

CONNECTED COMPONENTS OF SPECIAL CYCLES ON SHIMURA VARIETIES

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ABSTRACT. I use methods of Chai-Hida and ordinary p -Hecke correspondences to study the set of irreducible components of special fibers of special cycles of sufficiently low codimension in integral models of GSpin Shimura varieties, and apply this to prove irreducibility results for the special fibers of the moduli of polarized K3 surfaces. These results are also applied in joint work with Howard on the modularity of generating series of higher codimension cycles on GSpin Shimura varieties.

INTRODUCTION

The goal of this paper is to use p -Hecke correspondences—following in spirit a paper of de Jong [de 93]—to obtain control over irreducible components of the special fibers of certain cycles on GSpin Shimura varieties. More precisely, I show that, under some numerical constraints, each such component is the specialization of a unique component in the generic fiber. This plays a key role in the proof of the main result in [HM22] on the modularity of generating series of special cycles of higher codimension.

As a more immediate application, I study the moduli of polarized K3 surfaces. Fix a positive integer $d \geq 1$, and consider the moduli stack \mathcal{M}_{2d}° over \mathbb{Z} that parameterizes primitively polarized K3 surfaces of degree $2d$. I prove:

Theorem A. *For every prime p , the fiber $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$ is geometrically irreducible.*

When p is odd and $p^2 \nmid d$, this result was shown in [Mad15]. In effect, what’s happening in this paper is a reduction to this known case, by relating the ordinary locus of $\mathcal{M}_{2d/p^2}^\circ$ with that of \mathcal{M}_{2d}° using p -Hecke correspondences.

To state the somewhat technical main result I prove in this paper, we begin with a GSpin Shimura variety Sh_K associated with a quadratic lattice $V_{\mathbb{Z}}$ of signature $(n, 2)$. On this variety, we have an infinite collection of special cycles $Z_K(\Lambda) \rightarrow \mathrm{Sh}_K$ associated with positive definite lattices Λ , which in the complex analytic fiber parameterize Noether-Lefschetz type loci where the canonical family of polarized \mathbb{Z} -Hodge structures \mathbf{V} associated with $V_{\mathbb{Z}}$ pick up a collection of Hodge tensors generating a subspace isometric to Λ . As explained in the series of papers [Mad16; AGHM17; AGHM18; HM20; HM22], the Shimura variety admits a natural integral model \mathcal{S}_K over \mathbb{Z} , along with a cycle $\mathcal{Z}_K(\Lambda)$ extending $Z_K(\Lambda)$. Within $\mathcal{Z}_K(\Lambda)$, we have a certain open ‘non-degenerate’ locus ${}^\circ\mathcal{Z}_K(\Lambda)$, whose complex fiber is the subspace on which Λ spans a saturated submodule of the \mathbb{Z} -Hodge structure \mathbf{V} . Special cases of this construction include classical Noether-Lefschetz loci on the moduli of polarized K3 surfaces, where ${}^\circ\mathcal{Z}_K(\Lambda)$ parameterizes isometric embeddings of Λ as a direct summand in the primitive Picard lattice.

In any case, we are now ready for the main result:

Theorem B. *Let p be a prime where $V_{\mathbb{Z}}$ is self-dual, and suppose that $\mathrm{rank}(\Lambda) \leq \frac{n-4}{2}$. Then:*

- (1) ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}}$ is normal, flat, and equidimensional of dimension $n - \mathrm{rank}(\Lambda) + 1 = \dim \mathcal{S}_K - \mathrm{rank}(\Lambda)$.
- (2) The special fiber ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$ is geometrically normal and equidimensional of dimension $n - \mathrm{rank}(\Lambda)$.
- (3) The natural map

$$\pi_0({}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}) \xrightarrow{\cong} \pi_0({}^\circ\mathcal{Z}_K(\Lambda)_{\overline{\mathbb{Q}}}).$$

is a bijection.

The first two assertions are already explained in [HM20; HM22]. It is the last statement that is the focus of this paper.

The Hodge theoretic analogue of the essential idea of proof—which is by induction on the p -adic valuation of the discriminant of Λ —is easy to explain. Let’s do so in the context of Theorem A. Suppose that (X, ξ) is a primitively polarized K3 surface over \mathbb{C} of degree $2d$. Then the Betti cohomology $H^2(X, \mathbb{Z})$ is a pure Hodge structure of weight 2, and the Poincaré pairing endows it with the structure of a quadratic space over \mathbb{Z} that is isometric to

$U = \mathbf{H}^{\oplus 3} \oplus E_8^{\oplus 2}$. Here, \mathbf{H} is the hyperbolic plane, and E_8 is the root lattice corresponding to its eponymous Dynkin type.

We will distinguish one hyperbolic plane $\mathbf{H} \subset U$, and choose a hyperbolic basis e, f for it satisfying $e^2 = f^2 = 0$, $[e, f] = 1$. Let $U' = \mathbf{H}^\perp \subset U$ be its orthogonal complement, so that we can write U as the orthogonal direct sum

$$U = \mathbf{H} \perp U'.$$

We can choose the isometry

$$H^2(X, \mathbb{Z}) \xrightarrow{\sim} U$$

so that the Chern class of ξ maps to the element $e + df \in \mathbf{H}$. Within $H^2(X, \mathbb{Q})$ we have a lattice corresponding on the right hand side to the subspace

$$\langle p^{-1}e, pf \rangle \perp U' \subset U_{\mathbb{Q}}.$$

The basic point is that this lattice corresponds to an ‘isogenous’ K3 surface X' equipped with an isometry

$$H^2(X', \mathbb{Z}) \xrightarrow{\sim} \langle pe, p^{-1}f \rangle \perp U',$$

and that it admits a canonical primitive polarization ξ' of degree $2p^2d$, whose Chern class maps under the above isometry to the element $pe + p^2 \cdot (p^{-1}f)$.

If one does this carefully in families, one finds essentially a Hecke correspondence mapping the moduli of primitively polarized K3 surfaces of degree $2p^2d$ to that of degree $2d$, and this allows the direct comparison between their connected components.

Inspired by [de 93], I show that a similar argument can be made in the special fiber if we work over the ordinary locus, and this gives us the proof of Theorem B. One can then easily deduce Theorem A as a special case where $\text{rank}(\Lambda) = 1$ using the results of [Mad15]. Here’s the line of reasoning for the former theorem:

- The stack ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}}$ is a disjoint union of open subspaces ${}^\circ\mathcal{S}_{K_g^b, (p)}$ of integral models $\mathcal{S}_{K_g^b, (p)}$ of GSpin Shimura varieties associated with certain direct summands $V_{g, \mathbb{Z}}^b \subset V_{\mathbb{Z}}$. When Λ is *maximal* at p , one can delete the qualifier ‘of open subspaces’, and the compactifications from [Mad19] now yield (3) of Theorem B.
- When Λ is contained in a positive definite lattice $\tilde{\Lambda}$, with $[\tilde{\Lambda} : \Lambda] = p$, by restricting to the ordinary loci in the mod- p fiber, I use ordinary Hecke correspondences, combined with the Hodge-theoretic idea explained above, to show that there is a map

$$\pi : {}^\circ\mathcal{Z}_K^{\text{ord}}(\Lambda)_{\mathbb{F}_p} \rightarrow {}^\circ\mathcal{Z}_K^{\text{ord}}(\tilde{\Lambda})_{\mathbb{F}_p}$$

- I then use Serre-Tate deformation theory, a key argument of Chai-Hida for the irreducibility of monodromy of the ordinary Igusa tower, and the inductive hypothesis to show that π induces a bijection on geometric connected components.
- A comparison with the analogous construction in the generic fiber using classical Hecke correspondences now finishes the job.

Here is the organization of this paper:

- The first two sections, which deal with ordinary loci in integral models of Shimura varieties and ordinary Hecke correspondences, are covering ‘well-known’ ground, and are included for a lack of convenient reference that works in the generality that I need.
- Section 3 reviews the use of a slick argument of Chai and Hida for the control of the monodromy of Igusa towers, and applies it to the ordinary case we care about.
- In the first three sections, I work in the generality of Shimura varieties of Hodge type at primes with non-empty ordinary loci. In Section 5, I specialize to the case of GSpin Shimura varieties and—after some prerequisites on quadratic lattices and the group schemes associated with them covered in Section 4—apply the results from the three initial sections to prove the main results.

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helpful comments and apologize to them for only now getting around to implementing some of those suggestions. I also thank Pol van Hoften, Ayan Nath, and the referee of this article for helpful comments and corrections.

CONVENTIONS

- (1) We will fix a prime number p for the entirety of this paper.
- (2) Given a set X and any Grothendieck site, we will write \underline{X} for the locally constant sheaf over the site associated with the constant presheaf sending every object to X .
- (3) Given a topos, a pro-finite group H presented as an inverse limit of finite quotients H_n , and an object S in the topos, an \underline{H} -torsor $\pi : \mathcal{P} \rightarrow S$ is an inverse system

$$\{\pi_n : \mathcal{P}_n \rightarrow S\}_{n \in \mathbb{Z}_{\geq 1}},$$

where π_n is a torsor under H_n . In particular, this applies to the situation where D is a smooth group scheme over \mathbb{Z}_p with $H = D(\mathbb{Z}_p)$ and $H_n = D(\mathbb{Z}/p^n\mathbb{Z})$.

- (4) For any finite set of primes T , we will set

$$\mathbb{A}_f^T = \prod_{\ell \notin T} \mathbb{Q}_\ell,$$

the restricted product over all completions of \mathbb{Q} at finite places not in T . If $T = \{\ell\}$ is a singleton, we will write \mathbb{A}_f^ℓ instead.

- (5) For any local or global field F in characteristic 0, we will write \mathcal{O}_F for its ring of integers.
- (6) Suppose that X is a scheme or stack over a localization \mathcal{O} of \mathcal{O}_F in which p is not invertible. For any place $v|p \nmid N$ of F , we will write $X_{(v)}$ (resp. X_v) for the base-change of X over the localization $\mathcal{O}_{F,(v)}$ (resp. the completion $\mathcal{O}_{F,v}$) at v . For $M \in \mathcal{O}$, we will write $X[M^{-1}]$ for the base-change over $\mathcal{O}[M^{-1}]$.
- (7) Suppose that R is a ring and suppose that \mathbf{C} is an R -linear tensor category that is a faithful tensor subcategory of Mod_R , the category of R -modules. Suppose in addition that \mathbf{C} is closed under taking duals, symmetric and exterior powers in Mod_R . Then, for any object $D \in \text{Obj}(\mathbf{C})$, we will denote by D^\otimes the direct sum of the tensor, symmetric and exterior powers of D and its dual.
- (8) All formal schemes over \mathbb{Z}_p will be p -adic: That is, they are functors on p -nilpotent rings.
- (9) Given a connected reductive group G over a field F , we write $G^{\text{der}} \subset G$ for its derived subgroup, $\rho_G : G^{\text{sc}} \rightarrow G$ for the simply connected cover of its derived group and G^{ad} for its adjoint quotient.
- (10) Fix an algebraic closure \bar{F} for F . For any torus T over F , we set

$$X_*(T) = \text{Hom}(\mathbb{G}_{m,\bar{F}}, T_{\bar{F}}) ; X^*(T) = \text{Hom}(T_{\bar{F}}, \mathbb{G}_{m,\bar{F}})$$

for the cocharacter and character groups of T , respectively.

- (11) For a perfect field κ , we will write $W(\kappa)$ for its ring of Witt vectors and denote by $\sigma : W(\kappa) \rightarrow W(\kappa)$ the canonical lift of Frobenius.
- (12) Given a product $X_1 \times \cdots \times X_k$ in a category with finite products, we will write pr_i for the projection onto the i -th factor.

1. ORDINARY LOCI OF SHIMURA VARIETIES

We will fix a Shimura datum (G, X) . The purpose of this section is to recall and situate some results of Noot [Noo96] regarding the structure of the ordinary loci of integral models of Shimura varieties associated with this datum

1.1. Background on Shimura varieties of Hodge type.

1.1.1. Given $x \in X$, we have the associated homomorphism of \mathbb{R} -groups:

$$h_x : \mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}} \rightarrow G_{\mathbb{R}}.$$

We also have the associated (minuscule) cocharacter:

$$\mu_x : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\cong} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} G_{\mathbb{C}}.$$

The $G(\mathbb{R})$ -conjugacy class of h_x , and hence the $G(\mathbb{C})$ -conjugacy class $\{\mu_X\}_\infty$ of μ_x , is independent of the choice of x . Let $E \subset \mathbb{C}$ be the reflex field for (G, X) : This is the field of definition of $\{\mu_X\}_\infty$, and is a finite extension of \mathbb{Q} . This gives us a geometric conjugacy class $[\mu] \in (\mathrm{Hom}(\mathbb{G}_m, G)/G)(E)$. Given a place $v|p$ of E we obtain in turn a geometric conjugacy class $[\mu_v]$ defined over E_v

Construction 1.1.2. We will assume for the rest of the section that the Shimura datum is of Hodge type, so that it is equipped with an embedding

$$(G, X) \hookrightarrow (\mathrm{GSp}(H), S^\pm(H))$$

into a Siegel Shimura datum.¹ We fix a \mathbb{Z} -lattice $H_\mathbb{Z} \subset H$ on which the symplectic form is \mathbb{Z} -valued, and we take $K_p \subset G(\mathbb{Q}_p)$ to be the stabilizer of $H_p = \mathbb{Z}_p \otimes_\mathbb{Z} H_\mathbb{Z}$. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup of the form $K_p K^p$ for $K^p \subset G(\mathbb{A}_f^p)$ stabilizing the prime-to- p lattice $\widehat{\mathbb{Z}}^p \otimes_\mathbb{Z} H_\mathbb{Z}$, and let $K^\sharp \subset \mathrm{GSp}(H)(\mathbb{A}_f)$ be the stabilizer of $\widehat{\mathbb{Z}} \otimes_\mathbb{Z} H_\mathbb{Z}$. Then we obtain a finite unramified map of Shimura varieties (or rather stacks)

$$\mathrm{Sh}_K \stackrel{\mathrm{defn}}{=} \mathrm{Sh}_K(G, X) \rightarrow E \otimes_\mathbb{Q} \mathrm{Sh}_{K^\sharp} \stackrel{\mathrm{defn}}{=} \mathrm{Sh}_{K^\sharp}(\mathrm{GSp}(H), S^\pm(H)).$$

The Shimura variety Sh_{K^\sharp} has an integral model \mathcal{S}_{K^\sharp} over $\mathrm{Spec} \mathbb{Z}$ as a certain moduli stack of polarized abelian varieties. Using the symplectic representation H and the lattice $H_\mathbb{Z}$, we obtain a normal integral model \mathcal{S}_K for Sh_K over \mathcal{O}_E by taking the normalization of $\mathcal{O}_E \otimes_\mathbb{Z} \mathcal{S}_{K^\sharp}$ in Sh_K ; see for instance [KMS21, (1.3.5)].

1.1.3. Suppose that $G_{\mathbb{Z}_p}$ is a smooth group scheme over \mathbb{Z}_p with connected special fiber and with generic fiber $G_{\mathbb{Q}_p}$ such that $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$. Suppose also that we have a collection of tensors $\{s_\alpha\} \subset H_{\mathbb{Z}_p}^\otimes$ such that their pointwise stabilizer in $\mathrm{GL}(H_{\mathbb{Z}_p})$ is $G_{\mathbb{Z}_p}$.

Lemma 1.1.4. *The p -adic local system $\mathbf{H}_p = T_p(\mathcal{A})^\vee$ over $\mathcal{S}_K[1/p]$ is equipped with a canonical $G_{\mathbb{Z}_p}$ -structure. More precisely, there exist canonical tensors $\{s_{\alpha,p}\} \subset H^0(\mathcal{S}_K[1/p], \mathbf{H}_p^\otimes)$ such that, at each geometric point $x : \mathrm{Spec} F \rightarrow \mathcal{S}_K[1/p]$, there is an isomorphism $H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{p,x}$ carrying $\{s_\alpha\}$ to $\{s_{\alpha,p,x}\}$.*

Proof. See the discussion in [KMS21, (1.3.4)]. □

Remark 1.1.5. The de Rham realization $\mathbf{H}_{\mathrm{dR}} = H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{S}_K)$, when restricted to the generic fiber Sh_K , is also equipped with a canonical G -structure as a filtered vector bundle with integrable connection. That is, $\mathbf{H}_{\mathrm{dR}}|_{\mathrm{Sh}_K}$ is equipped with a two step Hodge filtration $\mathrm{Fil}_{\mathrm{Hdg}}^\bullet \mathbf{H}_{\mathrm{dR}}|_{\mathrm{Sh}_K}$ concentrated in degrees 0 and 1 and a Gauss-Manin connection, and we have tensors $\{s_{\alpha,\mathrm{dR}}\} \subset \mathrm{Fil}^0(\mathbf{H}_{\mathrm{dR}}^\otimes|_{\mathrm{Sh}_K})$ that are parallel for this connection. Moreover, for any point $x : \mathrm{Spec} L \rightarrow \mathrm{Sh}_K$ with L algebraically closed, there exists an isomorphism $L \otimes_\mathbb{Q} H \xrightarrow{\cong} \mathbf{H}_{\mathrm{dR},x}$ carrying $\{1 \otimes s_\alpha\}$ to $\{s_{\alpha,\mathrm{dR},x}\}$ such that the Hodge filtration pulls back to a filtration on $L \otimes_\mathbb{Q} H$ that is split by a cocharacter in the conjugacy class $[\mu^{-1}]$.

Lemma 1.1.6. *Fix a place $v|p$ of E . Suppose that we have a perfect field κ of characteristic p and a point $x_0 \in \mathcal{S}_{K,v}(k)$. Associated with this is the abelian variety \mathcal{A}_{x_0} , as well as the p -divisible group $\mathcal{G}_{x_0} = \mathcal{A}_{x_0}[p^\infty]$. Then \mathcal{G}_{x_0} is equipped with a canonical G -structure compatible with the $G_{\mathbb{Z}_p}$ -structure over $\mathcal{S}_K[1/p]$. More precisely, the following holds:*

- (1) *If $\mathbf{H}_{\mathrm{cris},x_0} = \mathbb{D}(\mathcal{A}_{x_0})(W(\kappa))$ is the contravariant Dieudonné module associated with \mathcal{A}_{x_0} , then there exist Frobenius invariant tensors*

$$\{s_{\alpha,\mathrm{cris},x_0}\} \subset \mathbf{H}_{\mathrm{cris},x_0}^\otimes$$

and an isomorphism $W(\kappa)[1/p] \otimes_\mathbb{Q} H \xrightarrow{\cong} \mathbf{H}_{\mathrm{cris},x_0}$ carrying $\{1 \otimes s_\alpha\}$ to $\{s_{\alpha,\mathrm{cris},x_0}\}$.

- (2) *If $L/W(\kappa)[1/p]$ is a p -adically complete discrete valuation field with residue field k and $x \in \mathcal{S}_K(\mathcal{O}_L)$ is a lift of x_0 with image $\bar{x} \in \mathrm{Sh}_K(\bar{L})$ for an algebraic closure \bar{L} of L , then the p -adic comparison isomorphism*

$$B_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathbf{H}_{p,\bar{x}} \xrightarrow{\cong} B_{\mathrm{cris}} \otimes_{W(\kappa)} \mathbf{H}_{\mathrm{cris},x_0}$$

carries $\{s_{\alpha,p,\bar{x}}\}$ to $\{s_{\alpha,\mathrm{cris},x_0}\}$.

Proof. This is shown in [KMS21, Prop. 1.3.7]. □

¹One can relax this to asking for (G, X) to be merely of abelian type, but I will restrict to the Hodge type case in this paper.

Remark 1.1.7. In the situation of assertion (1) of Lemma 1.1.6, via a choice of isomorphism $W(\kappa)[1/p] \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{cris},x_0}$, the σ -semilinear isomorphism $\mathbf{H}_{\text{cris},x_0} \xrightarrow{\cong} \mathbf{H}_{\text{cris},x_0}$ underlying its structure of an F -isocrystal corresponds to an element $b_{x_0} \in G(W(\kappa)[1/p])$ whose σ -conjugacy class $[b_{x_0}] \in B(G_{\mathbb{Q}_p})$ is independent of the choice of the isomorphism. Moreover, this class is in fact $[\mu_v^{-1}]$ -admissible; see [KMS21, Lemma 1.3.9].

Remark 1.1.8. In the situation of assertion (2) of Lemma 1.1.6, we also have the Berthelot-Ogus comparison isomorphism

$$L \otimes_{W(\kappa)} \mathbf{H}_{\text{cris},x_0} \xrightarrow{\cong} \mathbf{H}_{\text{dR},x}[p^{-1}]$$

This carries $\{1 \otimes s_{\alpha,\text{cris},x_0}\}$ to $\{s_{\alpha,\text{dR},x}[p^{-1}]\}$, and is compatible with the combination of the crystalline comparison isomorphism with the de Rham comparison isomorphism

$$B_{\text{dR}} \otimes_{\mathbb{Z}_p} \mathbf{H}_{p,\bar{x}} \xrightarrow{\cong} B_{\text{dR}} \otimes_{\mathcal{O}_L} \mathbf{H}_{\text{dR},x_0}.$$

In particular, the de Rham comparison isomorphism carries $\{s_{\alpha,p,\bar{x}}\}$ to $\{s_{\alpha,\text{dR},x}\} \subset \mathbf{H}_{\text{dR},x}^{\otimes}[1/p]$. Further, for any isomorphism $L \otimes_{\mathbb{Q}} H \xrightarrow{\cong} \mathbf{H}_{\text{dR},x}$ carrying $\{1 \otimes s_{\alpha}\}$ to $\{s_{\alpha,\text{dR},x}\}$, the Hodge filtration $\text{Fil}_{\text{Hdg}}^1 \mathbf{H}_{\text{dR},x}$ pulls back to a filtration on $L \otimes_{\mathbb{Q}} H$ that is split by a cocharacter in the geometric conjugacy class $[\mu_v^{-1}]$.

Definition 1.1.9. In the situation of Lemma 1.1.6, we will say that \mathcal{G}_{x_0} has a $G_{\mathbb{Z}_p}$ -**structure** if we have $\{s_{\alpha,\text{cris},x_0}\} \subset \mathbf{H}_{\text{cris},x_0}^{\otimes}$ and if there exists an isomorphism $W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{cris},x_0}$ carrying $\{1 \otimes s_{\alpha}\}$ to $\{s_{\alpha,\text{cris},x_0}\}$.

Remark 1.1.10. Suppose that $G_{\mathbb{Z}_p}$ is reductive, so that K_p is hyperspecial. Then, for any place $v|p$ of E , $\mathcal{S}_{K,(v)}$ is the integral canonical model for Sh_K over $\mathcal{O}_{E,(v)}$ constructed in [Kis10; KM16], and is in particular independent of the choice of symplectic representation H and the lattice $H_{\mathbb{Z}}$. Moreover, in this case, for all $x_0 \in \mathcal{S}_{K,(v)}(k)$ as above, \mathcal{G}_{x_0} is equipped with a $G_{\mathbb{Z}_p}$ -structure; see [Kis10, Corollary (1.4.3)] and [KM16, Theorem 2.5].

1.2. Deformation theory over the ordinary locus. Fix a place $v|p$ of E such that $\mathbb{Q}_p = E_v$, so that $[\mu_v]$ is defined over \mathbb{Q}_p . In this subsection, we will look at the deformation rings of $\mathcal{S}_{K,(v)}$ at its ordinary points, under some assumptions.

