and (5.23).

Let $P(m)$ be the assertion that the lower right horizontal arrow is bijective in the diagram above, and let $Q(m)$ be the assertion that the vertical arrow in the middle is bijective.

Proposition 5.25. *$P(m)$ is true whenever $p^2 \nmid m$.*

Proof. As observed in (5.4), under this hypothesis, we have $\mathcal{Z}^{\mathrm{pr}}(m) = \mathcal{Z}(m)$.

From (5.5), we find that $\mathcal{Z}(m)$ is a union of the normalizations of $\mathcal{S}_K$ in smaller Shimura subvarieties. Therefore [26, Corollary 4.1.11] shows that every connected component of its special fiber is the specialization of a unique connected component of its generic fiber. More precisely, the cited result shows that the proposition follows as long as we know that the special fiber of $\mathcal{Z}(m)$ is geometrically reduced; but this is fine, since $\mathcal{Z}(m)_{\mathbb{F}_p}$ is generically smooth. □

Proposition 5.26. *$P(m)$ implies $Q(m)$.*

Proof. By (5.21) and (5.23), $Q(m)$ is equivalent to: The cover $\mathcal{I}^{\circ}_{p,\lambda_0}(m) \to \mathcal{Z}^{\mathrm{ord}}(m)$ is relatively geometrically irreducible. For this, let $\mathcal{S}^{\mathrm{pr}}_{K_{m,\gamma,g}} \subset \mathcal{Z}^{\mathrm{pr}}(m)$ be the open and closed subscheme introduced in (5.5), associated with a pair $(\gamma, g) \in G(\mathbb{Z}_{(p)}) \times G(\mathbb{A}_f)$. Write $\mathcal{S}^{\mathrm{ord}}_{K_{m,\gamma,g}}$ for the intersection of this subscheme with $\mathcal{Z}^{\mathrm{ord}}(m)$. The condition $P(m)$ ensures that this is a union of connected components of $\mathcal{Z}^{\mathrm{ord}}(m)$.

Since $\mathcal{Z}^{\mathrm{pr}}(m)$ is a union of subschemes of the form $\mathcal{S}^{\mathrm{pr}}_{K_{m,\gamma,g}}$, to finish it is enough to show that the restriction

$$\mathcal{I}^{\circ}_{p,\lambda_0}(m)|_{\mathcal{S}^{\mathrm{ord}}_{K_{m,\gamma,g}}} \to \mathcal{S}^{\mathrm{ord}}_{K_{m,\gamma,g}}$$

is relatively geometrically irreducible.

Let $K_{m,\gamma,g,p} \subset G_{v_m}(\mathbb{Q}_p)$ be the p-primary part of $K_{m,\gamma,g}$, and consider the inverse limit

$$\mathrm{Sh}^p = \varprojlim_{K^{',p}} \mathrm{Sh}_{K_{m,\gamma,g,p}K^{',p}},$$

where $K^{',p}$ ranges over compact open subgroups of $G_{v_m}(\mathbb{A}_f^p)$. Then Sh^p is a pro-étale cover of $\mathrm{Sh}_{K_{m,\gamma,g}}$, equipped with an action of $G_{v_m}(\mathbb{A}_f^p)$. Moreover, it extends to a pro-étale cover $\mathcal{S}^p \to \mathcal{S}^{\mathrm{pr}}_{K_{m,\gamma,g}}$ once again equipped with an action of $G_{v_m}(\mathbb{A}_f^p)$. Let $\boldsymbol{f}$ be the tautological special endomorphism of the restriction of $\mathcal{A}$ over $\mathcal{Z}^{\mathrm{pr}}(m)$, satisfying $\boldsymbol{f} \circ \boldsymbol{f} = m \cdot \mathrm{id}$.

Over $\mathcal{S}^p$, we have a canonical G-structure preserving isomorphism

$$\xi_{\mathbb{A}_f^p} : \underline{H}_{\mathbb{A}_f^p} \xrightarrow{\simeq} \boldsymbol{H}_{\mathbb{A}_f^p}|_{\mathcal{S}^p}$$

carrying v_m to the prime-to-p adélic realization of the tautological endomorphism $\boldsymbol{f}$; see [24, Prop. 6.7].

Using (2.33), one can find a point $x_0 \in \mathcal{S}^{\mathrm{pr}}_{K_{m,\gamma,g}}(k)$ that is hypersymmetric for $\mathcal{S}_K$. This means that the natural map

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} I_{x_0} \to J_{x_0} \tag{5.26.1}$$

given by the action of I_{x_0} on $\mathcal{A}_{x_0}[p^\infty]$ is an isomorphism.