Assumption 1.2.1. The conjugacy class $[\mu_v]$ admits a representative $\mu_v : \mathbb{G}_{m,\mathbb{Z}_p} \rightarrow G_{\mathbb{Z}_p}$ whose centralizer $M_{\mathbb{Z}_p} \subset G_{\mathbb{Z}_p}$ is smooth with connected special fiber.

Remark 1.2.2. Under this assumption, we have a decomposition $H_{\mathbb{Z}_p} = H_{\mathbb{Z}_p}^1 \oplus H_{\mathbb{Z}_p}^0$ where $H_{\mathbb{Z}_p}^i$ is the eigenspace on which $\mu_v(z)$ acts via z^{-i} . Moreover, for all α , we have $\mu_v(z) \cdot s_{\alpha} = s_{\alpha}$.

Remark 1.2.3. If $G_{\mathbb{Z}_p}$ is reductive, then a representative μ_v always exists. To see this, one chooses a maximal torus $T \subset G_{\mathbb{Z}_p}$ contained in a Borel subgroup of $G_{\mathbb{Z}_p}$. There is a unique representative μ_v of the conjugacy class $[\mu_v]$ factoring through T and dominant with respect to the choice of Borel, and this does the job for us. The centralizer of μ_v is a Levi subgroup of $G_{\mathbb{Z}_p}$ and is thus reductive with connected fibers.

Definition 1.2.4. Given an algebraically closed field κ in characteristic p , we will say that a point $x_0 \in \mathcal{S}_{K,(v)}(\kappa)$ is μ_v -**ordinary** or simply **ordinary** if \mathcal{G}_{x_0} admits a $G_{\mathbb{Z}_p}$ -structure, and if there exists an isomorphism $W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{cris},x_0}$ with the following properties:

- (1) It carries $\{1 \otimes s_{\alpha}\}$ to $\{s_{\alpha,\text{cris},x_0}\}$.
- (2) The σ -semilinear endomorphism of $\mathbf{H}_{\text{cris},x_0}$ arising from its structure of a Dieudonné module conjugates under the isomorphism to the endomorphism $1 \otimes \mu_v(p)^{-1}$ of $W(\kappa) \otimes_{\mathbb{Z}_p} H_p$. In other words, it acts as p^i on $W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}^i$, for $i = 0, 1$.

Remark 1.2.5. If x_0 is an ordinary point, the abelian variety \mathcal{A}_{x_0} is ordinary in the usual sense: We have $\mathcal{G}_{x_0} \simeq \mathcal{G}_{x_0}^{\text{ét}} \times \mathcal{G}_{x_0}^{\text{mult}}$, where $\mathcal{G}_{x_0}^{\text{ét}}$ (resp. $\mathcal{G}_{x_0}^{\text{mult}}$) is an étale (resp. a multiplicative) p -divisible group. Under the isomorphism $W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{cris},x_0}$ as in the definition, $W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}^0$ (resp. $W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}^1$) maps isomorphically onto the Dieudonné module $\mathbf{H}_{\text{cris},x_0}^{\text{ét}}$ for $\mathcal{G}_{x_0}^{\text{ét}}$ (resp. $\mathbf{H}_{\text{cris},x_0}^{\text{mult}}$ for $\mathcal{G}_{x_0}^{\text{mult}}$).

Notation 1.2.6. Given an algebraically closed field κ in characteristic p , write $\text{Art}_{W(\kappa)}$ for the category of Artin local $W(\kappa)$ -algebras with residue field κ . Given $C \in \text{Art}_{W(\kappa)}$, write \mathfrak{m}_C for its maximal ideal. Note that $1 + \mathfrak{m}_C$ as a group under multiplication is p^n -torsion for n sufficiently large and so can be viewed as a \mathbb{Z}_p -module.

Definition 1.2.7. The **formal torus with character group** a finite free \mathbb{Z}_p -module M is the functor \widehat{T}_M on $\text{Art}_{W(\kappa)}$ given by

$$\widehat{T}_M(C) = \text{Hom}_{\mathbb{Z}_p}(M, 1 + \mathfrak{m}_C).$$

We will call $M^\vee = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ the **cocharacter group** for \widehat{T}_M .

Remark 1.2.8. Set

$$\mathcal{G}_0^{\text{ét}} = \underline{\text{Hom}}(H_{\mathbb{Z}_p}^0, \mathbb{Q}_p/\mathbb{Z}_p); \quad \mathcal{G}_0^{\text{mult}} = \underline{\text{Hom}}(H_{\mathbb{Z}_p}^1, \mu_{p^\infty}).$$

These are p -divisible groups over \mathbb{Z}_p .

Suppose that we have an ordinary point $x_0 \in \mathcal{S}_{K,v}(\kappa)$. Fix a choice of isomorphism witnessing the ordinarity of x_0 : This gives rise via Dieudonné theory to isomorphisms of p -divisible groups

$$\kappa \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\text{ét}} \xrightarrow{\cong} \mathcal{G}_{x_0}^{\text{ét}}; \quad \kappa \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\text{mult}} \xrightarrow{\cong} \mathcal{G}_{x_0}^{\text{mult}}.$$

By classical Serre-Tate ordinary theory [Kat81], the deformation functor $\text{Def}_{\mathcal{G}_{x_0}}$ on $\text{Art}_{W(\kappa)}$ is now isomorphic to the functor

$$\begin{aligned} \widehat{\text{Ext}}(\mathcal{G}_0^{\text{ét}}, \mathcal{G}_0^{\text{mult}}) : C &\mapsto \widehat{\text{Ext}}^1(C \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\text{ét}}, C \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\text{mult}}) \\ &\xrightarrow{\cong} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \otimes_{\mathbb{Z}_p} \widehat{H}^1(\text{Spec } C, \mu_{p^\infty}) \\ &\xrightarrow{\cong} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \otimes_{\mathbb{Z}_p} (1 + \mathfrak{m}_C). \end{aligned}$$

Here, $\widehat{H}^1(\text{Spec } C, \mu_{p^\infty})$ is the set of isomorphism classes of extensions of μ_{p^∞} by $\mathbb{Q}_p/\mathbb{Z}_p$ as fppf sheaves over $\text{Spec } C$ equipped with a trivialization over κ . The last isomorphism is obtained from Kummer theory.

In particular, $\text{Def}_{\mathcal{G}_{x_0}}$ is isomorphic to the formal torus \widehat{T} over $W(\kappa)$ with cocharacter group $\text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)$. This isomorphism is well-defined up to the action of $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ on the cocharacter group.

Remark 1.2.9. Let R be the coordinate ring of \widehat{T} and let $\mathcal{G}^{\text{univ}}$ be the universal deformation of \mathcal{G}_{x_0} over R . Dieudonné theory over such formally smooth rings associates with $\mathcal{G}^{\text{univ}}$ its Dieudonné module $M = \mathbb{D}(\mathcal{G}^{\text{univ}})(R)$, which is equipped with a topologically locally nilpotent integrable connection ∇ and a φ -semilinear map $F : M \rightarrow M$ that is horizontal for the connection. This can be described explicitly following de Jong [de 95, §4.3]: We have the tautological element

$$q^{\text{univ}} \in \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \otimes_{\mathbb{Z}_p} (1 + \mathfrak{m}_R)$$

mapping to

$$\text{dlog}(q^{\text{univ}}) \in \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \otimes_{\mathbb{Z}_p} \widehat{\Omega}_{R/W(\kappa)}^1 \subset \text{End}(H_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \widehat{\Omega}_{R/W(\kappa)}^1$$

We now have $M = R \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}$. The connection ∇ is given by $d \otimes 1 + \text{dlog}(q^{\text{univ}})$, and we have $F = \varphi \otimes \mu_v(p)^{-1}$.

Definition 1.2.10. For an ordinary algebraically closed point $x_0 \in \mathcal{S}_{K,v}(\kappa)$, the **canonical lift** of \mathcal{G}_{x_0} is the lift $\mathcal{G}_{x_0}^{\text{can}}$ over $W(\kappa)$ corresponding to the identity section of the formal torus \widehat{T} .

Definition 1.2.11. Let $\text{Lie } U_{\mu_v}^- \subset \text{Lie } G_{\mathbb{Z}_p}$ be the eigenspace on which the adjoint action of $\mu_p(z)$ is via z . We have

$$\text{Lie } U_{\mu_v}^- = \text{Lie } G \cap \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \subset \text{End}(H_{\mathbb{Z}_p})$$

where we are viewing $\text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)$ as the space of endomorphisms of $H_{\mathbb{Z}_p}$ whose kernel contains $H_{\mathbb{Z}_p}^0$ and whose image is contained in $H_{\mathbb{Z}_p}^0$. Let \widehat{T}_G be the formal torus with cocharacter group $\text{Lie } U_{\mu_v}^-$: This is a sub-formal torus of \widehat{T} .

Remark 1.2.12. Let R_G be the ring of functions of \widehat{T}_G : it is equipped with a canonical Frobenius lift φ_G given by the p -power map. The map $f : R \rightarrow R_G$ corresponding to $\widehat{T}_G \hookrightarrow \widehat{T}$ is compatible with Frobenius lifts. Via base-change, we now see from Remark 1.2.9 that the Dieudonné module for the universal p -divisible group over R_G is given by $M_G = R_G \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}$ with connection given by $\nabla_G = d \otimes 1 + \text{dlog}(f(q_{\text{univ}}))$ and φ_G -semilinear endomorphism $F_G = \varphi_G \otimes \mu_v(p)^{-1}$. Now, we have

$$f(q_{\text{univ}}) \in \text{Lie } U_{\mu_v}^- \otimes_{\mathbb{Z}_p} (1 + \mathfrak{m}_{R_G}) \subset \text{Lie } G \otimes_{\mathbb{Z}_p} (1 + \mathfrak{m}_{R_G}).$$

This shows that the tensors $s_{\alpha, R_G} = 1 \otimes s_{\alpha, \text{cris}, x_0} \in M_G^{\otimes}$ are parallel for the connection. They are also invariant under F_G .

Proposition 1.2.13. *Let $x_0 \in \mathcal{S}_{K,v}(\kappa)$ be ordinary, and let \widehat{U}_{x_0} be the deformation space for $\mathcal{S}_{K,v}$ at x_0 . If the image of \widehat{U}_{x_0} in $\text{Def}_{\mathcal{G}_{x_0}} \xrightarrow{\cong} \widehat{T}$ contains the canonical lift, then \widehat{U}_{x_0} is isomorphic to $\widehat{T}_G \subset \widehat{T}$.*

Proof. For the purposes of understanding complete local rings at algebraically closed points, after replacing K^p and $K^{\sharp,p}$ with finite index subgroups—and hence \mathcal{S}_K and \mathcal{S}_{K^\sharp} by finite étale covers—we can assume, without loss of generality, that $\text{Sh}_K \rightarrow E \otimes \text{Sh}_{K^\sharp}$ is a closed immersion [Kis10, Lemma (2.1.2)]. Therefore, \mathcal{S}_K is simply the normalization of its image in $\mathcal{O}_E \otimes \mathcal{S}_{K^\sharp}$, and so \widehat{U}_{x_0} is the normalization of its image in $\widehat{U}_{x_0^\sharp}$, the deformation space of \mathcal{S}_{K^\sharp} at the image $x_0^\sharp \in \mathcal{S}_{K^\sharp}(\kappa)$.

Now, by the Serre–Tate theorem for deformations of abelian varieties, we can identify $\widehat{U}_{x_0^\sharp}$ with a closed formal subscheme of the deformation space $\text{Def}_{\mathcal{G}_{x_0}}$ and hence of the formal torus \widehat{T} .

In summary, we have a canonical map $\widehat{U}_{x_0} \rightarrow \widehat{T}$ that is the normalization of its image.

By a theorem of Noot [Noo96, Theorem 3.7], there is a finite extension $L/W(\kappa)[1/p]$ such that the image of each irreducible component of $\widehat{U}_{x_0, \mathcal{O}_L} \stackrel{\text{defn}}{=} \text{Spf}(\mathcal{O}_L \otimes_{W(\kappa)} R_{x_0})$ in $\widehat{T}_{\mathcal{O}_L}$ is a translate by a torsion point of a formal sub-torus.

Consider the logarithm map

$$\ell : \widehat{T}(\mathcal{O}_L) = \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \otimes_{\mathbb{Z}_p} (1 + \mathfrak{m}_L) \xrightarrow{1 \otimes \log} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \otimes_{\mathbb{Z}_p} L.$$

I claim that we have

$$\ell(\widehat{U}_{x_0}(\mathcal{O}_L)) \subset \text{Lie } U_{\mu_v}^- \otimes_{\mathbb{Z}_p} L.$$

For this, we will need the following interpretation of the map ℓ : Given a lift $x \in \widehat{T}(\mathcal{O}_L)$, we obtain a p -divisible group \mathcal{G}_x . As in Remark 1.1.8, we have a canonical isomorphism $L \otimes_{W(\kappa)} \mathbf{H}_{\text{cris}, x_0} \xrightarrow{\cong} \mathbf{H}_{\text{dR}, x}[1/p]$, and we therefore have an isomorphism

$$L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{dR}, x}[1/p]$$

by our choice of fixed isomorphism. The Hodge filtration arising from this identification is of the form

$$\text{Fil}_x^1(L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}) = \exp(N_x)(L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}^1),$$

for some $N_x \in L \otimes_{\mathbb{Z}_p} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)$. It follows from a computation of Katz [DI81, A.3] that, at least up to sign, we have

$$N_x = \ell(x) \in \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \otimes_{\mathbb{Z}_p} L.$$

Now, by Remark 1.1.8, if $x \in \widehat{U}_{x_0}(\mathcal{O}_L)$, then $\text{Fil}_x^1(L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p})$ is split by a cocharacter conjugate to μ_v^{-1} . In particular, the parabolic subgroup $P_x \subset G_L$ stabilizing $\text{Fil}_x^1(L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p})$ is $G(L)$ -conjugate to the subgroup $P_L \subset G_L$ stabilizing $L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}^1$. That is, within the set $\text{Gr}(L)$ of subspaces of middle dimension in H_L , $\text{Fil}_x^1(L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p})$ is in the image of $G(L)$. However, the map

$$L \otimes_{\mathbb{Z}_p} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \xrightarrow{N \mapsto \exp(N)(L \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}^1)} \text{Gr}(L)$$

is injective, with image isomorphic to the L -points of the open Bruhat cell, and the pre-image of the image of $G(L)$ is precisely $L \otimes_{\mathbb{Z}_p} \text{Lie } U_{\mu_v}^-$. This proves the claim.

Now, it follows that the irreducible components of the image of $\widehat{U}_{x_0, \mathcal{O}_L}$ in $\widehat{T}_{G, \mathcal{O}_L}$ are all translates by torsion points of $\widehat{T}_{G, \mathcal{O}_L}$. By our hypothesis, one of these components must contain the identity section, and must therefore be isomorphic to $\widehat{T}_{G, \mathcal{O}_L}$.

This implies that the image of $\widehat{U}_{x_0} \rightarrow \widehat{T}$ already contains \widehat{T}_G . However, this image is irreducible (since \widehat{U}_{x_0} is irreducible), and so must be exactly \widehat{T}_G . Since \widehat{T}_G is already normal, we conclude that we must have $\widehat{U}_{x_0} \xrightarrow{\cong} \widehat{T}_G$. \square

Remark 1.2.14. Maintain the hypotheses of Proposition 1.2.13. Let $\widehat{U}_{x_0}^{\text{an}}$ be the rigid analytic space over $W(\kappa)[1/p]$ associated with \widehat{U}_{x_0} . The proposition shows that this is isomorphic to an open unit ball of dimension $\dim \text{Sh}_K$ contained in the rigid analytic space $\text{Sh}_{K, W(\kappa)[1/p]}^{\text{an}}$. Over the latter, the restriction of the vector bundle \mathbf{H}_{dR} is equipped with an integrable connection and parallel tensors $\{s_{\alpha, \text{dR}}\}$. On the other hand, under the isomorphism $\widehat{U}_{x_0} \xrightarrow{\cong} \widehat{T}_G$, the vector bundle with integrable connection M_G over the latter constructed in Remark 1.2.12 pulls

back to the restriction of \mathbf{H}_{dR} , and the tensors $\{s_{\alpha, R_G}\}$ constructed in *loc. cit.* pull back to horizontal tensors in $H^0(\widehat{U}_{x_0}^{\mathrm{an}}, \mathbf{H}_{\mathrm{dR}}^{\otimes})$. These tensors agree with the restrictions of $\{s_{\alpha, \mathrm{dR}}\}$: Indeed, since both collections are parallel for the connection, it suffices to verify that they agree at one point. One can do this at the point corresponding to the canonical lift via Remark 1.1.8.

The following assumption will be in force for the rest of this section.

Assumption 1.2.15. For all κ algebraically closed and all ordinary points $x_0 \in \mathcal{S}_{K,v}(\kappa)$, the image of \widehat{U}_{x_0} in $\mathrm{Def}_{\mathcal{G}_{x_0}}$ contains the canonical lift.

Remark 1.2.16. The above assumption—and hence the conclusion of Proposition 1.2.13—is always valid when $G_{\mathbb{Z}_p}$ is reductive. See for instance [SZ21, Theorem 6.5].

Corollary 1.2.17. *The ordinary points form an open substack $\mathcal{S}_{K,k(v)}^{\mathrm{ord}}$ of the special fiber $\mathcal{S}_{K,k(v)} = k(v) \otimes_{\mathcal{O}_{E,v}} \mathcal{S}_{K,v}$.*

Proof. With the notation of Proposition 1.2.13 and its proof, it is enough to show that a geometric generic point of $\mathrm{Spec} R_{x_0}/(p)$ is ordinary in our sense. Write $\mathbf{H}_{\mathrm{dR}, R_{x_0}}$ for the finite free R_{x_0} -module associated with $\mathbf{H}_{\mathrm{dR}}^{\otimes}|_{\mathrm{Spec} R_{x_0}}$. Via the isomorphism $R_{x_0} \xrightarrow{\sim} R_G$, R_{x_0} is equipped with a Frobenius lift φ , and we have an isomorphism

$$F : \varphi^* \mathbf{H}_{\mathrm{dR}, R_{x_0}}[p^{-1}] \xrightarrow{\sim} \mathbf{H}_{\mathrm{dR}, R_{x_0}}[p^{-1}].$$

By Remark 1.2.14, the restriction of $\{s_{\alpha, \mathrm{dR}}\}$ over $\mathrm{Spec} R_{x_0}[p^{-1}]$ gives a collection

$$\{s_{\alpha, \mathrm{dR}, R_{x_0}}\} \subset \mathbf{H}_{\mathrm{dR}, R_{x_0}}^{\otimes}$$

that is invariant under F , and moreover there is an isomorphism

$$(1.2.17.1) \quad R_{x_0} \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\sim} \mathbf{H}_{\mathrm{dR}, R_{x_0}}$$

carrying $\{1 \otimes s_{\alpha}\}$ to $\{s_{\alpha, \mathrm{dR}, R_{x_0}}\}$, and is such that F pulls back to the automorphism $1 \otimes \mu_v(p)^{-1}$ of $R_{x_0} \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p}$.

If F is an algebraic closure of the fraction field of $R_{x_0}/(p)$, we can lift $R_{x_0} \rightarrow F$ uniquely to a map $R_{x_0} \rightarrow W(F)$ that respects Frobenius lifts, and the base-change over $W(F)$ of (1.2.17.1) now witnesses the ordinarity of the corresponding F -valued point of $\mathcal{S}_{K,(v)}$. \square

Corollary 1.2.18. *Let $\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}}$ be the formal completion of $\mathcal{S}_{K,(v)}$ along the (open) ordinary locus $\mathcal{S}_{K,k(v)}^{\mathrm{ord}}$, and let $\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}, \mathrm{an}} \subset \mathrm{Sh}_{K,E_v}^{\mathrm{an}}$ be the corresponding rigid analytic tube. Then the restriction of $\{s_{\alpha, \mathrm{dR}}\}$ over $\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}, \mathrm{an}}$ extends canonically to a collection of parallel tensors $\{s_{\alpha, \mathrm{dR}}\} \subset H^0(\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}}, \mathrm{Fil}^0(\mathbf{H}_{\mathrm{dR}}^{\otimes}))$.*

Proof. Immediate from Remark 1.2.14. \square

1.3. The ordinary Igusa tower. Here, we review the story of the Igusa tower over the ordinary locus.

Remark 1.3.1. Set $\mathcal{G}_0 \stackrel{\mathrm{defn}}{=} \mathcal{G}_0^{\mathrm{ét}} \times \mathcal{G}_0^{\mathrm{mult}}$, and note that it can be equipped with a canonical $G_{\mathbb{Z}_p}$ -structure in the following sense: If we set $\mathbf{H}_0 = \mathbb{D}(\mathcal{G}_0)(\mathbb{Z}_p)$, then there is an isomorphism

$$\mathbf{H}_0 \xrightarrow{\sim} H_{\mathbb{Z}_p}^0 \oplus H_{\mathbb{Z}_p}^1 = H_{\mathbb{Z}_p}$$

well-defined up to the action of $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$, and we can use this to obtain F -invariant tensors $\{s_{\alpha, 0}\} \subset \mathbf{H}_0^{\otimes}$, which are carried to $\{s_{\alpha}\}$ under any choice of such isomorphism.

Definition 1.3.2. For an algebraically closed field κ in characteristic p , an automorphism of $\kappa \otimes_{\mathbb{Z}_p} \mathcal{G}_0$ is $G_{\mathbb{Z}_p}$ -**structure preserving** if the induced automorphism of $W(\kappa) \otimes_{\mathbb{Z}_p} \mathbf{H}_0$ fixes the tensors $\{s_{\alpha, 0}\}$.

For a p -complete \mathbb{Z}_p -algebra R , an automorphism α of $R \otimes_{\mathbb{Z}_p} \mathcal{G}_0$ is $G_{\mathbb{Z}_p}$ -**structure preserving** its restriction along any geometric point of $\mathrm{Spf} R$ is $G_{\mathbb{Z}_p}$ -structure preserving.

It is $M_{\mathbb{Z}_p}$ -**structure preserving** if in addition it preserves the direct product structure $R \otimes_{\mathbb{Z}_p} \mathcal{G}_0 = (R \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\mathrm{ét}}) \times (R \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\mathrm{mult}})$

Notation 1.3.3. For a p -complete ring R , write $\text{Aut}_G(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0)$ (resp. $\text{Aut}_M(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0)$) for the group of $G_{\mathbb{Z}_p}$ -structure (resp. $M_{\mathbb{Z}_p}$ -structure) preserving automorphisms. The functors

$$R \mapsto \text{Aut}_G(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0) ; R \mapsto \text{Aut}_M(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0)$$

will be denoted by $\underline{\text{Aut}}_G(\mathcal{G}_0)$ and $\underline{\text{Aut}}_M(\mathcal{G}_0)$, respectively.

Lemma 1.3.4. *Consider the formal functors*

$$\underline{\text{Aut}}_G(\mathcal{G}_0), \underline{\text{Aut}}_M(\mathcal{G}_0) : \text{Spf } R \mapsto \text{Aut}_G(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0).$$

- (1) *There is an isomorphism $\underline{\text{Aut}}_M(\mathcal{G}_0) \xrightarrow{\cong} M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ well-defined up to conjugation by an element of $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$.*
- (2) *Every $\alpha \in \text{Aut}_G(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0)$ carries $R \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\text{mult}}$ to itself.*
- (3) *If R is Noetherian, the natural map*

$$\text{Aut}_M(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0) \rightarrow \text{Aut}_G(R \otimes_{\mathbb{Z}_p} \mathcal{G}_0)$$

is an isomorphism.

Proof. The first assertion follows from the definitions and the fact that the automorphism sheaf $\underline{\text{Aut}}(\mathcal{G}_0^{\text{ét}} \times \mathcal{G}_0^{\text{mult}})$ of product structure preserving automorphisms of \mathcal{G}_0 is isomorphic up to conjugation by $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ to the locally constant pro-finite sheaf of groups $\underline{\text{GL}}(H_{\mathbb{Z}_p}^0) \times \underline{\text{GL}}(H_{\mathbb{Z}_p}^1)$.

Assertion (2) amounts to the easy fact that there are no non-trivial homomorphisms $\mu_{p^\infty} \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ over a p -nilpotent base.

Assertion (3) follows from the observation that, for any Noetherian ring R , the space of maps $\mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mu_{p^\infty}$ is trivial. \square

Remark 1.3.5. The restriction of the p -divisible group $\mathcal{G} = \mathcal{A}[p^\infty]$ over $\widehat{\mathcal{S}}_{K,v}^{\text{ord}}$ is canonically an extension

$$0 \rightarrow \mathcal{G}^{\text{mult}} \rightarrow \mathcal{G}|_{\widehat{\mathcal{S}}_{K,v}^{\text{ord}}} \rightarrow \mathcal{G}^{\text{ét}} \rightarrow 0,$$

where $\mathcal{G}^{\text{mult}}$ is of multiplicative type and $\mathcal{G}^{\text{ét}}$ is étale. If $x_0 \in \mathcal{S}_{K,k(v)}^{\text{ord}}(\kappa)$ is an algebraically closed point, then we have a canonical splitting $\mathcal{G}_{x_0} \xrightarrow{\cong} \mathcal{G}_{x_0}^{\text{ét}} \times \mathcal{G}_{x_0}^{\text{mult}}$.

Definition 1.3.6. For κ algebraically closed, an isomorphism $\kappa \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\cong} \mathcal{G}_{x_0}^{\text{ét}} \times \mathcal{G}_{x_0}^{\text{mult}} \simeq \mathcal{G}_{x_0}$ is $G_{\mathbb{Z}_p}$ -**structure preserving** if the associated isomorphism $W(\kappa) \otimes_{\mathbb{Z}_p} \mathbf{H}_0 \xrightarrow{\cong} \mathbf{H}_{\text{cris},x_0}$ carries $\{s_{\alpha,0}\}$ to $\{s_{\alpha,\text{cris},x_0}\}$

Definition 1.3.7. We define $\widehat{\text{Ig}}_{K,v}^{\text{ord}} \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$ to be the formal scheme parameterizing, for each $x : \text{Spf } R \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$, the set of isomorphisms $\eta : R \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\cong} \mathcal{G}_x^{\text{ét}} \times \mathcal{G}_x^{\text{mult}}$ such that:

- (1) η preserves the product structure on both sides;
- (2) The restriction of η over any algebraically closed point is $G_{\mathbb{Z}_p}$ -structure preserving.

Proposition 1.3.8. *The map $\widehat{\text{Ig}}_{K,v}^{\text{ord}} \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$ is a torsor under the locally constant pro-finite sheaf of groups $\underline{\text{Aut}}_G(\mathcal{G}_0) \simeq M_{\mathbb{Z}_p}(\mathbb{Z}_p)$.*

Proof. Given the natural action of $\underline{\text{Aut}}_M(\mathcal{G}_0)$ on $\widehat{\text{Ig}}_{K,v}^{\text{ord}}$, it suffices to check the assertion over the complete local ring R_{x_0} at any $\overline{\mathbb{F}}_p$ -point x_0 . In fact, it suffices to show that $\widehat{\text{Ig}}_{K,v}^{\text{ord}}(R_{x_0})$ is non-empty, and this can be deduced from Proposition 1.2.13 and Remark 1.2.12. \square

1.4. The generic fiber of the Igusa tower.

Notation 1.4.1. Let $P_{\mathbb{Z}_p}^- \subset G_{\mathbb{Z}_p}$ be the subgroup stabilizing the subspace $H_{\mathbb{Z}_p}^0 \subset H_{\mathbb{Z}_p}$: its generic fiber is a parabolic subgroup of $G_{\mathbb{Q}_p}$. We have a natural quotient map $P_{\mathbb{Z}_p}^- \rightarrow M_{\mathbb{Z}_p}$ whose kernel is the commutative unipotent subgroup $U_{\mathbb{Z}_p}^-$ with Lie algebra $\text{Lie } U_{\mu_v}^-$. Write $I_{G,p} \rightarrow \text{Sh}_K$ for the $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsor Sh_{K^p} .

Construction 1.4.2. We have two $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsors over the analytic space $\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$. First, we can take the analytification $\widehat{\text{Ig}}_{K,v}^{\text{ord,an}}$ of the Igusa tower. The second $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsor is obtained as follows: The filtration on $\mathcal{G}|_{\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}}$ from Remark 1.3.5 gives a corresponding filtration on the dual Tate module

$$T_p \left(\mathcal{G}|_{\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}} \right)^\vee \simeq \mathbf{H}_p|_{\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}}$$

yielding a short exact sequence $0 \rightarrow \mathbf{H}_p^{\text{ét}} \rightarrow \mathbf{H}_p|_{\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}} \rightarrow \mathbf{H}_p^{\text{mult}} \rightarrow 0$ of pro-finite étale sheaves over $\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$. This yields a reduction of structure group of $I_{G,v}|_{\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}}$ to a $P_{\mathbb{Z}_p}^-(\mathbb{Z}_p)$ -torsor, which we denote by $I_{P^-,v}^{\text{ord,an}}$. Pushing out via the quotient map $P_{\mathbb{Z}_p}^-(\mathbb{Z}_p) \rightarrow M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ now yields an $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsor $I_{M,v}^{\text{ord,an}} \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$.

Definition 1.4.3. Let $\widehat{\text{Ig}}_v^{\text{big}}$ be the sheaf over $\widehat{\mathcal{S}}_{K,v}^{\text{ord}}$ parameterizing for $x : \text{Spf } R \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$ pairs of isomorphisms $R \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\text{ét}} \xrightarrow{\simeq} \mathcal{G}_x^{\text{ét}}$ and $R \otimes_{\mathbb{Z}_p} \mathcal{G}_0^{\text{mult}} \xrightarrow{\simeq} \mathcal{G}_x^{\text{mult}}$. Its analytification $\widehat{\text{Ig}}_v^{\text{big,an}}$ is the sheaf parameterizing pairs (α, β) , where α (resp. β) is an isomorphism of pro-finite étale \mathbb{Z}_p -local systems $\underline{H}_{\mathbb{Z}_p}^0 \xrightarrow{\simeq} \mathbf{H}_p^{\text{ét}}$ (resp. $\underline{H}_{\mathbb{Z}_p}^1(1) = \underline{H}_{\mathbb{Z}_p}^1 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \xrightarrow{\simeq} \mathbf{H}_p^{\text{mult}}$). Here, $\mathbb{Z}_p(1) = T_p(\mathbb{G}_m)^\vee$ is the inverse cyclotomic tower. Note that $\widehat{\text{Ig}}_v^{\text{big,an}}$ is a $\text{GL}(H_{\mathbb{Z}_p}^0) \times \text{GL}(H_{\mathbb{Z}_p}^1)$ -torsor over $\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$.

Remark 1.4.4. By construction, $\widehat{\text{Ig}}_{K,v}^{\text{ord,an}}$ is a subsheaf of $\widehat{\text{Ig}}_v^{\text{big,an}}$.

Remark 1.4.5. Suppose that we have an algebraically closed point $x_0 \in \mathcal{S}_{K,k(v)}^{\text{ord}}(\kappa)$ and that $x \in \mathcal{S}_{K,v}^{\text{ord}}(W(\kappa))$ is its canonical lift. Write x also for the associated $\text{Spec } W(\kappa)[p^{-1}]$ -point of Sh_{K,E_v} . We have a canonical splitting $\mathbf{H}_{p,x} \xrightarrow{\simeq} \mathbf{H}_{p,x}^{\text{ét}} \oplus \mathbf{H}_{p,x}^{\text{mult}}$, which gives a reduction of structure for $I_{G,v,x}$ to an $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsor $I_{M,v,x}^{\text{can}} \subset I_{P^-,v,x}^{\text{ord,an}}$ that maps isomorphically onto $I_{M,v,x}^{\text{ord,an}}$. Explicitly, if \bar{L} is an algebraic closure of $L = W(\kappa)[1/p]$ and $\bar{x} \in \text{Sh}_K(\bar{L})$ is the associated geometric point, then $I_{M,v,x}^{\text{can}}(\bar{L})$ is the set of $G_{\mathbb{Z}_p}$ -structure preserving isomorphisms $H_{\mathbb{Z}_p} \xrightarrow{\simeq} \mathbf{H}_{p,\bar{x}}$ that carry $H_{\mathbb{Z}_p}^0$ onto $\mathbf{H}_{p,\bar{x}}^{\text{ét}}$ and $H_{\mathbb{Z}_p}^1$ onto $\mathbf{H}_{p,\bar{x}}^{\text{mult}}$.

Remark 1.4.6. Similarly, one sees that $\widehat{\text{Ig}}_{K,v,x}^{\text{ord,an}}(\bar{L})$ is also isomorphic to the same set, except that its $\text{Gal}(\bar{L}/L)$ -structure is twisted by having the Galois group act on $H_{\mathbb{Z}_p}^1$ via the inverse cyclotomic character. Indeed, a section of (the non-empty set) $\widehat{\text{Ig}}_{K,v,x}^{\text{ord}}(W(\kappa))$ is an isomorphism $W(\kappa) \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_x$ that preserves $G_{\mathbb{Z}_p}$ -structure over κ . Via the crystalline comparison isomorphism and assertion (2) of Lemma 1.1.6, the associated isomorphism

$$\underline{H}_{\mathbb{Z}_p}^0 \oplus \underline{H}_{\mathbb{Z}_p}^1(1) = T_p(\mathcal{G}_0)^\vee \xrightarrow{\simeq} \mathbf{H}_{p,x}$$

of p -adic local systems over $\text{Spec } L$ also preserves $G_{\mathbb{Z}_p}$ -structure. We now recover our desired assertion by restricting this isomorphism over $\text{Spec } \bar{L}$.

Construction 1.4.7. Let $\mathbb{Z}_p^{\text{cycl}}$ be the \mathbb{Z}_p^\times -torsor over $\text{Spec } \mathbb{Q}_p$ corresponding to the p -adic cyclotomic tower: this parameterizes isomorphisms of p -adic local systems $\mathbb{Z}_p \xrightarrow{\simeq} \mathbb{Z}_p(1)$. There is a canonical $M_{\mathbb{Z}_p}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ -equivariant map

$$(1.4.7.1) \quad I_{M,v}^{\text{ord,an}} \times \mathbb{Z}_p^{\text{cycl}} \rightarrow \widehat{\text{Ig}}_v^{\text{big,an}}$$

over $\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$, where \mathbb{Z}_p^\times acts trivially on the right hand side. This is constructed as follows: A section on the left over $x : \text{Spa}(R, R^+) \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$ yields isomorphisms

$$\alpha : \underline{H}_p^0 \xrightarrow{\simeq} \mathbf{H}_{p,x}^{\text{ét}} ; \beta' : \underline{H}_p^1 \xrightarrow{\simeq} \mathbf{H}_{p,x}^{\text{mult}} ; \zeta : \mathbb{Z}_p \xrightarrow{\simeq} \mathbb{Z}_p(1)$$

of p -adic sheaves over $\text{Spa}(R, R^+)$. We can now combine the last two to get an isomorphism $\beta' \otimes \zeta^{-1} : \underline{H}_p^1(1) \xrightarrow{\simeq} \mathbf{H}_{p,x}^{\text{mult}}$.

Proposition 1.4.8. *The map (1.4.7.1) factors through an isomorphism of $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsors*

$$I_{M,v}^{\text{ord,an}} \times_{\mathbb{Z}_p^\times} \mathbb{Z}_p^{\text{cycl}} \xrightarrow{\cong} \widehat{\text{Ig}}_{K,v}^{\text{ord,an}}$$

where on the left we are using the action of $\mathbb{Z}_p^\times = \mathbb{G}_m(\mathbb{Z}_p)$ on $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ via the central cocharacter $\mu_v^{-1} : \mathbb{G}_{m,\mathbb{Z}_p} \rightarrow M_{\mathbb{Z}_p}$.

Proof. First, note that, because of the centrality of μ_v^{-1} , the quotient on the left inherits the structure of an $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsor. The fact that (1.4.7.1) factors through this quotient is immediate from the construction, and amounts to the observation that $\mu_v(z)$ acts on $H_{\mathbb{Z}_p}^1$ via multiplication by z^{-1} . To see that it maps isomorphically onto the other $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsor $\widehat{\text{Ig}}_{K,v}^{\text{ord,an}}$, it suffices to test over a classical point in each connected component of $\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$. We can do so over the canonical lifts of the $\overline{\mathbb{F}}_p$ -points of $\mathcal{S}_{K,k(v)}^{\text{ord}}$ where Remarks 1.4.5 and 1.4.6 do the job. \square

2. HECKE CORRESPONDENCES

This section is a review of p -Hecke correspondences on Shimura varieties via isogenies of abelian varieties. We will consider both the generic fiber and the ordinary locus. The setup will be as in § 1.1.

2.1. Isogenies and p -Hecke correspondences in the generic fiber.

Definition 2.1.1. Suppose that L is an algebraically closed field in characteristic 0 and that we have $s, t \in \text{Sh}_K(L)$. A p -quasi-isogeny $f : \mathcal{A}_s \dashrightarrow \mathcal{A}_t$ **preserves G -structure** if the associated isomorphism $f^* : \mathbf{H}_{p,t}[p^{-1}] \xrightarrow{\cong} \mathbf{H}_{p,s}[p^{-1}]$ carries $\{s_{\alpha,p,t}\}$ to $\{s_{\alpha,p,s}\}$. The **type** of such a p -quasi-isogeny is the unique class

$$[[g]] \in G_{\mathbb{Z}_p}(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G_{\mathbb{Z}_p}(\mathbb{Z}_p) = K_p \backslash G(\mathbb{Q}_p) / K_p$$

such that, for any choice of trivialization $\alpha : H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{p,s}$ (resp. $\beta : H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{p,t}$) carrying $\{s_\alpha\}$ to $\{s_{\alpha,p,s}\}$ (resp. to $\{s_{\alpha,p,t}\}$), we have $\beta^{-1} \circ (f^*)^{-1} \circ \alpha \in [[g]] \subset G(\mathbb{Q}_p)$.

Definition 2.1.2. For $s, t : \text{Spec } R \rightarrow \text{Sh}_K$ arbitrary, a p -quasi-isogeny $\mathcal{A}_s \dashrightarrow \mathcal{A}_t$ **preserves G -structure (and has type $[[g]]$)** if its restriction over every geometric point of $\text{Spec } R$ does so.³

Notation 2.1.3. With $s, t : \text{Spec } R \rightarrow \text{Sh}_K$ as above, write $\text{QIsog}_G(s, t)$ for the space of p -quasi-isogenies from \mathcal{A}_s to \mathcal{A}_t preserving G -structure. Write $\text{Isog}_G(s, t)$ for its subspace consisting of honest isogenies $\mathcal{A}_s \rightarrow \mathcal{A}_t$ and $\text{QIsog}_{G,[[g]]}(s, t)$ for the subspace consisting of quasi-isogenies of type $[[g]]$. If $s = t$, we will write $\text{Aut}_G^\circ(s)$ instead of $\text{QIsog}_G(s, s)$. We will view each of $\text{QIsog}_G, \text{QIsog}_{G,[[g]]}, \text{Isog}_G$ as presheaves over $\text{Sh}_K \times \text{Sh}_K^4$ via the pair (s, t) .

Remark 2.1.4. If $G_{\mathbb{Z}_p}$ is reductive, then, for a choice of maximal torus $T \subset G_{\mathbb{Z}_p}$ as in Remark 1.2.3, the Cartan decomposition tells us that every double coset $[[g]]$ admits a unique representative of the form $\lambda(p)$ for some dominant cocharacter $\lambda \in X_*(T)$ defined over \mathbb{Z}_p . In this situation, we will also write $\text{QIsog}_{G,\lambda}(s, t)$ instead of $\text{QIsog}_{G,[[g]]}(s, t)$.

Construction 2.1.5. For any double coset $[[g]] \in K_p \backslash G(\mathbb{Q}_p) / K_p$ and any choice of $g \in [[g]] \subset G(\mathbb{Q}_p)$, set $K_g = K \cap gKg^{-1}$. We have two maps $s_g, t_g : \text{Sh}_{K_g} \rightarrow \text{Sh}_K$ where s_g arises from conjugation by g^{-1} and t_g from the natural inclusion, using which we can view Sh_{K_g} as a scheme over $\text{Sh}_K \times \text{Sh}_K$.

Proposition 2.1.6. *There is an isomorphism $\text{QIsog}_{G,[[g]]} \xrightarrow{\cong} \text{Sh}_{K_g}$ over $\text{Sh}_K \times \text{Sh}_K$.*

Proof. Let's begin by constructing an isomorphism such that we have a commuting diagram

$$\begin{array}{ccc} \text{QIsog}_{G,[[g]]} & \xrightarrow{\cong} & \text{Sh}_{K_g} \\ & \searrow t & \swarrow t_g \\ & & \text{Sh}_K \end{array}$$

²By this, we mean an element $f \in \text{Hom}(\mathcal{A}_s, \mathcal{A}_t) \otimes \mathbb{Q}$ such that, for some $m, n \geq 0$, $p^m f \in \text{Hom}(\mathcal{A}_s, \mathcal{A}_t)$ is an isogeny of degree p^n .

³For connected $\text{Spec } R$ it is enough to check at *some* geometric point.

⁴The product is over $\text{Spec } E$.

For this, set $K_{g,p} = K_p \cap gK_p g^{-1}$ and note that we have canonical bijections

$$K/K_g \xrightarrow{\cong} K_p/K_{g,p} \xrightarrow[\cong]{k_{K_{g,p}} \mapsto kgK_p} K_p g K_p / K_p.$$

From this one deduces: If $\mathrm{Sh}_{K^p} \rightarrow \mathrm{Sh}_K$ is the K_p -torsor parameterizing trivializations of \mathbf{H}_p with its $G_{\mathbb{Z}_p}$ -structure, then we have

$$\mathrm{Sh}_{K^p} \times^{G_{\mathbb{Z}_p}(\mathbb{Z}_p)} [K_p g K_p / K_p] \xrightarrow{\cong} \mathrm{Sh}_{K_g},$$

so that we have a canonical isomorphism

$$\mathrm{Sh}_{K^p} \times_{\mathrm{Sh}_{K,t_g}} \mathrm{Sh}_{K_g} \xrightarrow{\cong} \mathrm{Sh}_{K^p} \times [K_p g K_p / K_p].$$

On the other hand, suppose that we have $s, t : \mathrm{Spec} R \rightarrow \mathrm{Sh}_K$, a $G_{\mathbb{Z}_p}$ -structure preserving trivialization $\beta : \underline{H}_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{p,t}$ of p -adic local systems over $\mathrm{Spec} R$, and a $G_{\mathbb{Z}_p}$ -structure preserving p -quasi-isogeny $f : \mathcal{A}_s \dashrightarrow \mathcal{A}_t$ of type $[[g]]$. Consider the composition

$$\gamma : \mathbf{H}_{p,s} \xrightarrow{(f^*)^{-1}} \mathbf{H}_{p,t}[p^{-1}] \xrightarrow[\cong]{\beta^{-1}} \underline{H}_{\mathbb{Z}_p}[p^{-1}].$$

For any geometric point $y : \mathrm{Spec} L \rightarrow \mathrm{Spec} R$, since f has type $[[g]]$, we have

$$\gamma_y(\mathbf{H}_{p,s \circ y}) = g_1^{-1} H_{\mathbb{Z}_p} \subset H_{\mathbb{Q}_p}$$

for some coset $g_1 K_p \subset [[g]]$, which is locally constant on $\mathrm{Spec} R$. In this way, we get a map

$$\mathrm{Sh}_{K^p} \times_{\mathrm{Sh}_{K,t}} \mathrm{QIsog}_{G,[[g]]} \rightarrow \mathrm{Sh}_{K^p} \times [K_p g K_p / K_p] \xrightarrow{\cong} \mathrm{Sh}_{K^p} \times_{\mathrm{Sh}_{K,t_g}} \mathrm{Sh}_{K_g}.$$

One checks that it is K_p -equivariant and so descends to a map $\mathrm{QIsog}_{G,[[g]]} \rightarrow \mathrm{Sh}_{K_g}$.

To construct the inverse, suppose now that we have $t : \mathrm{Spec} R \rightarrow \mathrm{Sh}_K$, $\beta : \underline{H}_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{p,t}$ and a locally constant map $\gamma : \mathrm{Spec} R \rightarrow K_p g^{-1} K_p / K_p$. We want to show that this arises from a pair (s, f) , where $s \in \mathrm{Sh}_K(R)$ and $f \in \mathrm{QIsog}_{G,[[g]]}(s, t)$. Without loss of generality, we can assume that γ is constant and corresponds to a coset $g_1 K_p \subset [[g]]$. We now get a p -adic local system

$$\beta[p^{-1}](g_1 K_p) \subset \mathbf{H}_{p,t}[p^{-1}],$$

which is of the form $(f^*)^{-1}(T_p(\mathcal{A})^\vee)$ for a p -quasi-isogeny $\mathcal{A} \dashrightarrow \mathcal{A}_t$ of abelian schemes over $\mathrm{Spec} R$. One checks now that we have $\mathcal{A} \simeq \mathcal{A}_s$ where $s \in \mathrm{Sh}_K(R)$ is the image of (t, β, γ) under the map

$$\mathrm{Sh}_{K^p} \times [K_p g K_p / K_p] \xrightarrow{\cong} \mathrm{Sh}_{K^p} \times_{\mathrm{Sh}_{K,t_g}} \mathrm{Sh}_{K_g} \xrightarrow{\mathrm{pr}_2} \mathrm{Sh}_{K_g} \xrightarrow{s_g} \mathrm{Sh}_K.$$

This both completes the construction of the desired isomorphism and also shows that it is compatible with the maps s and s_g . \square

2.2. Ordinary isogenies and p -Hecke correspondences. Fix a place $v|p$ of E .

Definition 2.2.1. Suppose that κ is algebraically closed in characteristic p and that we have $s_0, t_0 \in \mathcal{S}_{K,(v)}(\kappa)$. A p -quasi-isogeny $f : \mathcal{A}_{s_0} \dashrightarrow \mathcal{A}_{t_0}$ **preserves G -structure** if the associated isomorphism $f^* : \mathbf{H}_{\mathrm{cris},t_0}[p^{-1}] \xrightarrow{\cong} \mathbf{H}_{\mathrm{cris},s_0}[p^{-1}]$ carries $\{s_{\alpha,\mathrm{cris},t_0}\}$ to $\{s_{\alpha,\mathrm{cris},s_0}\}$. In general, if we have $s, t : \mathrm{Spf} R \rightarrow \mathcal{S}_{K,v}$ for some p -complete ring R , then a p -quasi-isogeny $\mathcal{A}_s \dashrightarrow \mathcal{A}_t$ **preserves G -structure** if its restriction over every algebraically closed point of $\mathrm{Spf} R$ preserves G -structure.