Let x be the canonical lift of x_0, and fix a G-structure preserving isomorphism

$$\tau : W(k) \otimes_{\mathscr{O}_{E_v}} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{A}_x[p^\infty]$$

Note that $\mathcal{A}_{x_0}$ carries a special endomorphism $\boldsymbol{f}_{x_0}$. By modifying τ by an element of $M_\nu(\mathbb{Z}_p)$ if necessary, we can ensure that its inverse crystalline realization $\boldsymbol{\tau}^{-1}_{\mathrm{cris}}$ carries v_m^0 to the crystalline realization $\boldsymbol{f}_{\mathrm{cris},x_0}$ of $\boldsymbol{f}_{x_0}$. That is, we can ensure that τ corresponds to a lift $y_0 \in \mathcal{I}^M_{p,v_m^0}(k)$ of x_0.

Let $I_{x_0}(\boldsymbol{f}) \subset I_{x_0}$ be the commutant of $\boldsymbol{f}$. Then the isomorphism of (5.26.1), combined with conjugation by τ^{-1} induces an isomorphism

$$\varphi : \mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} I_{x_0}(\boldsymbol{f}) \xrightarrow{\simeq} M_{v_m^0}.$$

We now apply (1.5). In the notation there, we take S to be the restriction of $\mathcal{S}^p$ over $\mathcal{S}^{\mathrm{ord}}_{K_{m,\gamma,g},\overline{\mathbb{F}}_p}$, $H_{\mathbb{Z}_p} = M_{v_m^0}$, $\mathcal{P} \to S$ to be the $M_{v_m^0}$-torsor obtained from $\mathcal{I}^M_p(v_m^0)$, $Q = Q_{v_m^0}$, $T = \{p\}$, $G = G_{v_m}$ and $M_{\mathbb{Z}_{(p)}} = I_{x_0}(\boldsymbol{f})$.

The embedding $\psi : I_{x_0}(\boldsymbol{f})_{\mathbb{A}_f^p} \hookrightarrow G_{v_m,\mathbb{A}_f^p}$ is obtained as follows: By (2.30), given an $\mathbb{A}_f^p$-algebra R, an element of $I_{x_0}(\boldsymbol{f})(R)$ is given by an element $\eta \in \big(R \otimes_{\mathbb{Z}_{(p)}} \mathrm{End}(\mathcal{A}_{x_0})\big)^\times$, whose adélic realization $\boldsymbol{\eta}_{\mathbb{A}_f^p}$ preserves the tensors $\{\boldsymbol{s}_{\alpha,\mathbb{A}_f^p,x_0}\}$. The map

$$I_{x_0}(\boldsymbol{f})_{\mathbb{A}_f^p} \xrightarrow{\eta \mapsto \xi^{-1}_{\mathbb{A}_f^p,x_0} \circ \boldsymbol{\eta}_{\mathbb{A}_f^p} \circ \xi_{\mathbb{A}_f^p,x_0}} G_{\mathbb{A}_f^p}$$

is now the desired embedding.

Note that, with this notation, and using the last assertion of (5.16), the étale cover $\mathcal{P}_{\mathsf{X},1} \to S$ considered in (1.5) is precisely the restriction over S of $\mathcal{I}^{\circ}_{p,\lambda_0}(m)$. Therefore, to finish the proof of the proposition, we need to verify that the hypotheses of (1.5) are valid.

Hypothesis (1) can be deduced from the assumption that $P(m)$ holds, and the description of the connected components of Sh^p as in the proof of (5.7). Hypothesis (2) holds since any quadratic space over a local field of dimension greater than 4 is isotropic.

Hypothesis (3) follows from the validity of $P(m)$ and the fact that $G_{v_m}(\mathbb{A}_f^p)$ acts transitively on the connected components of $\mathrm{Sh}^p_{\overline{\mathbb{Q}}}$.