Notation 2.2.2. With $s, t : \mathrm{Spf} R \rightarrow \mathcal{S}_{K,v}$ as above, write $\mathrm{QIsog}_G(s, t)$ for the space of p -quasi-isogenies from \mathcal{A}_s to \mathcal{A}_t preserving G -structure, and $\mathrm{Isog}_G(s, t)$ for its subspace consisting of honest isogenies $\mathcal{A}_s \rightarrow \mathcal{A}_t$. If $s = t$, we will write $\mathrm{Aut}_G^\circ(s)$ instead of $\mathrm{QIsog}_G(s, s)$.

From here on, we will restrict ourselves to the completion along the ordinary locus $\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}}$ with Assumption 1.2.15 in force.

Definition 2.2.3. If we have $s, t : \mathrm{Spf} R \rightarrow \widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}}$ for some p -complete ring R , a quasi-isogeny $\mathcal{G}_s^{\acute{e}t} \times \mathcal{G}_s^{\mathrm{mult}} \dashrightarrow \mathcal{G}_t^{\acute{e}t} \times \mathcal{G}_t^{\mathrm{mult}}$ **preserves M -structure** if:

- (1) It preserves the product structure on both sides;
- (2) Its restriction over every algebraically closed point of $\mathrm{Spf} R$ preserves G -structure.

Remark 2.2.4. If $R = \kappa$ is an algebraically closed field, then any G -structure preserving quasi-isogeny $\mathcal{A}_{s_0} \dashrightarrow \mathcal{A}_{t_0}$ is automatically M -structure preserving. Here, we are using the property that, for all $x_0 \in \mathcal{S}_{K,k(v)}^{\text{ord}}(\kappa)$, we have a canonical isomorphism $\mathcal{G}_{x_0} \xrightarrow{\cong} \mathcal{G}_{x_0}^{\text{ét}} \times \mathcal{G}_{x_0}^{\text{mult}}$.

Notation 2.2.5. With $s, t : \text{Spf } R \rightarrow \mathcal{S}_{K,v}$ as above, write $\text{QIsog}_M(s, t)$ for the space of quasi-isogenies from $\mathcal{G}_s^{\text{ét}} \times \mathcal{G}_s^{\text{mult}}$ to $\mathcal{G}_t^{\text{ét}} \times \mathcal{G}_t^{\text{mult}}$ preserving M -structure, and $\text{Isog}_M(s, t)$ for its subspace consisting of honest isogenies $\mathcal{A}_s \rightarrow \mathcal{A}_t$.

Definition 2.2.6. If κ is algebraically closed and we have $f \in \text{QIsog}_M(s_0, t_0) = \text{QIsog}_G(s_0, t_0)$ for $s_0, t_0 \in \mathcal{S}_{K,k(v)}^{\text{ord}}(\kappa)$, the **type** of f is the double coset $[[m(f)]] \in M_{\mathbb{Z}_p}(\mathbb{Z}_p) \backslash M(\mathbb{Q}_p) / M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ determined as follows. Choose trivializations

$$\alpha : W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{cris}, s_0} ; \beta : W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{cris}, t_0}$$

witnessing the ordinarity of s_0 and t_0 : these are each well-defined up to multiplication by $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$. Then $[[m(f)]]$ is the class of $\beta \circ (f^*)^{-1} \circ \alpha^{-1} \in M(\mathbb{Q}_p)$.

Definition 2.2.7. If we have $s, t : \text{Spf } R \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$, $f \in \text{Isog}_G(s, t)$ ⁵ and $[[m]] \in M_{\mathbb{Z}_p}(\mathbb{Z}_p) \backslash M(\mathbb{Q}_p) / M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ then we will say that f **has type** $[[m]]$ if its restriction to each algebraically closed point of $\text{Spf } R$ has this property. We will write $\text{QIsog}_{G,[[m]]}(s, t)$ for the subspace consisting of such f . We also have the subspace $\text{QIsog}_{M,[[m]]}(s, t) \subset \text{QIsog}_{G,[[m]]}(s, t)$ consisting of those f that are also M -structure preserving.

Example 2.2.8. Suppose that R is an \mathbb{F}_p -algebra and that we have $s : \text{Spec } R \rightarrow \mathcal{S}_{K,k(v)}^{\text{ord}}$. Hitting s with the absolute Frobenius of R gives another point $s^{(1)} \in \mathcal{S}_{K,k(v)}^{\text{ord}}(R)$ and we have the relative Frobenius map $F_s : \mathcal{A}_s \rightarrow \mathcal{A}_{s^{(1)}}$. This is a p -isogeny, and is in fact M -structure preserving with type $[\mu_v(p)^{-1}]$.

Remark 2.2.9. Definition 2.2.7 gives something non-empty only when $[[m]]$ is contained in $M(\mathbb{Q}_p) \cap \text{End}(H_{\mathbb{Z}_p})$. This will suffice for our purposes, but one can extend the definition usefully to all double cosets $[[m]]$ in the following way: Choose $r \geq 0$ such that $[[p^r m]]$ is contained in $M(\mathbb{Q}_p) \cap \text{End}(H_{\mathbb{Z}_p})$, and define $\text{QIsog}_{G,[[m]]}(s, t)$ to be the subspace of $\text{QIsog}_G(s, t)$ consisting of f such that $p^r f$ lies in $\text{QIsog}_{G,[[p^r m]]}(s, t)$. By definition, multiplication-by- p^r , gives an isomorphism $\text{QIsog}_{G,[[m]]}(s, t) \xrightarrow{\cong} \text{QIsog}_{G,[[p^r m]]}(s, t)$. We have similar subspaces with G replaced with M in the subscript.

Notation 2.2.10. Fix a double coset $[[m]] \in M_{\mathbb{Z}_p}(\mathbb{Z}_p) \backslash M(\mathbb{Q}_p) / M_{\mathbb{Z}_p}(\mathbb{Z}_p)$. Let

$$\widehat{\text{QIsog}}_{G,[[m]]}^{\text{ord}} \xrightarrow{(s,t)} \widehat{\mathcal{S}}_{K,v}^{\text{ord}} \times \widehat{\mathcal{S}}_{K,v}^{\text{ord}} ; \widehat{\text{QIsog}}_{M,[[m]]}^{\text{ord}} \xrightarrow{(s,t)} \widehat{\mathcal{S}}_{K,v}^{\text{ord}} \times \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$$

be the formal presheaves given by $\text{QIsog}_{G,[[m]]}(s, t)$ and $\text{QIsog}_{M,[[m]]}(s, t)$, respectively.

Remark 2.2.11. If $G_{\mathbb{Z}_p}$ is reductive, then so is $M_{\mathbb{Z}_p}$, and for the choice of maximal torus $T \subset G_{\mathbb{Z}_p}$ as in Remark 1.2.3, the Cartan decomposition tells us that every double coset $[[m]]$ admits a unique representative of the form $\lambda(p)$ for some cocharacter $\lambda \in X_*(T)$ defined over \mathbb{Z}_p and dominant *with respect to* $M_{\mathbb{Z}_p}$. In this situation, we will replace the $[[m]]$ in the subscript with λ , and write $\text{QIsog}_{G,\lambda}(s, t)$ and $\text{QIsog}_{M,\lambda}(s, t)$ instead. This notation will also be used with λ replaced by any other cocharacter in its $M_{\mathbb{Z}_p}$ -conjugacy class.

Construction 2.2.12. Suppose that we have $s, t : \text{Spf } R \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$, $f \in \text{QIsog}_{M,[[m]]}(s, t)$ and a section

$$\beta : R \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\cong} \mathcal{G}_t^{\text{ét}} \times \mathcal{G}_t^{\text{mult}}$$

of $\widehat{\text{Ig}}_{K,v}^{\text{ord}}(t)$. For any section $\alpha \in \widehat{\text{Ig}}_{K,v}^{\text{ord}}(s)$, the composition

$$R \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\alpha} \mathcal{G}_s^{\text{ét}} \times \mathcal{G}_s^{\text{mult}} \dashrightarrow \mathcal{G}_t^{\text{ét}} \times \mathcal{G}_t^{\text{mult}} \xrightarrow{\beta^{-1}} R \otimes_{\mathbb{Z}_p} \mathcal{G}_0$$

⁵It is important to work with honest isogenies here, since we will have use for their deformation theory. Quasi-isogenies on the other hand always deform uniquely.

⁶The product is of formal stacks over $\text{Spf } \mathcal{O}_{E,v}$.

is a self-quasi-isogeny that preserves M -structure, and so yields a locally constant map $\mathrm{Spf} R \rightarrow M(\mathbb{Q}_p)$. The induced map to $M(\mathbb{Q}_p)/M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is independent of the choice of α and lands in $M_{\mathbb{Z}_p}(\mathbb{Z}_p)mM_{\mathbb{Z}_p}(\mathbb{Z}_p)/M_{\mathbb{Z}_p}(\mathbb{Z}_p)$. In this way, we obtain a canonical map

$$(2.2.12.1) \quad \widehat{\mathrm{I}}\mathfrak{g}_{K,v}^{\mathrm{ord}} \times_{\widehat{\mathcal{S}}_{K,v,t}^{\mathrm{ord}}} \widehat{\mathrm{Q}}\mathrm{Isog}_{M,[[m]]}^{\mathrm{ord}} \rightarrow \widehat{\mathrm{I}}\mathfrak{g}_{K,v}^{\mathrm{ord}} \times \underline{M_{\mathbb{Z}_p}(\mathbb{Z}_p)mM_{\mathbb{Z}_p}(\mathbb{Z}_p)/M_{\mathbb{Z}_p}(\mathbb{Z}_p)}.$$

By noting that taking inverses of quasi-isogenies flips source and target and switches a type $[[m]]$ to its inverse $[[m^{-1}]]$, we also obtain a map

$$(2.2.12.2) \quad \widehat{\mathrm{I}}\mathfrak{g}_{K,v}^{\mathrm{ord}} \times_{\widehat{\mathcal{S}}_{K,v,s}^{\mathrm{ord}}} \widehat{\mathrm{Q}}\mathrm{Isog}_{M,[[m]]}^{\mathrm{ord}} \rightarrow \widehat{\mathrm{I}}\mathfrak{g}_{K,v}^{\mathrm{ord}} \times \underline{M_{\mathbb{Z}_p}(\mathbb{Z}_p)m^{-1}M_{\mathbb{Z}_p}(\mathbb{Z}_p)/M_{\mathbb{Z}_p}(\mathbb{Z}_p)}.$$

Notation 2.2.13. For a double coset $[[m]] = M_{\mathbb{Z}_p}(\mathbb{Z}_p)mM_{\mathbb{Z}_p}(\mathbb{Z}_p)$, set

$$\widehat{\mathrm{I}}\mathfrak{g}_{[[m]]}^{\mathrm{ord}} \stackrel{\mathrm{defn}}{=} \widehat{\mathrm{I}}\mathfrak{g}_{K,v}^{\mathrm{ord}} \times^{M_{\mathbb{Z}_p}(\mathbb{Z}_p)} \underline{M_{\mathbb{Z}_p}(\mathbb{Z}_p)mM_{\mathbb{Z}_p}(\mathbb{Z}_p)/M_{\mathbb{Z}_p}(\mathbb{Z}_p)}.$$

Proposition 2.2.14. *The maps (2.2.12.1) and (2.2.12.2) descend to maps sitting in commuting diagrams of formal stacks*

$$\begin{array}{ccc} \widehat{\mathrm{Q}}\mathrm{Isog}_{M,[[m]]}^{\mathrm{ord}} & \longrightarrow & \widehat{\mathrm{I}}\mathfrak{g}_{[[m]]^{-1}}^{\mathrm{ord}} \\ \searrow s & & \swarrow \\ & \widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}} & \end{array} \quad ; \quad \begin{array}{ccc} \widehat{\mathrm{Q}}\mathrm{Isog}_{M,[[m]]}^{\mathrm{ord}} & \longrightarrow & \widehat{\mathrm{I}}\mathfrak{g}_{[[m]]}^{\mathrm{ord}} \\ \searrow t & & \swarrow \\ & \widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}} & \end{array}$$

where the diagonal maps are finite étale over $\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}}$.⁷

Proof. That the map descends amounts to checking K_p -equivariance, which is straightforward.

By construction, $\widehat{\mathrm{Q}}\mathrm{Isog}_{M,[[m]]}^{\mathrm{ord}}$ is isomorphic—for some $k \geq 1$ sufficiently large—to an open and closed substack of the formal stack $\widehat{\mathrm{Isog}}_{\mathfrak{g}_{p^k}}^{\mathrm{ord}}$ over $\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}} \times \widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}}$ whose fiber over a point (s, t) parameterizes product structure preserving p -isogenies

$$\mathcal{G}_s^{\mathrm{ét}} \times \mathcal{G}_s^{\mathrm{mult}} \rightarrow \mathcal{G}_t^{\mathrm{ét}} \times \mathcal{G}_t^{\mathrm{mult}}$$

of degree bounded by p^k . Via the projection onto the factor via s (resp. t) this formal stack is simply parameterizing pairs of finite flat subgroup schemes of $(\mathcal{G}_s^{\mathrm{ét}}, \mathcal{G}_s^{\mathrm{mult}})$ (resp. $(\mathcal{G}_t^{\mathrm{ét}}, \mathcal{G}_t^{\mathrm{mult}})$) of combined rank at most p^k . This is represented by a finite étale stack over $\widehat{\mathcal{S}}_{K,v}^{\mathrm{ord}}$. \square

Construction 2.2.15. Fix an algebraically closed field κ of characteristic p . For $m \in M(\mathbb{Q}_p) \cap \mathrm{End}(H_{\mathbb{Z}_p})$, write m^0, m^1 for the induced endomorphisms of $H_{\mathbb{Z}_p}^0$ and $H_{\mathbb{Z}_p}^1$. We then get two maps

$$_ \circ m^1, m^0 \circ _ : \mathrm{Lie} U_{\mu_v}^- \rightarrow \mathrm{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)$$

which in turn give us maps $\psi_m^1, \psi_m^0 : \widehat{T}_G \rightarrow \widehat{T}$ of formal tori over $W(\kappa)$.

Let $\widehat{T}_{G,m} \subset \widehat{T}_G \times \widehat{T}_G$ ⁸ be the diagonalizable formal group sitting in an exact sequence

$$1 \rightarrow \widehat{T}_{G,m} \xrightarrow{(s_m, t_m)} \widehat{T}_G \times \widehat{T}_G \xrightarrow{\psi_m^1 \circ \mathrm{pr}_1 - \psi_m^0 \circ \mathrm{pr}_2} \widehat{T}.$$

Remark 2.2.16. The maps $_ \circ m^1, m^0 \circ _$ are injective, and after inverting p one differs from the other by conjugation by m . This implies that the maps s_m and t_m are both finite flat maps that are purely inseparable mod- p : Indeed, the corresponding maps of character groups are isomorphisms of \mathbb{Q}_p -vector spaces after inverting p .

⁷We will see in the next subsection that the top arrows are in fact isomorphisms.

⁸The product here is over $\mathrm{Spf} W(\kappa)$.

Proposition 2.2.17. *Suppose that we have $(s_0, t_0, f) \in \widehat{\text{QIsog}}_{\mathcal{G}, [m]}^{\text{ord}}(\kappa)$, and let $\widehat{U}_{(s_0, t_0, f)}$ be the deformation functor for $\widehat{\text{QIsog}}_{\mathcal{G}, [m]}^{\text{ord}}$ on $\text{Art}_W(\kappa)$. Then we can choose isomorphisms $\widehat{T}_G \xrightarrow{\cong} \widehat{U}_{s_0}$ and $\widehat{T}_G \xrightarrow{\cong} \widehat{U}_{t_0}$ as in Proposition 1.2.13 such that we have a commuting diagram*

$$\begin{array}{ccc} \widehat{T}_{G,m} & \xrightarrow{(s_m, t_m)} & \widehat{T}_G \times \widehat{T}_G \\ \cong \downarrow & & \downarrow \cong \\ \widehat{U}_{(s_0, t_0, f)} & \xrightarrow{(s, t)} & \widehat{U}_{s_0} \times \widehat{U}_{t_0} \end{array}$$

where the left vertical arrow is also an isomorphism.

Proof. Choose isomorphisms

$$\mathcal{G}_{t_0} \xleftarrow[\cong]{\beta} \kappa \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow[\cong]{\alpha} \mathcal{G}_{s_0}$$

witnessing the ordinarity of t_0 and s_0 such that the G -structure preserving self-isogeny

$$\beta^{-1} \circ f \circ \alpha : \kappa \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \rightarrow \kappa \otimes_{\mathbb{Z}_p} \mathcal{G}_0$$

corresponds to $m \in M(\mathbb{Q}_p) = \text{Aut}_G^\circ(\mathcal{G}_0)$. By Proposition 1.2.13, we can use α and β to get isomorphisms $\widehat{T}_G \xrightarrow{\cong} \widehat{U}_{s_0}$ and $\widehat{T}_G \xrightarrow{\cong} \widehat{U}_{t_0}$.

Over $\widehat{T}_G \times \widehat{T}_G \simeq \widehat{U}_{s_0} \times \widehat{U}_{t_0}$, we have two extensions \mathcal{G}_s and \mathcal{G}_t of $\mathcal{G}_0^{\text{ét}}$ by $\mathcal{G}_0^{\text{mult}}$, and the locus $\widehat{U}_{(s_0, t_0, f)}$ is where the isogeny f lifts to an isogeny between these extensions. Now, the étale part $f^{\text{ét}}$ (resp. multiplicative part f^{mult}) of f lifts uniquely, and is given by

$$\begin{aligned} \mathcal{G}_0^{\text{ét}} &= \underline{\text{Hom}}(H_{\mathbb{Z}_p}^0, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{-\circ m^0} \underline{\text{Hom}}(H_{\mathbb{Z}_p}^0, \mathbb{Q}_p/\mathbb{Z}_p) = \mathcal{G}_0^{\text{ét}} \\ (\text{resp. } \mathcal{G}_0^{\text{mult}} &= \underline{\text{Hom}}(H_{\mathbb{Z}_p}^1, \mu_{p^\infty}) \xrightarrow{-\circ m^1} \underline{\text{Hom}}(H_{\mathbb{Z}_p}^1, \mu_{p^\infty}) = \mathcal{G}_0^{\text{mult}}). \end{aligned}$$

We now obtain two further extensions over $\widehat{T}_G \times \widehat{T}_G$: One by pulling back \mathcal{G}_t along $f^{\text{ét}}$ and the other by pushing forward along f^{mult} . The locus where f lifts is the same as the locus where these two extensions are isomorphic. To finish, one just checks that the first extension is obtained from the map $\psi_m^0 : \widehat{T}_G \rightarrow \widehat{T}$, and the second from the map $\psi_m^1 : \widehat{T}_G \rightarrow \widehat{T}$, where we are using Serre-Tate ordinary theory to identify \widehat{T} with the deformation space of such extensions. \square

Corollary 2.2.18. *The map $\widehat{\text{QIsog}}_{\mathcal{G}, [m]}^{\text{ord}} \rightarrow \widehat{\text{QIsog}}_{M, [m]}^{\text{ord}}$ is a finite flat homeomorphism.*

Proof. By Remark 2.2.4, the map is an isomorphism on algebraically closed points. Moreover, by definition, $\widehat{\text{QIsog}}_{\mathcal{G}, [m]}^{\text{ord}}$ is an open and closed substack of the formal stack over $\widehat{\mathcal{S}}_{K,v}^{\text{ord}} \times \widehat{\mathcal{S}}_{K,v}^{\text{ord}}$ parameterizing isogenies between $\text{pr}_1^* \mathcal{A}$ and $\text{pr}_2^* \mathcal{A}$. Therefore, it is enough now to know that the map is a finite flat homeomorphism on completions at algebraically closed points. This follows from Proposition 2.2.17 and Remark 2.2.16. \square

2.3. Comparison between the two notions of p -Hecke correspondences. Here, we will look at the relationship between p -Hecke correspondences over the generic fiber and over the ordinary locus.

Remark 2.3.1. Suppose that $s_0, t_0 \in \mathcal{S}_{K, k(v)}^{\text{ord}}(\kappa)$ are algebraically closed points, that $L/W(\kappa)[1/p]$ is a finite extension and that $s, t \in \mathcal{S}_{K,v}(\mathcal{O}_L)$ are lifts of s_0, t_0 . Write s_η, t_η for the associated L -valued points of Sh_K . Viewing s, t as $\text{Spf } \mathcal{O}_L$ -valued points of $\widehat{\mathcal{S}}_{K,v}^{\text{ord}}$, we obtain the space $\text{QIsog}_G(s, t)$. On the other hand, we also have the space $\text{QIsog}_G(s_\eta, t_\eta)$ from § 2.1.

Proposition 2.3.2. *There is a canonical isomorphism*

$$\text{Isog}_G(s, t) \xrightarrow{\cong} \text{Isog}_G(s_\eta, t_\eta).$$

Furthermore, if s and t are the canonical lifts of s_0 and t_0 , respectively, then the natural map $\text{QIsog}_G(s, t) \rightarrow \text{QIsog}_G(s_0, t_0)$ is also an isomorphism.

Proof. Write $\text{Isog}(\mathcal{A}_s, \mathcal{A}_t)$ (resp. $\text{Isog}(\mathcal{A}_{s_\eta}, \mathcal{A}_{t_\eta})$) for the space of p -isogenies from \mathcal{A}_s to \mathcal{A}_t (resp. between their generic fibers). By the Néronian property of abelian schemes, the natural map

$$\text{Isog}(\mathcal{A}_s, \mathcal{A}_t) \rightarrow \text{Isog}(\mathcal{A}_{s_\eta}, \mathcal{A}_{t_\eta})$$

is a bijection. Therefore, we only have to check that the two *a priori* different notions of G -structure preservation, one using the crystalline realization (on the source) and the other using the étale realization (on the target) are compatible. This follows from assertion (2) of Proposition 1.1.6.

The second assertion follows from Serre-Tate ordinary deformation theory, which shows that the natural map

$$\text{QIsog}(\mathcal{A}_s, \mathcal{A}_t) \rightarrow \text{QIsog}(\mathcal{A}_{s_0}, \mathcal{A}_{t_0})$$

is a bijection when s and t are canonical lifts. \square

Remark 2.3.3. Let ι be the inverse of the first isomorphism from Proposition 2.3.2. Given $[[g]] \in K_p \backslash G(\mathbb{Q}_p) / K_p$, one sees, using Proposition 1.4.8, that we have

$$\iota \left(\text{QIsog}_{G, [[g]]}(s_\eta, t_\eta) \right) \subset \bigsqcup_{[[m]] \in \mathcal{S}([[g]])} \text{QIsog}_{G, [[m]]}(s, t)$$

where $\mathcal{S}([[g]]) \subset M_{\mathbb{Z}_p}(\mathbb{Z}_p) \backslash M(\mathbb{Q}_p) / M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ consists of those double cosets which admit representatives m in the image of $K_p g K_p \cap P^-(\mathbb{Q}_p)$.

Remark 2.3.4. We can globalize Proposition 2.3.2. Over $\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}} \times \widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}$, we have the analytification $\widehat{\text{Isog}}_G^{\text{ord,an}}$, and we also have the restriction of the rigid analytification $\text{Isog}_G^{\text{an}}$ of the stack of G -structure preserving isogenies in the generic fiber. These are both finite étale over each factor of the product, and so Proposition 2.3.2 shows that we actually have an isomorphism

$$\text{Isog}_G^{\text{an}}|_{\widehat{\mathcal{S}}_{K,v}^{\text{ord,an}} \times \widehat{\mathcal{S}}_{K,v}^{\text{ord,an}}} \xrightarrow{\cong} \widehat{\text{Isog}}_G^{\text{ord,an}}.$$

Corollary 2.3.5. *The top horizontal arrows in Proposition 2.2.14 are isomorphisms.*

Proof. It is enough to check that the horizontal arrow in the second diagram is an isomorphism. Since both source and target are finite étale over $\widehat{\mathcal{S}}_{K,v}^{\text{ord}}$, it is enough to know that the map is an isomorphism on $\kappa = \overline{\mathbb{F}}_p$ -points. Fix $t_0 \in \mathcal{S}_{K,k(v)}^{\text{ord}}(\kappa)$, and let $\text{QIsog}_{G,t_0}^{\text{ord}}(\kappa)$ be the fiber of $t : \widehat{\text{QIsog}}_G^{\text{ord}}(\kappa) \rightarrow \widehat{\mathcal{S}}_{K,v}^{\text{ord}}(\kappa)$ over t_0 . For a fixed choice of isomorphism $\beta : W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_{\text{cris},t_0}$ as in Definition 2.2.6, we obtain a map

$$\begin{aligned} \text{QIsog}_{G,t_0}^{\text{ord}}(\kappa) &\rightarrow M(\mathbb{Q}_p) / M_{\mathbb{Z}_p}(\mathbb{Z}_p) \\ f &\mapsto m(f) M_{\mathbb{Z}_p}(\mathbb{Z}_p), \end{aligned}$$

where $m(f) \in M(\mathbb{Q}_p)$ is the class of $\beta \circ (f^*)^{-1} \circ \alpha^{-1}$ for any choice of isomorphism α as in the same definition. It is now enough to know that this map is an isomorphism.

For this, let $t \in \mathcal{S}_K(W(\kappa))$ be the canonical lift of t_0 and let $t_\eta \in \text{Sh}_K(L)$ with $L = W(\kappa)[p^{-1}]$ be its generic fiber. We will now show that the map is injective. Indeed, if we have $f \in \text{QIsog}_G(s_0, t_0)$ and $\tilde{f} \in \text{QIsog}_G(\tilde{s}_0, t_0)$ such that $m(f) M_{\mathbb{Z}_p}(\mathbb{Z}_p) = m(\tilde{f}) M_{\mathbb{Z}_p}(\mathbb{Z}_p)$, then, since $m(\tilde{f})^{-1} m(f) \in M_{\mathbb{Z}_p}(\mathbb{Z}_p)$, one finds that $\tilde{f}^{-1} \circ f \in \text{QIsog}_G(s_0, \tilde{s}_0)$ is an isomorphism $\mathcal{A}_{s_0} \xrightarrow{\cong} \mathcal{A}_{\tilde{s}_0}$. If s, \tilde{s} are the canonical lifts, the second assertion in Proposition 2.3.2 shows that this isomorphism lifts to an element in $\text{QIsog}_G(s, \tilde{s})$ that is an isomorphism of abelian schemes. One now finds from Proposition 2.1.6 that this is only possible if s and \tilde{s} , along with the corresponding lifts of f and \tilde{f} , yield the same point of the fiber of $t : \text{QIsog}_G \rightarrow \text{Sh}_K$ over t_η .

To show surjectivity, fix an algebraic closure \bar{L} for L , and let $\text{QIsog}_{G,t_\eta}(\bar{L})$ be the fiber of $t : \text{QIsog}_G(\bar{L}) \rightarrow \text{Sh}_K(\bar{L})$ over t_η . We then have a commutative diagram

$$\begin{array}{ccc} \text{QIsog}_{G,t_\eta}(\bar{L}) & \xrightarrow[\simeq]{f \mapsto g(f)G_{\mathbb{Z}_p}(\mathbb{Z}_p)} & G(\mathbb{Q}_p)/G_{\mathbb{Z}_p}(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \text{QIsog}_{G,t_0}^{\text{ord}}(\kappa) & \xrightarrow{f \mapsto m(f)M(\mathbb{Q}_p)} & M(\mathbb{Q}_p)/M_{\mathbb{Z}_p}(\mathbb{Z}_p) \end{array}$$

Here, the top arrow is the isomorphism obtained by taking the fibers of the isomorphism from Proposition 2.1.6 over $t_\eta \in \text{Sh}_K(\bar{L})$ in the second factor and taking the disjoint union over all $[[g]]$. The vertical arrow on the left is obtained from Remark 2.3.4 and the reduction map.

It is enough to show that the right vertical map is surjective. Unwinding definitions, and using Proposition 1.4.8, one gets the following description of this map: Write $t_{\bar{\eta}}$ for the points t_η viewed as a \bar{L} -valued point. Using the Iwasawa decomposition

$$G(\mathbb{Q}_p) = U_{\mu_v}^-(\mathbb{Q}_p)M(\mathbb{Q}_p)G_{\mathbb{Z}_p}(\mathbb{Z}_p),$$

we can write $g \in G(\mathbb{Q}_p)$ in the form $n^-(g)m(g)k(g)$, where the coset $m(g)M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is canonically determined. The right vertical map sends $gG_{\mathbb{Z}_p}(\mathbb{Z}_p)$ to $m(g)M_{\mathbb{Z}_p}(\mathbb{Z}_p)$. This finishes the proof of surjectivity. \square

3. AN ARGUMENT OF CHAI AND HIDA

The purpose of this section is to abstract some ideas due to Chai and Hida on a ‘pure thought’ study of the monodromy of Igusa towers over Shimura varieties, and apply them to the particular situation of ordinary loci. All the key ideas can already be found in [Hid11] and [CO11]. It is also possible that the main statement here can be deduced from the very general results of van Hoften and Xiao [vX23].

3.1. The abstract setup.

Proposition 3.1.1. *Let H be a connected reductive group over \mathbb{Q} such that $H_{\mathbb{Q}_p}$ contains a maximal torus that splits over a cyclic extension of \mathbb{Q}_p (this hypothesis holds in particular when H is unramified at p). Then H satisfies weak approximation with respect to $\{p, \infty\}$; that is, $H(\mathbb{Q})$ is dense in $H(\mathbb{Q}_p) \times H(\mathbb{R})$.*

Proof. This is essentially contained in [PR94]. If H is semi-simple and simply connected, the result follows directly from Theorem 7.8 of *op. cit.* In general, we find from Proposition 2.11 of *loc. cit.* that there are quasi-trivial⁹ tori T_1 and T_2 over \mathbb{Q} , and an integer $m \geq 1$ such that there is a central isogeny: $(H^{\text{sc}})^m \times T_1 \rightarrow H^m \times T_2$. In fact, the proof of this result shows that we can choose T_1 and T_2 to have the same splitting field as the maximal central torus of H . It is easy to see H satisfies weak approximation with respect to $\{p, \infty\}$ whenever $H^m \times T_2$ does, so we can replace H by the latter group and assume that it admits a central cover $H_1 \rightarrow H$ where H_1 is a product of a semi-simple, simply connected group with a quasi-trivial torus.

Let F be the kernel of $H_1 \rightarrow H$: It is a central sub-group of H_1 , and so, by our hypothesis, splits over a cyclic extension of \mathbb{Q}_p . The result now follows from Proposition 7.10 and Corollary 2 in Ch. 7 of *op. cit.* \square

Corollary 3.1.2. *Let the notation be as in the hypotheses of Proposition 3.1.1 above, and let $H_{\mathbb{Z}_{(p)}}$ be a smooth group scheme over $\mathbb{Z}_{(p)}$ with generic fiber H . For any $\mathbb{Z}_{(p)}$ -algebra R set $H(R) = H_{\mathbb{Z}_{(p)}}(R)$. With the hypotheses as in the proposition, for any integer $n \geq 1$, the map*

$$H(\mathbb{Z}_{(p)}) \rightarrow H(\mathbb{Z}/p^n\mathbb{Z})$$

is surjective.

Proof. Let $P_n = \ker(H(\mathbb{Z}_p) \rightarrow H(\mathbb{Z}/p^n\mathbb{Z}))$. By (3.1.1), $H(\mathbb{Q})P_n = H(\mathbb{Q}_p)$. We now have:

$$H(\mathbb{Z}_p) = H(\mathbb{Q}_p) \cap H(\mathbb{Z}_p) = H(\mathbb{Q})P_n \cap H(\mathbb{Z}_p) = H(\mathbb{Z}_{(p)})P_n.$$

The corollary now follows, since the map $H(\mathbb{Z}_p) \rightarrow H(\mathbb{Z}/p^n\mathbb{Z})$ is surjective by the smoothness of $H_{\mathbb{Z}_{(p)}}$. \square

⁹This means that the Galois representation attached to the character group is a permutation representation.

Notation 3.1.3. Suppose that S is an algebraic or formal algebraic stack, and that we have an $H(\mathbb{Z}_p)$ -torsor over S given by a sequence $\mathcal{P} = \{\mathcal{P}_n\}_{n \geq 1}$ of compatible finite étale $H(\mathbb{Z}/p^n\mathbb{Z})$ -torsors $\mathcal{P}_n \rightarrow S$. If X is an $H_{\mathbb{Z}_p}$ -equivariant scheme over \mathbb{Z}_p , for every $n \geq 1$, we set

$$\mathcal{P}_{X,n} = \mathcal{P}_n \times^{H(\mathbb{Z}/p^n\mathbb{Z})} X(\mathbb{Z}/p^n\mathbb{Z}).$$

3.1.4. Suppose now that G is a reductive group over \mathbb{Q} , T is a finite set of primes containing p , and that S is a scheme over $\mathbb{Z}_{(p)}$ equipped with an action of $G(\mathbb{A}_f^T)$. Assume that this action lifts to one on the $H(\mathbb{Z}_p)$ -torsor \mathcal{P} that commutes with the $H(\mathbb{Z}_p)$ -action. Suppose further that there is another reductive group J over \mathbb{Q} with the following properties:

- There exists an embedding

$$\psi : J_{\mathbb{A}_f^T} \hookrightarrow G_{\mathbb{A}_f^T}.$$

- There exists a (necessarily smooth) model $J_{\mathbb{Z}_{(p)}}$ for J over $\mathbb{Z}_{(p)}$, and an isomorphism

$$\varphi : J_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \xrightarrow{\sim} H_{\mathbb{Z}_p}$$

In particular, we have an embedding

$$\Phi : J(\mathbb{Z}_{(p)}) \xrightarrow{m \mapsto (\varphi(m), \psi(m))} H(\mathbb{Z}_p) \times G(\mathbb{A}_f^T),$$

inducing for every $n \geq 1$ a map

$$\Phi_n : J(\mathbb{Z}_{(p)}) \xrightarrow{m \mapsto (\varphi_n(m), \psi(m))} H(\mathbb{Z}/p^n\mathbb{Z}) \times G(\mathbb{A}_f^T).$$

Notation 3.1.5. Suppose that $Q \subset H_{\mathbb{Z}_p}$ is a closed \mathbb{Z}_p -subgroup scheme such that $X = H_{\mathbb{Z}_p}/Q$ is represented by a scheme. Let $J_{\mathbb{Z}_{(p)}}^{\text{sc}}$ (resp. $H_{\mathbb{Z}_p}^{\text{sc}}$) be the normalization of $J_{\mathbb{Z}_{(p)}}$ (resp. $H_{\mathbb{Z}_p}$) in J^{sc} (resp. H^{sc}), and let \tilde{Q} be the pre-image of Q in $H_{\mathbb{Z}_p}^{\text{sc}}$. Let $Z_{H,p} \subset H_{\mathbb{Z}_p}$ be the Zariski closure of the center $Z_H \subset H$.

Proposition 3.1.6. *Suppose that the following conditions hold:*

- (1) $\rho_G(G^{\text{sc}}(\mathbb{A}_f^T))$ acts trivially on $\pi_0(S)$;
- (2) For all $\ell \notin T$, $G_{\mathbb{Q}_\ell}$ is isotropic;
- (3) $G(\mathbb{A}_f^T)$ acts transitively on $\pi_0(S)$;
- (4) $\Phi(J(\mathbb{Z}_{(p)}))$ fixes a point $\varpi \in \pi_0(\mathcal{P})$;
- (5) Q contains $Z_{H,p}$;
- (6) The $\mathbb{Z}_{(p)}$ -group $J_{\mathbb{Z}_{(p)}}^{\text{sc}}$ (equivalently, the \mathbb{Z}_p -group $H_{\mathbb{Z}_p}^{\text{sc}}$) and the \mathbb{Z}_p -group \tilde{Q} are smooth with connected special fiber.

Then, for every $n \in \mathbb{Z}_{\geq 1}$, the map $\pi_0(\mathcal{P}_{X,n}) \rightarrow \pi_0(S)$ is a bijection.

Proof. We will need the following consequence of the Kneser-Tits conjecture (see [PR94, Theorem 7.6]): For any simply connected isotropic group D over \mathbb{Q}_ℓ , $D(\mathbb{Q}_\ell)$ does not admit any finite index sub-groups.

This, combined with hypotheses (1) and (2), implies that $\rho_G(\tilde{G}(\mathbb{Q}_\ell))$ acts trivially on $\pi_0(\mathcal{P}_{X,n})$ as well. Let ϖ be as in hypothesis (4), and let $F_\varpi \subset \pi_0(\mathcal{P}_{X,n})$ be the fiber over the image of ϖ in $\pi_0(S)$. By hypothesis (3), it is enough to show that F_ϖ is a singleton: Any other fiber is a translate of this by an element of $G(\mathbb{A}_f^T)$.

Hypothesis (4) implies that the subgroup

$$\tilde{H}_n := \{\varphi_n(m) : m \in J(\mathbb{Z}_{(p)}), \psi(m) \in \rho_G(\tilde{G}(\mathbb{A}_f^T))\} \subset H(\mathbb{Z}/p^n\mathbb{Z})$$

fixes the image of ϖ in $\pi_0(\mathcal{P}_{X,n})$ (which we once again denote by ϖ). Here, $\varphi_n : J(\mathbb{Z}_{(p)}) \rightarrow H(\mathbb{Z}/p^n\mathbb{Z})$ is obtained by reduction-mod- p^n from the map φ .

It is now enough to show that \tilde{H}_n acts transitively on the fiber $F_\varpi \subset \pi_0(\mathcal{P}_{X,n})$. For this, it is enough to know that it surjects onto $X(\mathbb{Z}/p^n\mathbb{Z})$ via the map induced by φ_n . Note, however that \tilde{H}_n contains $\rho_J(J^{\text{sc}}(\mathbb{Z}_{(p)}))$. Therefore, it is enough to show that the latter surjects onto $X(\mathbb{Z}/p^n\mathbb{Z})$.

First, note that the natural map

$$H_{\mathbb{Z}_p}^{\text{sc}}/\tilde{Q} \rightarrow H_{\mathbb{Z}_p}/Q$$

is an isomorphism of fppf sheaves over \mathbb{Z}_p . Indeed, it is a monomorphism by definition, and hypothesis (5) implies that it is also surjective.

Now, hypothesis (6) ensures that $H_{\mathbb{Z}_p}^{\text{sc}}$ and \tilde{Q} are smooth over \mathbb{Z}_p with connected fibers. Therefore, by Lang's theorem [Lan56], we have

$$\mathcal{X}(\mathbb{Z}/p^n\mathbb{Z}) = H^{\text{sc}}(\mathbb{Z}/p^n\mathbb{Z})/\tilde{Q}(\mathbb{Z}/p^n\mathbb{Z}).$$

It now follows from (3.1.2) that $\rho_J(J^{\text{sc}}(\mathbb{Z}(p)))$ maps surjectively onto $\mathcal{X}(\mathbb{Z}/p^n\mathbb{Z})$ via φ_n . \square

3.2. Hecke action on connected components of Shimura varieties. Let (G, X) be a Shimura datum. Write $g \mapsto g^{\text{ad}}$ for the natural map $G \rightarrow G^{\text{ad}}$, and set

$$G(\mathbb{Q})_+ = \{g \in G(\mathbb{Q}) : g^{\text{ad}} \in G(\mathbb{R})^0\}$$

where $G(\mathbb{R})^0 \subset G(\mathbb{R})$ is the topological connected component of the identity.

3.2.1. Fix a compact open subgroup $K_p \subset G(\mathbb{Q}_p)$. For any sufficiently small compact open $K \subset G(\mathbb{A}_f)$ of the form $K_p K^p$ with $K^p \subset G(\mathbb{A}_f^p)$, we obtain a Shimura variety $\text{Sh}_K = \text{Sh}_K(G, X)$ over the reflex field E with

$$\text{Sh}_K(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K).$$

We will be interested in the inverse limit

$$\text{Sh}_{K_p} = \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \text{Sh}_{K_p K^p},$$

which is a scheme over \mathbb{Q} . There is a natural action of $G(\mathbb{A}_f^p)$ on Sh_{K_p} obtained over \mathbb{C} via the right multiplication action on $G(\mathbb{A}_f)$.

Lemma 3.2.2. *Let $\pi_0(\text{Sh}_{K_p, \overline{\mathbb{Q}}})$ be the set of connected components of $\text{Sh}_{K_p, \overline{\mathbb{Q}}}$. Suppose that $G(\mathbb{Q})_+$ is dense in $G(\mathbb{Q}_p)$; for instance, this is the case if G satisfies the hypotheses of Proposition 3.1.1. Then:*

- (1) $G(\mathbb{A}_f^p)$ acts transitively on $\pi_0(\text{Sh}_{K_p, \overline{\mathbb{Q}}})$.
- (2) $\rho_G(\tilde{G}(\mathbb{A}_f^p))$ acts trivially on $\pi_0(\text{Sh}_{K_p, \overline{\mathbb{Q}}})$.

Proof. By [Del79, (2.1.3)], the set of connected components of $\text{Sh}_K(\mathbb{C})$ is a torsor under the group

$$\overline{\pi}_0 \pi(G)/K \stackrel{\text{defn}}{=} G(\mathbb{A}_f)/\rho_G(\tilde{G}(\mathbb{A}_f))G(\mathbb{Q})_+K,$$

where $G(\mathbb{Q})_+ \subset G(\mathbb{Q})$ is the stabilizer of a connected component of X , the action being induced from that of $G(\mathbb{A}_f)$ on itself via right multiplication. This implies that $\pi_0(\text{Sh}_{K_p, \overline{\mathbb{Q}}})$ is a torsor under

$$G(\mathbb{A}_f)/\rho_G(\tilde{G}(\mathbb{A}_f))G(\mathbb{Q})_+K_p.$$

Assertion (2) follows immediately from this; and assertion (1) follows from the additional observation that, under our hypotheses, $G(\mathbb{Q})_+K_p = G(\mathbb{Q}_p)$. \square

3.3. Kisin's analogue of Tate's theorem. Suppose now that (G, X) is of Hodge type and fix a place $v|p$ of E .

Construction 3.3.1. By Construction 1.1.2, for a given choice of symplectic representation H with lattice $H_{\mathbb{Z}}$, we obtain an integral model \mathcal{S}_K over \mathcal{O}_E , and hence a model $\mathcal{S}_{K, (v)}$ over $\mathcal{O}_{E, (v)}$. If we vary K^p , then we obtain a tower of such models $\mathcal{S}_{K_p K^p, (v)}$ where the transition maps $\mathcal{S}_{K_p K^p, (v)} \rightarrow \mathcal{S}_{K_p \tilde{K}^p, (v)}$ for $K^p \subset \tilde{K}^p$ are finite étale, and we can take the inverse limit

$$\mathcal{S}_{K_p, (v)} \stackrel{\text{defn}}{=} \varprojlim_{K^p} \mathcal{S}_{K_p K^p, (v)}.$$

The $G(\mathbb{A}_f^p)$ -action on Sh_{K_p} extends to one on $\mathcal{S}_{K_p, (v)}$: More precisely, over $\mathcal{S}_{K_p, (v)}$, we have a canonical isomorphism of \mathbb{A}_f^p -local systems

$$\epsilon^p : \underline{\mathbb{A}}_f^p \otimes_{\mathbb{Q}} H \xrightarrow{\sim} \hat{V}^p(\mathcal{A})^\vee,$$

where on the right hand side we have the dual of the prime-to- p adélic Tate module of \mathcal{A} , and a functorial point x of $\mathcal{S}_{K_p, (v)}$ gives rise to a pair $(\mathcal{A}_x, \epsilon_x)$, where \mathcal{A}_x is an abelian scheme and ϵ_x is a trivialization as above of its dual prime-to- p adélic Tate module. For $g \in G(\mathbb{A}_f^p)$, the functorial point $g \cdot x$ will be the unique one giving rise to the pair $(\mathcal{A}_x, \epsilon_x \circ g^{-1})$.

Construction 3.3.2. Let the notation be as in Lemma 1.1.6, but assume that κ is the algebraic closure of a finite field k , and, for any $m \in \mathbb{Z}_{\geq 1}$, let $k_m \subset \kappa$ be the unique degree m extension of k . Suppose also that x_0 is defined over k and write $x_{0,m}$ for the corresponding k_m -point of \mathcal{S}_{K_p, K^p} . Let $J_{x_{0,m}}^\circ = J_{\mathcal{G}_{x_{0,m}}}^\circ$ be the algebraic group over \mathbb{Q}_p such that $J_{x_0}^\circ(\mathbb{Q}_p) = \text{QIsog}_G(x_{0,m}, x_{0,m})$. In more detail, if $\mathbf{H}_{\text{cris}, x_{0,m}}$, then the G -structure up to isogeny from Lemma 1.1.6 gives us a subgroup $\text{GL}(\mathbf{H}_{x_{0,m}})$ that can be identified with $G_{W(k_m)[1/p]}$, and, for any \mathbb{Q}_p -algebra R , we have

$$J_{x_0}^\circ(R) = G(W(k_m) \otimes_{\mathbb{Z}_p} R) \cap (\underline{\text{Aut}}_F^\circ(\mathbf{H}_{\text{cris}, x_{0,m}}))(R) \subset \text{GL}(R \otimes_{W(k_m)} \mathbf{H}_{\text{cris}, x_{0,m}}),$$

where $\underline{\text{Aut}}_F^\circ(\mathbf{H}_{\text{cris}, x_{0,m}})$ is the algebraic group over \mathbb{Q}_p obtained as the group scheme of invertible elements in the ring $\text{End}_F(\mathbf{H}_{\text{cris}, x_{0,m}}[1/p])$ of endomorphisms of the F -isocrystal $\mathbf{H}_{\text{cris}, x_{0,m}}[1/p]$.