Hypothesis (5) is clear, and hypothesis (6) was shown in (5.16). It remains to check hypothesis (4), which would follow from knowing that the diagonally embedded subgroup

$$\Phi = (\varphi, \psi) : I_{x_0}(\boldsymbol{f})(\mathbb{Z}_{(p)}) \hookrightarrow M_{v_m^0}(\mathbb{Z}_p) \times G_{v_m}(\mathbb{A}_f^p)$$

fixes $y_0 \in \mathcal{I}^M_{p,v_m^0}(k)$. For any $\eta \in I_{x_0}(\boldsymbol{f})(\mathbb{Z}_{(p)})$, $\Phi(\eta)(y_0)$ corresponds to the point with underlying abelian variety $\mathcal{A}_{x_0}$, but with the prime-to-p level structure $\xi_{\mathbb{A}_f^p,x_0}$ replaced by

$$\xi_{\mathbb{A}_f^p,x_0} \circ \psi(\eta) = \xi_{\mathbb{A}_f^p,x_0} \circ \xi^{-1}_{\mathbb{A}_f^p,x_0} \circ \boldsymbol{\eta}_{\mathbb{A}_f^p} \circ \xi_{\mathbb{A}_f^p,x_0} = \boldsymbol{\eta}_{\mathbb{A}_f^p} \circ \xi_{\mathbb{A}_f^p,x_0};$$

and the trivialization τ replaced with

$$\tau \circ \varphi(\eta) = \tau \circ \tau^{-1} \circ \eta[p^\infty] \circ \tau = \eta[p^\infty] \circ \tau.$$

Therefore, the isomorphism $\eta : \mathcal{A}_{x_0} \xrightarrow{\simeq} \mathcal{A}_{\Phi(\eta)(x_0)} = \mathcal{A}_{x_0}$ is G-admissible, level structure preserving, and also carries the trivialization τ to the trivialization $\Phi(\eta)(\tau)$. It also fixes the special endomorphism $\boldsymbol{f}_{x_0}$. Combining this with (4.8) and the canonical lift shows that $\Phi(\eta)(y_0) = y_0$, and completes the proof of the proposition. □

Proposition 5.27. *For any $m \in \mathbb{Z}^{>0}_{(p)}$, there is a canonical bijection:*

$$\pi_0\big(\mathcal{Z}^{\mathrm{pr}}(m)_{\overline{\mathbb{F}}_p}\big) \xrightarrow{\simeq} \pi_0\big(\mathcal{Z}^{\mathrm{pr}}(m)_{\overline{\mathbb{Q}}}\big).$$

Proof. Using induction on $\mathrm{ord}_p(m)$, this follows from (5.25), (5.26) and the following assertion, which is clear from (5.24.1): $P(m)$ and $Q(m)$ together imply $P(p^2m)$. □

Proof of Theorem 1. Let $\mathsf{M}^{\circ}_{2d,\mathbb{Z}_{(p)}}$ be the moduli stack of primitively polarized K3 surfaces over $\mathbb{Z}_{(p)}$ of degree $2d$ (see [25, §3]).

Let N be the self-dual quadratic $\mathbb{Z}$-lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$, where U is the hyperbolic plane. Choose a hyperbolic basis e, f for the first copy of U. Set

$$L_d = \langle e - df \rangle^{\perp} \subset N.$$

This is a quadratic space of signature $(19, 2)$. We can choose our quadratic space V, and self-dual $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}}$ such that V has signature $(20, 2)$ and such that there exists an isometric embedding as a direct summand

$$L_{d,\mathbb{Z}_{(p)}} \hookrightarrow V_{\mathbb{Z}_{(p)}}.$$

Associated with the lattice $V_{\mathbb{Z}_{(p)}}$ and a suitable neat level subgroup $K^p \subset \mathrm{GSpin}(V)(\mathbb{A}_f^p)$, we have the integral model $\mathcal{S}_K$ over $\mathbb{Z}_{(p)}$.

Let $\mathsf{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ be the open smooth locus of $\mathsf{M}_{2d,\mathbb{Z}_{(p)}}$: This is a fiber-by-fiber dense subspace. In particular, it suffices to show that $\mathsf{M}^{\mathrm{sm}}_{2d,\overline{\mathbb{F}}_p}$ is irreducible.

By the theory of [25, §5], extended to the case $p = 2$ in [16, Prop. A. 12] (see also the erratum at [23]), there is a finite étale cover $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,K}$ of $\mathsf{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$, and an étale period map[19]

$$\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,K} \to \mathcal{Z}^{\mathrm{pr}}(2d)$$

[19]See erratum (**??**) in the appendix below

that is in fact an open immersion, since it is one in the generic fiber; see [25, Cor. 5.15].

Combined with (5.27), this shows that every irreducible component of $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,K,\overline{\mathbb{F}}_p}$ is the specialization of a unique irreducible component of $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,K,\overline{\mathbb{Q}}}$. From this, we deduce the same assertion for the fibers of $\mathsf{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$. However, it is well-known that the moduli stack is irreducible over $\mathbb{C}$. For instance, this follows from the Torelli theorem; see the proof of [25, Prop. 5.3]. □

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