Let $\underline{\text{Aut}}^\circ(\mathcal{A}_{x_{0,m}})$ be the algebraic group over \mathbb{Q} obtained as the group scheme of invertible elements in $\text{End}(\mathcal{A}_{x_{0,m}})_{\mathbb{Q}}$; then we have a natural map of \mathbb{Q}_p -group schemes

$$\mathbb{Q}_p \otimes_{\mathbb{Q}} \underline{\text{Aut}}^\circ(\mathcal{A}_{x_{0,m}}) \rightarrow \underline{\text{Aut}}_F^\circ(\mathbf{H}_{\text{cris}, x_{0,m}})$$

We now define an algebraic group $I_{x_{0,m}}^\circ$ over \mathbb{Q} such that for any \mathbb{Q} -algebra R , we have

$$I_{x_{0,m}}^\circ(R) = \underline{\text{Aut}}^\circ(\mathcal{A}_{x_{0,m}})(R) \cap J_{x_{0,m}}^\circ(\mathbb{Q}_p \otimes_{\mathbb{Q}} R) \subset \underline{\text{Aut}}_F^\circ(\mathbf{H}_{\text{cris}, x_{0,m}})(\mathbb{Q}_p \otimes_{\mathbb{Q}} R).$$

Proposition 3.3.3. *With the notation as above:*

- (1) $I_{x_{0,m}}^\circ$ is a connected reductive group over \mathbb{Q} .
- (2) For m sufficiently divisible, the natural map

$$\mathbb{Q}_p \otimes_{\mathbb{Q}} I_{x_{0,m}}^\circ \rightarrow J_{x_{0,m}}^\circ$$

is an isomorphism of algebraic groups over \mathbb{Q}_p .

- (3) The action of $I_{x_{0,m}}^\circ$ on the prime-to- p adélic Tate module $\widehat{V}^p(\mathcal{A}_{x_0})$, via the trivialization

$$\epsilon_{x_0} : \mathbb{A}_f^p \otimes_{\mathbb{Z}(p)} H_{(p)} \xrightarrow{\cong} \widehat{V}^p(\mathcal{A}_{x_0})^\vee$$

gives rise to an embedding

$$\mathbb{A}_f^p \otimes_{\mathbb{Q}} I_{x_{0,m}}^\circ \hookrightarrow G_{\mathbb{A}_f^p}.$$

Proof. The reductivity of $I_{x_{0,m}}^\circ$ is a consequence of the fact that its real points are a compact Lie group modulo scalars: see (2.1.3) of [KMS21].

We can define a subgroup $I_{x_{0,m}}^{\circ, p} \subset I_{x_{0,m}}^\circ$ as the largest \mathbb{Q} -subgroup whose action on $\widehat{V}^p(\mathcal{A}_{x_0})^\vee$ gives rise, via ϵ_{x_0} , to an embedding $\mathbb{A}_f^p \otimes_{\mathbb{Q}} I_{x_{0,m}}^{\circ, p} \hookrightarrow G_{\mathbb{A}_f^p}$.

Then Corollary 2.2.10 of [KMS21] shows that, for m sufficiently divisible, the natural map $\mathbb{Q}_p \otimes_{\mathbb{Q}} I_{x_{0,m}}^{\circ, p} \rightarrow J_{x_{0,m}}^\circ$ is an isomorphism, which *a fortiori*, implies that $I_{x_{0,m}}^{\circ, p} = I_{x_{0,m}}^\circ$, and so verifies assertions (2) and (3). \square

3.4. Monodromy over the ordinary loci of Shimura varieties. We will now put ourselves in the situation of § 1.3 so that we have the formal Igusa tower $\widehat{\text{Ig}}_{K_p, K^p, v}^{\text{ord}} \rightarrow \widehat{\mathcal{S}}_{K_p, K^p, v}^{\text{ord}}$ with special fiber $\text{Ig}_{K_p, K^p, k(v)}^{\text{ord}} \rightarrow \mathcal{S}_{K_p, K^p, k(v)}^{\text{ord}}$. Taking the limit over K^p gives an $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -torsor

$$\text{Ig}_{K_p, k(v)}^{\text{ord}} \rightarrow \mathcal{S}_{K_p, k(v)}^{\text{ord}}.$$

Proposition 3.4.1. *Suppose that the following conditions hold:*

- (1) $M_{\mathbb{Z}_p}$ and $M_{\mathbb{Z}_p}^{\text{sc}}$ are smooth over \mathbb{Z}_p with connected special fiber.
- (2) M admits a maximal torus splitting over a cyclic extension of \mathbb{Q}_p (see the hypotheses of Proposition 3.1.1).
- (3) $Q \subset M_{\mathbb{Z}_p}$ is a closed subgroup scheme containing $Z_{M_{\mathbb{Z}_p}}$ with \widehat{Q} smooth with connected special fiber.
- (4) If $\overline{\mathbb{Z}_p} \subset \overline{\mathbb{Q}_p}$ is the subring of algebraic p -adic integers, then the map

$$\pi_0(\mathcal{S}_{K_p, \overline{\mathbb{F}_p}}^{\text{ord}}) \rightarrow \pi_0(\mathcal{S}_{K_p, \overline{\mathbb{Z}_p}}) \simeq \pi_0(\text{Sh}_{K_p, \overline{\mathbb{Q}_p}})$$

is a bijection.

- (5) For every prime $\ell \neq p$, $G_{\mathbb{Q}_\ell}$ is isotropic.

(6) There exists a finite field $k \subset \overline{\mathbb{F}}_p$, a compact open $K^p \subset G(\mathbb{A}_f^p)$ and $x_0 \in \mathcal{S}_{K^p K_p}^{\text{ord}}(k)$ such that the map

$$\text{Aut}_G^\circ(\mathcal{G}_{x_0}) \rightarrow \text{Aut}_G^\circ(\overline{\mathbb{F}}_p \otimes_k \mathcal{G}_{x_0})$$

is a bijection.

Then for $\mathbf{X} = M_{\mathbb{Z}_p}/Q$, and every $n \in \mathbb{Z}_{\geq 1}$, the natural map

$$\pi_0(\text{Ig}_{K^p, k(v), \mathbf{X}, n}^{\text{ord}}) \rightarrow \pi_0(\mathcal{S}_{K^p, k(v)}^{\text{ord}})$$

is a bijection.

Proof. This is an application of Proposition 3.1.6. In the notation there, we will take $S = \mathcal{S}_{K^p, (v)}^{\text{ord}}$, $H = J_0$, $\mathcal{P} = \widehat{\mathcal{P}}$, $G = G$, and $T = \{p\}$.

By Lemma 3.2.2 and Assumption (4), $\tilde{G}(\mathbb{A}_f^p)$ acts trivially on $\pi_0(\widehat{\mathcal{C}})$, which verifies Assumption 1 from (3.1.6).

Assumption (2) in (3.1.6) is Assumption (5) here.

Assumption (3) in (3.1.6) follows from Assumption (4) here and Lemma 3.2.2.

Assumptions (5) and (6) in (3.1.6) follow from Assumptions (1) and (3) here.

To finish, it remains to find the $\mathbb{Z}_{(p)}$ -group scheme $J_{\mathbb{Z}_{(p)}}$ as in (3.1.4) with generic fiber J satisfying Assumption (4).

Let x_0 be as in Assumption (6). By replacing k with a suitable extension, we can assume that $I_{x_0}^\circ = I_{x_0, 1}^\circ$, $J_{x_0}^\circ = J_{x_0, 1}^\circ$ are such that we have the isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Q}} I_{x_0}^\circ \xrightarrow{\cong} J_{x_0}^\circ$$

given to us by Proposition 3.3.3.

We will take $J = I_{x_0}^\circ$ and $J_{\mathbb{Z}_{(p)}}$ to be the Zariski closure of J in the $\mathbb{Z}_{(p)}$ -group $\underline{\text{Aut}}(\mathcal{A}_{x_0})_{(p)}$ obtained as the group scheme of invertible elements in the $\mathbb{Z}_{(p)}$ -algebra $\text{End}(\mathcal{A}_{x_0}) \otimes \mathbb{Z}_{(p)}$. We can also interpret $J_{\mathbb{Z}_{(p)}}$ as the largest subgroup of J acting on \mathcal{G}_{x_0} via *automorphisms* instead of self-quasi-isogenies.

Choose a lift $\eta \in \text{Ig}_{K^p, k(v)}^{\text{ord}}(\overline{\mathbb{F}}_p)$ of x_0 . This gives a G -structure preserving isomorphism $\overline{\mathbb{F}}_p \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\cong} \overline{\mathbb{F}}_p \otimes_k \mathcal{G}_{x_0}$, which we also denote by η . Assumption 6 now gives us isomorphisms

$$\varphi : J_{\mathbb{Q}_p} \xrightarrow{\cong} J_{x_0}^\circ \xrightarrow{\cong} \text{Aut}_G^\circ(\overline{\mathbb{F}}_p \otimes_k \mathcal{G}_{x_0}) \xrightarrow{\cong} \text{Aut}_G^\circ(\overline{\mathbb{F}}_p \otimes_{\mathbb{Z}_p} \mathcal{G}_0) \xrightarrow{\cong} M_{\mathbb{Q}_p},$$

where the penultimate isomorphism is obtained via conjugation by η , and the last one is from Lemma 1.3.4. This isomorphism maps $\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} J_{\mathbb{Z}_{(p)}}$ onto $M_{\mathbb{Z}_p}$.

Now, (3) of Proposition 3.3.3 shows that the action of $M(\mathbb{A}_f^p)$ on the prime-to- p adélic Tate module of $\mathcal{A}_{\bar{x}_0}$, along with the trivialization $\epsilon_{\bar{x}_0}$, gives an embedding $\psi : J_{\mathbb{A}_f^p} \hookrightarrow G_{\mathbb{A}_f^p}$.

In this way, we get a map

$$\Phi : J(\mathbb{Z}_{(p)}) \xrightarrow{(\varphi, \psi)} M_{\mathbb{Z}_p}(\mathbb{Z}_p) \times G(\mathbb{A}_f^p).$$

To verify Assumption (4), it is now enough to show that for all $m \in J(\mathbb{Z}_{(p)})$, $\Phi(m)$ fixes η . For this, note that $\Phi(m)(\eta)$ corresponds to the same underlying abelian variety $\mathcal{A}_{\bar{x}_0}$ but with $\epsilon_{\bar{x}_0}$ replaced by $\epsilon_{\bar{x}_0} \circ \psi(m)^{-1}$ and η replaced by the point of $\widehat{\text{Ig}}_{K, v}^{\text{ord}}$ above $\Phi(m)(\bar{x}_0)$ corresponding to the isomorphism $\varphi(m) \circ \eta$.

This means that the isomorphism

$$m : \mathcal{A}_{\bar{x}_0} \xrightarrow{\cong} \mathcal{A}_{\Phi(m)(\bar{x}_0)} = \mathcal{A}_{\bar{x}_0}$$

is G -structure preserving and carries η to the isomorphism $\varphi(m) \circ \eta$. Therefore, arguing as in the proof of Corollary 2.3.5, one sees that $\eta = \Phi(m)(\eta)$, thus finishing the proof of the proposition. \square

Definition 3.4.2. We will say that x_0 is **hypersymmetric** if it satisfies Assumption (6) above. The definition is originally due to Chai [CO11] in the case where $G = \text{GSp}(H)$.

Remark 3.4.3. Suppose that \mathcal{A}_{x_0} is hypersymmetric as an abelian variety; that is, suppose that the natural map

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{End}(\mathcal{A}_{x_0}) \rightarrow \text{End}(\overline{\mathbb{F}}_p \otimes_k \mathcal{G}_{x_0}).$$

is an isomorphism. Then it is immediate that x_0 is hypersymmetric in the sense of the definition above. In fact, it is enough to assume that \mathcal{A}_{x_0} is isogenous to a hypersymmetric abelian variety: any abelian variety isogenous to a hypersymmetric one is itself hypersymmetric.

Remark 3.4.4. Any ordinary elliptic curve \mathcal{E} is hypersymmetric in the above sense: The right hand side is $\mathbb{Z}_p \times \mathbb{Z}_p$, and so it is enough to know that $\text{End}(\mathcal{E})$ has rank at least 2 as a \mathbb{Z} -module (since the image of the map in question is saturated), which is clear, since Frobenius does not act as a scalar.

Remark 3.4.5. Suppose that we have an imaginary quadratic extension L/\mathbb{Q} and a map $T_L \stackrel{\text{defn}}{=} \text{Res}_{L/\mathbb{Q}} \mathbb{G}_m \rightarrow G$ whose real fiber yields an element of X . Suppose also that L is split at p and that $G_{\mathbb{Z}_p}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ contains the image of $(\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$. Then the mod- v reduction of the CM points arising from T_L are all hypersymmetric points of $\mathcal{S}_{K_p, k(v)}^{\text{ord}}$ contains many hypersymmetric points: Indeed, the symplectic representation H , viewed as a representation of T_L , must be isomorphic to a direct sum of the tautological representation on L , for weight reasons. Therefore, if x_0 is a point of \mathcal{S}_K that is the reduction of a special point arising from T_L , then \mathcal{A}_{x_0} is isogenous to a power of the CM elliptic curve associated with that point, which is ordinary, since we have assumed that p is split in L . By Remark 3.4.4, x_0 is hypersymmetric.

Remark 3.4.6. In [Fun22], one finds a somewhat general criterion for when a Newton stratum in a Shimura variety at a place of hyperspecial level contains a hypersymmetric point. Specialized to the ordinary case, it says that the ordinary locus contains a hypersymmetric point precisely when G admits a subgroup $I \subset G$ that is the centralizer of an elliptic element and whose Dynkin diagram, when viewed as a $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -equivariant graph, is isomorphic to that of $M_{\mathbb{Z}_p}$.

4. GROUP SCHEMES ASSOCIATED WITH QUADRATIC LATTICES

In this section, we will prove some technical results about group schemes associated with quadratic lattices that will be employed to prove our main irreducibility results. The reader can refer back to the results here as necessary.

4.1. Applications of Witt's extension theorem. Fix a **self-dual** quadratic space (N, Q) over \mathbb{Z}_p : This is a quadratic form

$$Q : N \rightarrow \mathbb{Z}_p$$

on a finite free \mathbb{Z}_p -module N that is such that the associated bilinear form

$$[x, y]_Q = Q(x + y) - Q(x) - Q(y)$$

on N is non-degenerate. Note that when $p = 2$ this forces n to be even.

We have the reductive \mathbb{Z}_p -group scheme $\text{GSpin}(N)$, sitting in two short exact sequences of reductive groups

$$\begin{aligned} 1 &\rightarrow \mathbb{G}_m \rightarrow \text{GSpin}(N) \rightarrow \text{SO}(N) \rightarrow 1; \\ 1 &\rightarrow \text{Spin}(N) \rightarrow \text{GSpin}(N) \xrightarrow{\nu} \mathbb{G}_m \rightarrow 1. \end{aligned}$$

Here, $\nu : \text{GSpin}(N) \rightarrow \mathbb{G}_m$ is the spinor norm. For more details, see [Mad16, § 1].

In the first part of this section, we will see that the various lemmas from § 2 of [Mad16] hold in quite some generality, without in particular the hypothesis that $p > 2$.

Lemma 4.1.1. *Suppose that F is a field over \mathbb{Z}_p . Suppose that we have two proper subspaces $W_1, W_2 \subset N_F$ such that there is an isometry $f : W_1 \xrightarrow{\cong} W_2$ of quadratic spaces (with their inherited quadratic forms) over F . Suppose that W_1 has codimension at least 2 in N_F . Then there exists $h \in \text{SO}(N)(F)$ such that $h(w_1) = f(w_1)$ for all $w_1 \in W_1$.*

Proof. Let us make some preliminary observations:

- If W_1 is isotropic, then the lemma reduces to the transitivity of the action of $\text{SO}(N)(F)$ on the Grassmannian parameterizing isotropic subspaces of N_F of fixed rank.
- By Witt's extension theorem [Bou06, §4, Théorème 1], there exists an element $g \in \text{O}(N)(F)$ such that $g(w_1) = f(w_1)$ for all $w_1 \in W_1$. In particular, if we knew that there is an element of $\text{O}(N)(F) \setminus \text{SO}(N)(F)$ restricting to the identity on W_1 , then we would be done.

Now, suppose that W_1 is itself self-dual with the inherited quadratic form. Then we have $N_F = W_1 \oplus W_1^\perp$ where

$$W_1^\perp = \{n \in N_F : [n, w] = 0 \text{ for all } w \in W_1\}$$

is also self-dual. We can now pick any element of $O(W_1^\perp)(F) \setminus SO(W_1^\perp)(F)$ to extend the identity on W_1 to an element $g' \in O(N)(F) \setminus SO(N)(F)$.

Next, suppose that $W_1 = F \cdot u_1$ where $Q(u_1) \neq 0$. After scaling if necessary we can assume that $Q(u_1) = 1$. If 2 is invertible in F , then this is a special case of the second paragraph. Assume therefore that F has characteristic 2: in this case $\dim N_F > 2$ is even, and we can find $e_1 \in N_F$ such that $Q(e_1) = 0$ and $[e_1, u_1]_Q = 1$, so that $u_1 = e_1 + e'_1$, where $Q(e'_1) = 0$ and $[e_1, e'_1]_Q = 1$. Now, e_1, e'_1 span a self-dual proper subspace of N_F , and we return to the situation from the previous paragraph.

Suppose now that W_1 is *non-degenerate*: this means that the projective quadric defined by the restriction of Q to W_1 is a smooth F -scheme. If 2 is invertible in F , then this is equivalent to saying that W_1 is self-dual, and we return to a previously considered case. Otherwise, we have the possibility that there exists $u_1 \in W_1$ such that $Q(u_1) = 1$, and such that $(F \cdot u_1)^\perp = W_1$. Choose a direct sum decomposition $W_1 = F \cdot u_1 \oplus V_1$, and set $V_2 = f(V_1) \subset W_2$. Then $V_1 \subset N_F$ is self-dual, and so by the second paragraph of the proof, there is an $h' \in SO(N)(F)$ with $h'(v_1) = f(v_1)$ for all $v_1 \in V_1$. If $h'(u_1) = f(u_1)$, then we are done. Otherwise, note that V_2^\perp has dimension at least 4. Therefore, by the previous paragraph, we can find $h'' \in SO(V_2^\perp)(F)$ such that $h''(h'(u_1)) = f(u_1)$. We now take h to be the composition of h' with the element of $SO(N)(F)$ that restricts to the identity on V_1 and to h'' on V_1^\perp .

Finally, let W_1 be arbitrary, and, for $i = 1, 2$, let $V_i \subset W_i$ be the subspace consisting of the isotropic vectors in the radical $W_i \cap W_i^\perp$. Then we have $f(V_1) = V_2$. Choose Witt decompositions

$$N_F = V_1 \oplus U_1 \oplus V_1' = V_2 \oplus U_2 \oplus V_2'.$$

Note that we have $W_i = V_i \oplus (U_i \cap W_i)$ for $i = 1, 2$ and that $U_i \cap W_i \subset U_i$ is a non-degenerate subspace of a self-dual quadratic space. With this, we can reduce to the situation in the previous paragraph. \square

Lemma 4.1.2. *Suppose that R is a \mathbb{Z}_p -algebra, and that $W \subset N_R$ is a direct summand. For any R -algebra S set*

$$A_W(S) = \{\varphi \in \text{Hom}_S(W_S, N_S) : [\varphi(w), w]_Q = 0, \text{ for all } w \in W_S\}$$

Then:

- (1) $A_W(R)$ is locally free over R ;
- (2) For any R -algebra S , we have

$$S \otimes_R A_W(R) = A_W(S) \subset \text{Hom}_S(W_S, N_S).$$

The map

$$S \otimes_{\mathbb{Z}_p} \text{Lie SO}(N) \xrightarrow{X \mapsto X|_{W_S}} A_W(S)$$

is surjective.

Proof. Consider the surjective map of finite locally free S -modules:

$$\pi_Q : \text{Hom}_S(W_S, N_S) \rightarrow \text{Hom}_S(W_S, W_S^\vee)$$

induced by the dual surjection $N_S \rightarrow W_S^\vee$. Then we see that $A_W(S)$ is the pre-image under π_Q of the locally free sub-module consisting of maps φ such that $\varphi(w)(w) = 0$ for all $w \in W_S$.¹⁰ From this, the first two assertions of the lemma are immediate.

For the third and final assertion, it now suffices to prove it under the hypothesis that $R = F$ is an algebraically closed field, where we are essentially in the situation of Lemma 2.2 of [Mad16]. The only modification one needs in that proof is to consider the case where $W \subset N_F$ is non-degenerate but not self-dual—once again, a characteristic 2 phenomenon—which, as in the proof of Lemma 4.1.1 above, reduces to the case where $W = F \cdot u$ with $Q(u) = 1$. That is, given a vector $v \in W^\perp \subset N_F$, we must find $X \in F \otimes_{\mathbb{Z}_p} \text{Lie SO}(N)$, such that $X(u) = v$. For this, we can assume that v as in *loc. cit.*, we can assume that $u = e + f$, where e, f are isotropic vectors spanning a hyperbolic plane $U \subset N_F$; then

$$\varphi : U \xrightarrow{e \mapsto e+v-u : f \mapsto f} N_F$$

is an element of $A_U(F)$ satisfying $\varphi(u) = v$, and so we reduce the requisite surjectivity statement to the case where W is itself self-dual, which is covered by the argument in [Mad16]. \square

¹⁰Concretely, if W_S is free of rank n , then we can identify this sub-module with the space of $n \times n$ anti-symmetric matrices with zeros along the diagonal.

Lemma 4.1.3. *Suppose that we have a surjection $R \rightarrow \bar{R}$ of \mathbb{Z}_p -algebras with square-zero kernel I . Suppose that we have direct summands $W_1, W_2 \subset N_R$ such that there is an isomorphism $f : W_1 \xrightarrow{\sim} W_2$ of quadratic spaces (with their inherited quadratic forms) over R . Suppose also that there exists $g' \in \mathrm{SO}(N)(R)$ such that*

$$g'(w_1) - f(w_1) \in I \otimes_{\mathbb{Z}_p} N \subset N_R, \text{ for all } w_1 \in W.$$

Then there exists $g \in \mathrm{SO}(N)(R)$ such that $g(w_1) = f(w_1)$ for all $w_1 \in W_1$.

Proof. This is shown exactly as in [Mad16, Lemma 2.8]: For $G = \mathrm{SO}(N)$, the point is to show that there exists $X \in I \otimes_{\mathbb{Z}_p} \mathrm{Lie} G$ such that $X(w_1) = w_1 - g'^{-1}f(w_1)$ for all $w_1 \in W_1$ using Lemma 4.1.2 above, and to then replace g' with $g' \circ (1 - \tilde{X})$, where $\tilde{X} \in I \otimes_{\mathbb{Z}_p} \mathrm{Lie} G$ is a lift of X . \square

4.1.4. Suppose that we have a direct summand $W \subset N$. Let $Q_W \subset \mathrm{GSpin}(N)$ be the \mathbb{Z}_p -group scheme such that for every \mathbb{Z}_p -algebra R , we have

$$Q_W(R) = \{g \in \mathrm{GSpin}(N)(R) : g \cdot w = w, \text{ for all } w \in W_R\}.$$

Set $\tilde{Q}_W = \mathrm{Spin}(N) \cap Q_W = \ker \nu|_{Q_W}$, and let $\bar{Q}_W \subset \mathrm{SO}(N)$ be the image of Q_W .

Lemma 4.1.5. *Suppose that $\mathrm{rank}(W) \leq \mathrm{rank}(N) - 2$. Then:*

(1) *We have two short exact sequences of group schemes:*

$$\begin{aligned} 1 \rightarrow \mathbb{G}_m \rightarrow Q_W \rightarrow \bar{Q}_W \rightarrow 1; \\ 1 \rightarrow \tilde{Q}_W \rightarrow Q_W \xrightarrow{\nu} \mathbb{G}_m \rightarrow 1. \end{aligned}$$

(2) *The group schemes Q_W and \bar{Q}_W are smooth over \mathbb{Z}_p with geometrically connected fibers.*

(3) *The map $\nu|_{Q_W}$ is smooth and so \tilde{Q}_W is also smooth over \mathbb{Z}_p .*

This lemma is an immediate consequence of the following result about algebraic groups over algebraically closed fields.

Lemma 4.1.6. *Let N_0 be a non-degenerate quadratic space over a field k , and let $W_0 \subset N_0$ be a k -subspace of codimension at least 2. Define subgroups*

$$Q_{W_0} \subset \mathrm{GSpin}(N_0); \quad \tilde{Q}_{W_0} \subset \mathrm{Spin}(N_0); \quad \bar{Q}_{W_0} \subset \mathrm{SO}(N_0)$$

as above. Then $Q_{W_0}, \tilde{Q}_{W_0}, \bar{Q}_{W_0}$ are smooth connected k -algebraic groups.

Proof. It is enough to show this for \tilde{Q}_{W_0} : Indeed, we have short exact sequences of group schemes:

$$\begin{aligned} 1 \rightarrow \mathbb{G}_m \rightarrow Q_{W_0} \rightarrow \bar{Q}_{W_0} \rightarrow 1; \\ 1 \rightarrow \tilde{Q}_{W_0} \rightarrow Q_{W_0} \xrightarrow{\nu} \mathbb{G}_m \rightarrow 1. \end{aligned}$$

The surjectivity of $\nu|_{Q_{W_0}}$ follows from the surjectivity of its restriction to the central \mathbb{G}_m in Q_{W_0} .

Let $W'_0 \subset W_0$ be the radical for the restriction of the symmetric bilinear form to W_0 , let $V_0 \subset W'_0$ be the subspace consisting of all the isotropic vectors (we have $V_0 = W'_0$ unless k has characteristic 2), and let $U_0 = (V_0)^\perp$ be its orthogonal complement. Let $P_0 \subset \mathrm{Spin}(N_0)$ be the parabolic subgroup stabilizing V_0 ; its Levi quotient can be identified with

$$\mathrm{GL}(V_0) \times \mathrm{Spin}(U_0/V_0).$$

Let M_0 of the Levi quotient consisting of elements of the form $(1, g)$, where g restricts to the identity on W_0/V_0 , and let $P'_0 \subset P_0$ be its pre-image. Note that $M_0 = \mathrm{Spin}(W_0/V_0)$, where W_0/V_0 is a non-degenerate quadratic space. In particular, it is connected, and so is its pre-image P'_0 .

We have $P'_0 \simeq U(P_0) \rtimes M_0$, where $U(P_0) \subset P_0$ is the unipotent radical. It can now be checked that, under any such isomorphism, \tilde{Q}_{W_0} is mapped isomorphically onto the (smooth, connected) subgroup

$$V_0 \rtimes M_0 \subset U(P_0) \rtimes M_0$$

where $V_0 \subset U(P_0)$ consists of those elements g whose restriction to W_0 is the identity. \square

Lemma 4.1.7. *Suppose that R is a \mathbb{Z}_p -algebra, that W has codimension at least 2 in N , and that we have another embedding of quadratic spaces over R , $j : W_R \hookrightarrow N_R$ mapping onto a direct summand. Then the functor on R -algebras*

$$S \mapsto \{g \in \mathrm{GSpin}(N)(S) : g^{-1}(w) = j(w) \text{ for all } w \in W_S\}$$

is represented by a Q_W -torsor over $\mathrm{Spec} R$. In particular, if $R = \mathbb{Z}_p$, then the torsor is trivial, and so we always have $g \in \mathrm{GSpin}(N)(\mathbb{Z}_p)$ such that $g^{-1}(w) = j(w)$ for all $w \in W$.

Proof. Clearly, the functor is represented by a closed subscheme of $\mathrm{GSpin}(N)_R$. Moreover, Q_W acts on this subscheme by left multiplication, and this action is simply transitive whenever the S -points of the subscheme are non-empty.

It is now enough to show that the functor admits sections over any strictly henselian local ring of R . But this is clear from Lemmas 4.1.1 and 4.1.3, which together show that the subscheme in question is faithfully flat and smooth over R .

The second assertion now from Lang's theorem [Lan56], combined with the fact that Q_W is smooth with connected special fiber: this implies that every Q_W -torsor over $\mathrm{Spec} \mathbb{Z}_p$ admits a section. \square

4.2. Certain subspaces of quadric Grassmannians.

Notation 4.2.1. Let $\{\lambda_0\}$ be the conjugacy class of minuscule cocharacters of $\mathrm{GSpin}(N)$ characterized by the following properties:

- For the left multiplication action on the Clifford algebra $C(N)$, λ_0 has weights $0, -1$.
- Via the action of λ_0 , N acquires a weight space decomposition of the form:

$$N = N^1 \oplus N^0 \oplus N^{-1},$$

where $N^{\pm 1}$ are complementary isotropic lines and $N^0 \subset N$ is the subspace orthogonal to both.

For any such choice of λ_0 , the stabilizer $P_{\lambda_0} \subset \mathrm{GSpin}(N)$ of the isotropic lines is a parabolic subgroup, and the quotient $\mathrm{GSpin}(N)/P_{\lambda_0}$ is isomorphic to the Grassmannian over \mathbb{Z}_p parameterizing isotropic lines in N . For this reason, we will denote this Grassmannian by Par_{λ_0} : It depends only on the conjugacy class $\{\lambda_0\}$.

Definition 4.2.2. Given a \mathbb{Z}_p -algebra R , we will call an isotropic line $J \subset N_R$ **W -generic** if the following conditions hold:

- $J + W_R \subset N_R$ is a local direct summand of rank $\mathrm{rank}(W) + 1$; equivalently, J maps isomorphically onto a local direct summand of N_R/W_R .
- The map

$$W_R \xrightarrow{t \mapsto [t, \cdot]_Q} \mathrm{Hom}_R(J, R)$$

is surjective.

W -generic isotropic lines are parameterized by an open subscheme $\mathrm{Par}_{\lambda_0}^\circ(W)$ of Par_{λ_0}

Definition 4.2.3. Let $\mathbb{P}(W)$ be the projective space over \mathbb{Z}_p parameterizing hyperplanes in W . Then there is a natural map $\mathrm{Par}_{\lambda_0}^\circ(W) \rightarrow \mathbb{P}(W)$ sending an isotropic line $J \in \mathrm{Par}_{\lambda_0}^\circ(W)(R)$ to the kernel of the associated surjection from W_R to $\mathrm{Hom}_R(J, R)$. Fix $U \in \mathbb{P}(W)(\mathbb{Z}_p)$, and let

$$\mathrm{Par}_{\lambda_0}^\circ(W, U) \subset \mathrm{Par}_{\lambda_0}^\circ(W)$$

be the fiber above it.

Lemma 4.2.4. $\mathrm{Par}_{\lambda_0}^\circ(W, U)$ is a smooth scheme over \mathbb{Z}_p .

Proof. Note that we can identify this scheme with the open subscheme of the projective \mathbb{Z}_p -scheme M^{loc} parameterizing isotropic lines in $U^\perp \subset N$; see [Mad16, (2.10)]. The singular locus of any geometric fiber over a field k of this scheme can be identified with the projective space of isotropic lines contained in the radical of U_k ; but no line arising from a point of $\mathrm{Par}_{\lambda_0}^\circ(W, U)(k)$ can be contained in $U_k \subset W_k$, given that it has to be complementary to W_k . \square

4.2.5. We will assume now that the quadratic form induced on $W_{\mathbb{Q}_p}$ is non-degenerate, and that $\text{Par}_{\lambda_0}^\circ(W, U)(\mathbb{Z}_p)$ is non-empty, and we will choose a point J_0 in it. Let $H_{\mathbb{Z}_p} \subset \text{GSpin}(N)$ be the closed \mathbb{Z}_p -subgroup scheme with

$$H_{\mathbb{Z}_p}(R) = \{g \in \text{GSpin}(N)(R) : g \cdot w = w, \text{ for all } w \in W_R\}$$

for any \mathbb{Z}_p -algebra R . Let $Q \subset H_{\mathbb{Z}_p}$ be the stabilizer of J_0 : Since Q has to preserve the non-degenerate pairing of J_0 against W , it follows that Q actually fixes J_0 pointwise, and is thus the pointwise stabilizer of $J_0 \oplus W \subset N$. In other words, in the notation of (4.1.4), we have $H_{\mathbb{Z}_p} = Q_W$ and $Q = Q_{J_0 \oplus W}$.

Lemma 4.2.6. *Let \tilde{Q} and $H_{\mathbb{Z}_p}^{\text{sc}}$ be defined as above Proposition 3.1.6. Suppose that $\text{rank}(W) \leq \text{rank}(N) - 3$.*

- (1) $H_{\mathbb{Z}_p}$ is a smooth \mathbb{Z}_p -group scheme with reductive generic fiber.
- (2) $H_{\mathbb{Q}_p}$ admits a maximal torus that splits over a quadratic extension of \mathbb{Q}_p .
- (3) Q is also a smooth \mathbb{Z}_p -group scheme.
- (4) The quotient $H_{\mathbb{Z}_p}/Q$ is represented by a smooth scheme over \mathbb{Z}_p , and there is an isomorphism

$$H_{\mathbb{Z}_p}/Q \xrightarrow{\cong} \text{Par}_{\lambda_0}^\circ(W, U)$$

of smooth \mathbb{Z}_p -schemes.

- (5) The \mathbb{Z}_p -group schemes $H_{\mathbb{Z}_p}^{\text{sc}}$ and \tilde{Q} are smooth with connected special fibers.
- (6) Let $\nu : \text{GSpin}(N) \rightarrow \mathbb{G}_m$ be the spinor norm; then its restriction to Q is surjective. In particular, the map on \mathbb{F}_p -points

$$\nu : Q(\mathbb{F}_p) \rightarrow \mathbb{F}_p^\times$$

is surjective.

Proof. Assertions (1), (3), (5), and the first part of assertion (6) are immediate from Lemma 4.1.5. Note that the reductivity of the generic fiber $H_{\mathbb{Q}_p}$ is a consequence of the hypothesis that $W_{\mathbb{Q}_p}$ is a non-degenerate quadratic space, so that we can identify

$$H_{\mathbb{Q}_p} = \text{GSpin}(W_{\mathbb{Q}_p}^\perp) \subset \text{GSpin}(N_{\mathbb{Q}_p}).$$

As for (2), note that $W_{\mathbb{Q}_p}^\perp$ is isometric to the orthogonal sum of an anisotropic quadratic space of dimension ≤ 4 and copies of the hyperbolic plane. In particular, it suffices to show (2) under the additional assumption that $V \stackrel{\text{defn}}{=} W_{\mathbb{Q}_p}^\perp$ is anisotropic. Here, an easy case-by-case analysis using the classification from [Shi10, §25] does the job.

The second part of (6) is immediate from (5) and Lang's theorem on connected groups over finite fields [Lan56]. It only remains to prove (4), which, by Lemma 4.2.4, comes down to showing that the map

$$H_{\mathbb{Z}_p} \xrightarrow{g \mapsto g \cdot J_0} \text{Par}_{\lambda_0}^\circ(W, U)$$

induces an isomorphism of fppf sheaves $H_{\mathbb{Z}_p}/Q \xrightarrow{\cong} \text{Par}_{\lambda_0}^\circ(W, U)$. This follows from Lemma 4.1.7, which shows that, for any \mathbb{Z}_p -algebra R , and any $J \in \text{Par}_{\lambda_0}^\circ(W, U)(R)$, the scheme of sections g of $H_{\mathbb{Z}_p}(S)$ for R -algebras S such that

$$g(S \otimes_R J_0) = J$$

is a torsor over $\text{Spec } R$ under the group scheme Q . □

Remark 4.2.7. The map

$$\text{GSpin}(N)(\mathbb{Z}_p) \xrightarrow{h \mapsto h\lambda_0(p)} \text{GSpin}(N)(\mathbb{Z}_p) \xrightarrow{\lambda_0(p)} \text{GSpin}(N)(\mathbb{Z}_p)/\text{GSpin}(N)(\mathbb{Z}_p)$$

yields a bijection

$$\text{GSpin}(N)(\mathbb{Z}_p)/(\text{GSpin}(N)(\mathbb{Z}_p) \cap \lambda_0(p) \text{GSpin}(N)(\mathbb{Z}_p)\lambda_0(p)^{-1}) \xrightarrow{\cong} \text{GSpin}(N)(\mathbb{Z}_p)\lambda_0(p) \text{GSpin}(N)(\mathbb{Z}_p)/\text{GSpin}(N)(\mathbb{Z}_p).$$

On the other hand, using the minuscule-ness of λ_0 , one finds that

$$\text{GSpin}(N)(\mathbb{Z}_p) \cap \lambda_0(p) \text{GSpin}(N)(\mathbb{Z}_p)\lambda_0(p)^{-1} \subset \text{GSpin}(N)(\mathbb{Z}_p)$$

is simply the pre-image of the \mathbb{F}_p -points of the parabolic subgroup of $\text{GSpin}(N)$ fixing the isotropic line N^1 . In particular, we have a bijection

$$\text{GSpin}(N)(\mathbb{Z}_p)/(\text{GSpin}(N)(\mathbb{Z}_p) \cap \lambda_0(p) \text{GSpin}(N)(\mathbb{Z}_p)\lambda_0(p)^{-1}) \xrightarrow[\cong]{h \mapsto hN_{\mathbb{F}_p}^1} \text{Par}_{\lambda_0}(\mathbb{F}_p),$$

where Par_{λ_0} is the quadric Grassmannian parameterizing isotropic lines in N .

Remark 4.2.8. Let Lat be the set of self-dual \mathbb{Z}_p -lattices in $N[p^{-1}]$. Consider the map

$$\text{GSpin}(N)(\mathbb{Z}_p) \xrightarrow{g \mapsto g\lambda_0(p)N} \text{Lat}.$$

Its image is exactly the set Lat_{λ_0} of lattices of the form hN for $h \in \text{GSpin}(N)(\mathbb{Z}_p)\lambda_0(p)\text{GSpin}(N)(\mathbb{Z}_p)$ and is in bijection with $\text{GSpin}(N)(\mathbb{Z}_p)\lambda_0(p)\text{GSpin}(N)(\mathbb{Z}_p)/\text{GSpin}(N)(\mathbb{Z}_p)$. By the discussion in Remark 4.2.7, we find that there is a bijection

$$(4.2.8.1) \quad \text{Par}_{\lambda_0}(\mathbb{F}_p) \xrightarrow{\cong} \text{Lat}_{\lambda_0}$$

This bijection is in fact canonical and can be described as follows: Suppose that we have an isotropic line $\tilde{N}(-1) \in \text{Par}_{\lambda_0}(\mathbb{F}_p)$. Pick any isotropic lift $N(-1) \subset N$ of $\tilde{N}(-1)$ and choose a complementary line $N(1) \subset N$. The lattice

$$\tilde{N} = p^{-1}N(-1) \oplus N(0) \oplus pN(1),$$

where $N(0) = N(-1)^\perp \cap N(1)^\perp$ is now the associated point in Lat_{λ_0} .

Lemma 4.2.9. *Suppose that we have another \mathbb{Z}_p -lattice $\tilde{W} \subset W$ with $[W : \tilde{W}] = p$. Choose any hyperplane $U \in \mathbb{P}(W)(\mathbb{Z}_p)$ such that $\tilde{W} = U + pW$. Suppose that we have*

$$\text{rank}(W) \leq \frac{\text{rank}(N) - 3}{2}.$$

Then:

- (1) $\text{Par}_{\lambda_0}^\circ(W, U)(\mathbb{F}_p)$ is non-empty¹¹ and, via (4.2.8.1), maps onto the set of self-dual \mathbb{Z}_p lattices $\tilde{N} \subset N[p^{-1}]$ such that $\tilde{N} \cap W[p^{-1}] = \tilde{W} \subset W[p^{-1}]$.
- (2) Given \tilde{N} as in (1), $N \subset N[p^{-1}]$ is the unique self-dual lattice such that
 - $N = h\tilde{N}$ for some $h \in \text{GSpin}(N)(\mathbb{Z}_p)\lambda_0(p)^{-1}\text{GSpin}(N)(\mathbb{Z}_p)$.
 - $N \cap W[p^{-1}] = \tilde{W} \subset W[p^{-1}]$

Proof. In this proof, if Y is a \mathbb{Z}_p -module, we will write \bar{Y} for the \mathbb{F}_p -vector space $Y \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

We will need the following observation: If Y is a self-dual quadratic space over \mathbb{Z}_p of rank $r \geq 3$, then \bar{Y} is isotropic. This means that the space of isotropic lines in \bar{Y} is birationally isomorphic to $\mathbb{P}_{\mathbb{F}_p}^{r-2}$, and so admits many \mathbb{F}_p -rational points. These in turn can be lifted to isotropic lines in Y via Hensel's lemma. Moreover, these lines can be chosen to lie outside the quadric associated with any proper direct summand of Y .

Fix a direct sum decomposition

$$W = U \oplus \mathbb{Z}_p w_0,$$

so that

$$\tilde{W} = U \oplus \mathbb{Z}_p p w_0.$$

Let $V_1 = U^\perp \subset N$ be the orthogonal complement to U . The radical of \bar{V}_1 is isomorphic to that of \bar{U} , which, by our hypotheses, shows that \bar{V}_1 admits a non-degenerate quadratic subspace of rank at least 3. In particular, by the second paragraph, we can find an isotropic line $\tilde{N}(-1) \subset N$ that is orthogonal to U , has trivial intersection with W , and such that there exists $y \in N(-1)$ with

$$[y, w_0]_Q \in \mathbb{Z}_p^\times.$$

Any such $\tilde{N}(-1)$ corresponds to a point in $\text{Par}_{\lambda_0}^\circ(W, U)(\mathbb{Z}_p)$, which is therefore non-empty.

Let $\tilde{N} \in \text{Lat}_{\lambda_0}$ be associated with an isotropic line $\tilde{N}(-1) \in \text{Par}_{\lambda_0}(\mathbb{F}_p)$. We need to show that we have $\tilde{N} \cap W[p^{-1}] = \tilde{W}$ precisely when $\tilde{N}(-1)$ belongs to $\text{Par}_{\lambda_0}^\circ(W, U)(\mathbb{F}_p)$.

Suppose first that $\tilde{N}(-1)$ is in $\text{Par}_{\lambda_0}^\circ(W, U)(\mathbb{F}_p)$; then we can lift it to a line $N(-1)$ in $\text{Par}_{\lambda_0}^\circ(W, U)(\mathbb{Z}_p)$, and, using the observation from the second paragraph once again, we can find an isotropic line $N(1) \subset N$ that is orthogonal to W , and which is complementary to $N(-1)$.

¹¹Note that this set only depends on the image of U in $N_{\mathbb{F}_p}$, which is simply the image of \tilde{W} and so is independent of the choice of U .

Let $w_0 \in W$ be as above. Then one sees that $N(-1)$ must pair non-degenerately with w_0 , and so we have $w_0(1) \notin pN(1)$, where $w_0(i)$ is the component of w_0 in $N(i)$. Moreover, by construction, we have

$$W \subset N(1)^\perp = N(0) \oplus N(1); U \subset N(-1)^\perp \cap N(1)^\perp = N(0).$$

This shows that we must have

$$\tilde{N} \cap W[p^{-1}] = (N(0) \oplus pN(1)) \cap W = U \oplus \mathbb{Z}_p \cdot pw_0 = \tilde{W} \subset W.$$

For the converse, suppose that $\tilde{N} \cap W[p^{-1}] = \tilde{W}$. Then in particular, we find that, given a decomposition $N = N(-1) \oplus N(0) \oplus N(1)$ as above with

$$\tilde{N} = p^{-1}N(-1) \oplus N(0) \oplus pN(1),$$

we must have $U = U[p^{-1}] \cap \tilde{N}$, and $w_0(1) \in N(1) \setminus pN(1)$ (where, once again, this is the projection of w_0 onto $N(1)$). The first condition implies that \bar{U} is contained in $\bar{N}(-1) \oplus \bar{N}(0)$, but does not contain $\bar{N}(-1)$ (the latter because $p^{-1}u \notin \tilde{N}$ for $u \in U \setminus pU$). This, together with the second condition, shows that $\bar{N}(-1)$ belongs to $\text{Par}_{\lambda_0}^\circ(W, U)(\mathbb{F}_p)$.

For assertion (2), suppose that we have a decomposition

$$\tilde{N} = \tilde{N}(-1) \oplus \tilde{N}(0) \oplus \tilde{N}(1)$$

arising from a cocharacter $\lambda : \mathbb{G}_m \rightarrow \text{GSpin}(\tilde{N})$ in the conjugacy class λ_0 such that $\lambda(p)^{-1}\tilde{N} \cap W[p^{-1}] = W$. Let $\tilde{w}_0(i)$ be the projection of pw_0 onto $\tilde{N}(i)$. Then since $p^{-1}\tilde{w}_0(i) \in p^{-i}\tilde{N}(i)$ by hypothesis, we see that

$$\tilde{w}_0(-1) \in p^2\tilde{N}(-1); \tilde{w}_0(0) \in p\tilde{N}(0); \tilde{w}_0(1) \in \tilde{N}(1).$$

This shows that the image of $\tilde{N}(1)$ in $\tilde{N}_{\mathbb{F}_p}$ can only be the isotropic line spanned by the image of pw_0 . This means that $\tilde{\lambda}(p)^{-1}\tilde{N} = N$. \square

5. CYCLES ON GSPIN SHIMURA VARIETIES

In this section, we apply the above considerations to the special case of GSpin Shimura varieties associated with quadratic spaces over \mathbb{Q} , and show that the certain irreducible special cycles in their generic fibers continue to have irreducible reduction over \mathbb{F}_p . Combined with the methods of [Mad15], this yields a proof of the irreducibility of the moduli of primitively polarized K3 surfaces of fixed degree—and that of lattice polarized K3 surfaces—in any characteristic.

5.1. Special cycles on orthogonal and GSpin Shimura varieties. We review the story of special cycles on GSpin (and orthogonal) Shimura varieties associated with quadratic spaces over \mathbb{Q} of signature $(n, 2)$. Details can be found in [Mad16], [HM20], though the presentation here most closely hews to that found in [HM22, §2.1, 2.2, 3.2].

5.1.1. The starting point is a quadratic space (V, Q) over \mathbb{Q} with signature $(n, 2)$ for some $n \geq 4$. The quadratic form Q gives rise to a symmetric pairing

$$[x, y]_Q = Q(x + y) - Q(x) - Q(y)$$

on V .

Associated with this is the reductive group $G = \text{GSpin}(V)$ over \mathbb{Q} , as well as a Hermitian symmetric domain X that parameterizes the space of oriented negative definite planes in $V_{\mathbb{R}}$. The pair (G, X) is a Shimura datum of Hodge type with reflex field \mathbb{Q} ; a choice of symplectic representation is given by the Clifford algebra $H := C(V)$, on which G acts via left multiplication.

We will assume that the quadratic space has been chosen so that it admits a lattice $V_{\mathbb{Z}} \subset V$ on which the quadratic form Q is \mathbb{Z} -valued and is such that the completion at p , $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ is a *self-dual* lattice. In this situation, $G_{\mathbb{Q}_p}$ admits a reductive model

$$G_{\mathbb{Z}(p)} = \text{GSpin}(V_{\mathbb{Z}(p)}).$$

Associated with $V_{\mathbb{Z}}$, we have a compact open subgroup $K_{V_{\mathbb{Z}}} = \prod_{\ell} K_{V_{\mathbb{Z}_\ell}} \subset G(\mathbb{A}_f)$, where, for each prime ℓ , $K_{V_{\mathbb{Z}_\ell}} \subset G(\mathbb{Q}_\ell)$ is the largest compact open subgroup contained in $C(V_{\mathbb{Z}_\ell})^\times$ and acting trivially on $V_{\mathbb{Z}_\ell}^\vee/V_{\mathbb{Z}_\ell}$.

Construction 5.1.2 (Integral models). For a compact open subgroup $K \subset G(\mathbb{A}_f)$, we have the associated Shimura variety (or more precisely stack) $\mathrm{Sh}_K := \mathrm{Sh}_K(G, X)$ over \mathbb{Q} . Suppose that K is of the form $K^p K_{V_{\mathbb{Z}_p}}$, and let S_K be the set of primes such that, for $\ell \notin S_K$, we have $K_\ell = K_{V_{\mathbb{Z}_\ell}}$. We then have a normal integral model \mathcal{S}_K over $\mathbb{Z}[S_K^{-1}]$ characterized as follows: At any prime $\ell \notin S_K$ such that $V_{\mathbb{Z}_\ell}$ is self-dual, $\mathcal{S}_{K, \mathbb{Z}(\ell)}$ is the smooth integral canonical model for Sh_K constructed in [Kis10] and [KM16]. For arbitrary $\ell \notin S_K$, we have the following characterization: Choose an isometric embedding $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^\sharp$ of quadratic lattices with the following properties:

- (1) The embedding maps onto a direct summand of $V_{\mathbb{Z}}^\sharp$.
- (2) $V_{\mathbb{Z}}^\sharp$ has signature $(n^\sharp, 2)$.
- (3) $V_{\mathbb{Z}_\ell}^\sharp$ is self-dual.

Such an embedding always exists; see [HM22, Lemma B.1.1]. Let $\mathcal{S}_{K^\sharp, \mathbb{Z}(\ell)}$ be the smooth integral canonical model for the Shimura variety Sh_{K^\sharp} associated with V^\sharp . Then $\mathcal{S}_{K, \mathbb{Z}(\ell)}$ is the normalization of $\mathcal{S}_{K^\sharp, \mathbb{Z}(\ell)}$ in Sh_K .

Note that this is a localization of Construction 1.1.2: One can choose $V_{\mathbb{Z}}^\sharp$ such that it is self-dual at all finite places, in which case Sh_{K^\sharp} admits an integral canonical model \mathcal{S}_{K^\sharp} over \mathbb{Z} , and \mathcal{S}_K is obtained by taking the normalization of this model in Sh_K and inverting the primes in S_K .

5.1.3. Let Σ_K be the set of primes ℓ such that either $\ell \in S_K$ or $V_{\mathbb{Z}_\ell}$ is *not* self-dual: this excludes p by our hypotheses. The lattice

$$H_{\mathbb{Z}} = C(V_{\mathbb{Z}}) \subset H = C(V)$$

gives us an abelian scheme $\mathcal{A} \rightarrow \mathcal{S}_K[\Sigma_K^{-1}]$. The lattice $V_{\mathbb{Z}} \subset V$ gives rise to canonical sub-sheaves

$$\mathbf{V}_? \subset \underline{\mathrm{End}}(\mathbf{H}_?)$$

for $? = B, \ell, \mathrm{dR}, \mathrm{cris}$. For every morphism $x : T \rightarrow \mathcal{S}_K[\Sigma_K^{-1}]$, we have a canonical \mathbb{Z} -submodule

$$V(x) \subset \mathrm{End}(\mathcal{A}_x)$$

whose cohomological realizations are sections of $\mathbf{V}_?$ for appropriate values of $?$. The space $V(x)$ has a canonical positive definite quadratic form

$$Q : V(x) \rightarrow \mathbb{Z}$$

characterized by the identity $Q(f)\mathrm{id}_{\mathcal{A}_x} = f \circ f \in \mathrm{End}(\mathcal{A}_x)$. This is the module of **special endomorphisms** of x . All of this is explained in quite a bit more detail in [HM20].

Remark 5.1.4 (Rationality of the Shimura cocharacter). The Shimura cocharacter conjugacy class $[\mu]$ is defined over \mathbb{Q} , and in fact has a representative over \mathbb{Q} (because of our hypothesis that $n \geq 4$). For our purposes, we only need to note that the representatives $\mu : \mathbb{G}_m \rightarrow G_{\mathbb{Z}_p}$ for this class are exactly those that yield weight decompositions

$$(5.1.4.1) \quad H_{\mathbb{Z}_p} = H_{\mathbb{Z}_p}^1 \oplus H_{\mathbb{Z}_p}^0 ; \quad V_{\mathbb{Z}_p} = V_{\mathbb{Z}_p}^{-1} \oplus V_{\mathbb{Z}_p}^0 \oplus V_{\mathbb{Z}_p}^1$$

where $\mu(z)$ acts on $H_{\mathbb{Z}_p}^i$ via z^{-i} and where $V_{\mathbb{Z}_p}^{\pm 1} \subset V_{\mathbb{Z}_p}$ are complementary isotropic lines on which $\mu(z)$ acts via $z^{\mp 1}$ and $V_{\mathbb{Z}_p}^0$ is their mutual orthogonal complement.

Remark 5.1.5 (Ordinary locus of \mathcal{S}_K). By Remark 5.1.4, we see that Assumption 1.2.1 is valid, and so by Remark 1.2.5, we see that, for every $x_0 \in \mathcal{S}_K^{\mathrm{ord}}(\kappa)$, the abelian scheme \mathcal{A}_{x_0} is ordinary. Conversely, if \mathcal{A}_{x_0} is ordinary for $x_0 \in \mathcal{S}_K(\kappa)$, we have $x_0 \in \mathcal{S}_K^{\mathrm{ord}}(\kappa)$. Knowing this amounts to showing that the other Newton strata of $\mathcal{S}_{K, \mathbb{F}_p}$ cannot contain points parameterizing ordinary abelian varieties. However, for any point x_0 with \mathcal{A}_{x_0} ordinary, one sees that $\underline{\mathrm{End}}(\mathbf{H}_{x_0})[p^{-1}]$ must have slopes $-1, 0, 1$ as an F -isocrystal. This in turn implies that $\mathbf{V}_{x_0}[p^{-1}]$ must also have slopes $-1, 0, 1$. This can only happen if x_0 is in $\mathcal{S}_K^{\mathrm{ord}}(\kappa)$. One can deduce this directly from the classification of F -crystals of K3 type, which shows that the non-ordinary Newton strata all parameterize points with other slopes; see for example [Mad16, §6.27].

5.1.6. Let Λ be a positive definite lattice over \mathbb{Z} . Set

$$\mathbf{L}(\Lambda) = \{\mathbb{Z}\text{-lattices } \Lambda' \subset \Lambda_{\mathbb{Q}} : \Lambda \subset \Lambda' \subset (\Lambda')^{\vee} \subset \Lambda^{\vee}\},$$

We will consider the stack $\mathcal{Z}_K(\Lambda) \rightarrow \mathcal{S}_K[\Sigma_K^{-1}]$ associating with $x : T \rightarrow \mathcal{S}_K[\Sigma_K^{-1}]$ the set

$$\mathcal{Z}_K(\Lambda)(x) = \{\text{isometric embeddings } \iota : \Lambda \rightarrow V(x)\}.$$

The next result can be found in [HM22, §3.2].

Lemma 5.1.7. *The stack $\mathcal{Z}_K(\Lambda)$ is finite and unramified over $\mathcal{S}_K[\Sigma_K^{-1}]$. Moreover, if $\Lambda' \in \mathbf{L}(\Lambda)$, then there is a natural closed immersion $\mathcal{Z}_K(\Lambda') \hookrightarrow \mathcal{Z}_K(\Lambda)$ obtained by restricting an isometric embedding to $\Lambda \subset \Lambda'$.*

Remark 5.1.8. If we fix a basis $\Lambda \simeq \mathbb{Z}^m$, we obtain a map of $\mathcal{S}_K[\Sigma_K^{-1}]$ -stacks $\mathcal{Z}_K(\Lambda) \rightarrow \underline{\text{End}}(\mathcal{A})^m$, where $\underline{\text{End}}(\mathcal{A})$ is the endomorphism stack of \mathcal{A} over $\mathcal{S}_K[\Sigma_K^{-1}]$. This latter stack is locally finite and unramified over $\mathcal{S}_K[\Sigma_K^{-1}]$ by standard facts about endomorphisms of abelian schemes, and $\mathcal{Z}_K(\Lambda)$ is realized as an open and closed substack of $\underline{\text{End}}(\mathcal{A})^m$.

Remark 5.1.9. Fix an embedding $\Lambda_{\mathbb{Q}} \hookrightarrow V$, and let $V^b = \Lambda_{\mathbb{Q}}^{\perp} \subset V$. Let $G^b \subset G$ be the subgroup acting trivially on $\Lambda_{\mathbb{Q}}$: this is isomorphic to $\text{GSpin}(V^b)$. Set

$$\Upsilon(\Lambda) = \{g \in G(\mathbb{A}_f) : \Lambda \subset V \cap gV_{\mathbb{Z}}\}.$$

For each $g \in \Upsilon(\Lambda)$, we obtain a quadratic lattice $V_{g,\mathbb{Z}}^b = V^b \cap gV_{\mathbb{Z}} \subset V^b$ of signature $(n - \text{rank}(\Lambda), 2)$. We can associate with this a GSpin Shimura variety $\text{Sh}_{K_g^b}$ mapping to Sh_K , where $K_g^b = gKg^{-1} \cap G^b(\mathbb{A}_f)$. Then there is a canonical isomorphism of Sh_K -stacks

$$(5.1.9.1) \quad \bigsqcup_{g \in G^b(\mathbb{Q}) \backslash \Upsilon(\Lambda)/K} \text{Sh}_{K_g^b} \xrightarrow{\cong} \mathcal{Z}_K(\Lambda).$$

See the proof of [HM22, Lemma 3.2.3].

Remark 5.1.10. Consider the open substack

$${}^{\circ}\mathcal{Z}_K(\Lambda) = \mathcal{Z}_K(\Lambda) \setminus \bigcup_{\Lambda' \in \mathbf{L}(\Lambda), \Lambda \subsetneq \Lambda'} \mathcal{Z}_K(\Lambda') \subset \mathcal{Z}_K(\Lambda)$$

This is the locus where the embedding $\Lambda \rightarrow V(x)$ maps isomorphically onto a direct summand of $V(x)$.

Set $Z_K(\Lambda) = \mathcal{Z}_K(\Lambda)_{\mathbb{Q}}$ and ${}^{\circ}Z_K(\Lambda) = {}^{\circ}\mathcal{Z}_K(\Lambda)_{\mathbb{Q}}$. Let ${}^{\circ}\Upsilon(\Lambda) \subset \Upsilon(\Lambda)$ be the subset consisting of elements g such that Λ is a direct summand of $V \cap gV_{\mathbb{Z}}$. Then the decomposition (5.1.9.1) restricts to an isomorphism

$$(5.1.10.1) \quad \bigsqcup_{g \in G^b(\mathbb{Q}) \backslash {}^{\circ}\Upsilon(\Lambda)/K} \text{Sh}_{K_g^b} \xrightarrow{\cong} {}^{\circ}Z_K(\Lambda).$$

Using this, we see that, for every $\Lambda' \in \mathbf{L}(\Lambda)$, the morphism

$$Z_K(\Lambda') \rightarrow Z_K(\Lambda)$$

is an open and closed immersion, and there is an isomorphism of \mathbb{Q} -stacks

$$\bigsqcup_{\Lambda' \in \mathbf{L}(\Lambda)} {}^{\circ}Z_K(\Lambda') \xrightarrow{\cong} Z_K(\Lambda).$$

Remark 5.1.11. Suppose that $\Lambda' \in \mathbf{L}(\Lambda)$ is such that $p \nmid [\Lambda' : \Lambda]$. Then the map $\mathcal{Z}_K(\Lambda')_{\mathbb{Z}_{(p)}} \rightarrow \mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}}$ is an open and closed immersion. It is enough to know that it is étale after base-change to \mathbb{F}_p , and this follows from the fact that the deformation theory of $\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$ as a stack over $\mathcal{S}_{K,\mathbb{F}_p}$ depends only on $\Lambda_{\mathbb{Z}_p}$. This follows for instance from the Serre-Tate theorem telling us that the deformation theory of abelian schemes in characteristic p is equivalent to that of the corresponding p -divisible groups.

Proposition 5.1.12. *Suppose that $\text{rank}(\Lambda) \leq (n - 4)/2$.*

- (1) ${}^{\circ}\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}}$ is normal, flat, and equidimensional of dimension $n - \text{rank}(\Lambda) + 1 = \dim \mathcal{S}_K - \text{rank}(\Lambda)$.
- (2) The special fiber ${}^{\circ}\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$ is geometrically normal and equidimensional of dimension $n - \text{rank}(\Lambda)$.
- (3) Let ${}^{\circ}\mathcal{Z}_K^{\text{ord}}(\Lambda)_{\mathbb{F}_p} \subset {}^{\circ}\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$ be the pre-image of $\mathcal{S}_{K,\mathbb{F}_p}^{\text{ord}}$. Then $\mathcal{Z}_K^{\text{ord}}(\Lambda)_{\mathbb{F}_p}$ is open and dense in $\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$.

Proof. See [HM22, Proposition 3.2.4] for the first two assertions. The last one follows from [HM20, Proposition 7.1.21]: This shows that the ordinary locus of a certain open substack $\mathcal{Z}_K^{\text{Pr}}(\Lambda)_{\mathbb{F}_p} \subset {}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$ is dense within it; but, as argued in the proof of [HM22, Proposition 3.2.4], $\mathcal{Z}_K^{\text{Pr}}(\Lambda)_{\mathbb{F}_p}$ is in turn dense in ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$. \square

Remark 5.1.13. Suppose that we are in the situation of Proposition 5.1.12. For each $g \in {}^\circ\Upsilon(\Lambda)$, we obtain from (5.1.10.1), an open and closed immersion $\text{Sh}_{K_g^b} \hookrightarrow {}^\circ\mathcal{Z}_K(\Lambda)$. Set ${}^\circ\mathcal{S}_{K_g^b, (p)}$ to be the Zariski closure of $\text{Sh}_{K_g^b}$ in ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}}$. By the normality of the latter stack, we see that we have a decomposition

$$(5.1.13.1) \quad \bigsqcup_{g \in G^b(\mathbb{Q}) \backslash {}^\circ\Upsilon(\Lambda)/K} {}^\circ\mathcal{S}_{K_g^b, (p)} \xrightarrow{\cong} {}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}}$$

into open and closed (normal) substacks.

Note that, if $\mathcal{S}_{K_g^b}$ is the integral model for $\text{Sh}_{K_g^b}$ from Construction 5.1.2, then the map ${}^\circ\mathcal{S}_{K_g^b, (p)} \rightarrow \mathcal{S}_{K, (p)}$ lifts to an open immersion ${}^\circ\mathcal{S}_{K_g^b, (p)} \hookrightarrow \mathcal{S}_{K_g^b, (p)}$. Indeed, this follows from the construction of $\mathcal{S}_{K_g^b, (p)}$ as the normalization of $\mathcal{S}_{K, (p)}$ in $\text{Sh}_{K_g^b}$, and Lemma 5.1.14 below.

Lemma 5.1.14. *Let $f : X \rightarrow Y$ be a quasi-finite separated map of normal Deligne-Mumford stacks flat over $\mathbb{Z}_{(p)}$. If the generic fiber $f_{\mathbb{Q}}$ is an open immersion, then so is f .*

Proof. By [LM00, Théorème 16.5], we can write f as the composition of a finite map $\bar{f} : \bar{X} \rightarrow Y$ of Deligne-Mumford stacks with an open immersion $j : X \hookrightarrow \bar{X}$. By replacing \bar{X} with the flat closure of its generic fiber, we can also assume that \bar{X} is flat over $\mathbb{Z}_{(p)}$. By the normality of Y , the finite map $\bar{X} \rightarrow Y$ must be an isomorphism onto a union of connected components, and hence f is indeed an open immersion. \square

5.2. The ordinary loci of special cycles. We will now look at the ordinary locus ${}^\circ\mathcal{Z}_K^{\text{ord}}(\Lambda)_{\mathbb{F}_p}$ in more detail. For the rest of this section, we will maintain the assumption $\text{rank}(\Lambda) \leq \frac{n-4}{2}$ from Proposition 5.1.12. We will also assume $\Lambda \neq 0$.

Remark 5.2.1. For every point $x_0 \in \mathcal{S}_{K, \mathbb{F}_p}(\kappa)$, we have a p -adic counterpart of the space of special endomorphisms: Namely the F -invariant vectors $\mathbf{V}_{\text{cris}, x_0}^{F=1} \subset \mathbf{V}_{\text{cris}, x_0}$. These are precisely the elements of $\mathbf{V}_{\text{cris}, x_0}$ that are also crystalline realizations of endomorphisms of \mathcal{G}_{x_0} . When x_0 is ordinary, there exist isomorphisms $V_{\mathbb{Z}_p}^0 \xrightarrow{\cong} \mathbf{V}_{\text{cris}, x_0}^{F=1}$ well-defined up to an element of $M_{\mathbb{Z}_p}(\mathbb{Z}_p)$. In particular, for any point $(x_0, \iota) \in {}^\circ\mathcal{Z}_K^{\text{ord}}(\Lambda)_{\mathbb{F}_p}(\kappa)$, ι the crystalline realization of ι gives an isometric embedding $\Lambda_{\mathbb{Z}_p} \hookrightarrow \mathbf{V}_{\text{cris}, x_0}^{F=1}$ onto a direct summand.

Remark 5.2.2. For the remainder of this subsection, fix $g \in {}^\circ\Upsilon(\Lambda)$. We can apply the constructions of (4.1.4) with $N = V_{\mathbb{Z}_p}$ and $W = W_g \stackrel{\text{defn}}{=} g_p^{-1}\Lambda_{\mathbb{Z}_p} \subset N$. We set $G_{g, \mathbb{Z}_p}^b = Q_{W_g}$ in the notation there: This is a smooth group scheme over \mathbb{Z}_p , clearly a subgroup of $G_{\mathbb{Z}_p}$, and conjugation by g_p identifies its generic fiber with the subgroup $G_{\mathbb{Q}_p}^b \subset G_{\mathbb{Q}_p}$. Moreover, via this identification, we have $G_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p) = K_{g, p}^b \subset G^b(\mathbb{Q}_p)$.

Lemma 5.2.3. *With the above notation:*

- (1) *We can choose the decomposition (5.1.4.1) so that the corresponding cocharacter factors through G_{g, \mathbb{Z}_p}^b .*
- (2) *All such decompositions—and hence all such cocharacters—are conjugate under $G_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)$.*
- (3) *The centralizer of any such cocharacter is a smooth subgroup scheme $M_{g, \mathbb{Z}_p}^b \subset G_{g, \mathbb{Z}_p}^b$ such that $\tilde{M}_{g, \mathbb{Z}_p}^b$ ¹² is also smooth.*

Proof. This amounts to knowing that, if $U_{\mathbb{Z}_p}$ is the hyperbolic plane over \mathbb{Z}_p , then there is an isometric embedding $U_{\mathbb{Z}_p} \hookrightarrow N$ that is orthogonal to W_g .

The numerical condition on $\text{rank}(\Lambda)$ ensures that we have $n \geq 6$, $\dim V^b \geq \frac{n}{2} + 4$ and $\dim V^b - \text{rank}(\Lambda) \geq 6$. In particular, for any $g \in {}^\circ\Upsilon(\Lambda)$, the quadric of isotropic lines in W_g^\perp is a flat local complete intersection over \mathbb{Z}_p of relative dimension ≥ 5 , whose special fiber is a rational variety. Moreover, the singular locus of the special fiber consists of isotropic lines that are contained in the radical of $W_{g, \mathbb{F}_p}^\perp$ and has codimension ≥ 5 ; see [Mad16, Lemma 2.11].

¹²This is the Zariski closure of the derived subgroup of the generic fiber.

Using this geometric information, we see that we can first pick any isotropic line $V_{\mathbb{Z}_p}^{-1} \subset W_g^\perp$ in the smooth locus, and then choose a complementary isotropic line $V_{\mathbb{Z}_p}^1$ also in the smooth locus of the quadric associated with W_g^\perp .

That all such decompositions are conjugate under $G_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)$ follows from Lemma 4.1.7, applied with W the orthogonal direct sum $U_{\mathbb{Z}_p} \oplus W_g^\perp$.

The group M_{g, \mathbb{Z}_p}^b is an extension of $\mathbb{G}_{m, \mathbb{Z}_p}$ by the pointwise stabilizer of the direct summand $V_{\mathbb{Z}_p}^{-1} \oplus W_g \oplus V_{\mathbb{Z}_p}^1$. Therefore, assertion (3) follows from Lemma 4.1.5. \square

Remark 5.2.4. Choose any cocharacter $\mu_g : \mathbb{G}_{m, \mathbb{Z}_p} \rightarrow G_{g, \mathbb{Z}_p}^b$ as in Lemma 5.2.3. Then Lemma 5.2.3 tells us that Assumption 1.2.1 is valid, and so we can now consider the ordinary locus $\mathcal{S}_{K_g^b, \mathbb{F}_p}^{\text{ord}} \subset \mathcal{S}_{K_g^b}$. By (2) of the lemma, this is independent of the choice of cocharacter.

Lemma 5.2.5. *We have $\mathcal{S}_{K_g^b, \mathbb{F}_p}^{\text{ord}} = {}^\circ\mathcal{S}_{K_g^b, \mathbb{F}_p} \times_{{}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}} {}^\circ\mathcal{Z}_K^{\text{ord}}(\Lambda)_{\mathbb{F}_p}$. Moreover, Assumption 1.2.15 holds: the canonical lift of any point of $\mathcal{S}_{K_g^b, \mathbb{F}_p}^{\text{ord}}$ lifts to $\mathcal{S}_{K_g^b}$.*

Proof. We want to show that, for any algebraically closed point $x_0 \in {}^\circ\mathcal{S}_{K_g^b}(\kappa)$ with \mathcal{A}_{x_0} ordinary, we can find a G_{g, \mathbb{Z}_p}^b -structure preserving trivialization $W(\kappa) \otimes_{\mathbb{Z}_p} H_{\mathbb{Z}_p} \xrightarrow{\sim} \mathbf{H}_{\text{cris}, x_0}$ such that the Frobenius endomorphism on the right hand side conjugates to the endomorphism $1 \otimes \mu_g(p)^{-1}$ of $W(\kappa) \otimes_{\mathbb{Z}_p} H_p$.

First, I claim that we can find a $G_{\mathbb{Z}_p}$ -structure preserving such isomorphism. This amounts to knowing that the only Newton stratum in $\mathcal{S}_{K, \mathbb{F}_p}$ that can contain a point x_0 with \mathcal{A}_{x_0} an ordinary is the μ_p -ordinary stratum.

We can now use Lemma 4.1.7 to modify this to a G_{g, \mathbb{Z}_p}^b -structure preserving one.

The validity of Assumption 1.2.15 is a consequence of the fact that all endomorphisms of \mathcal{A}_{x_0} deform to endomorphisms of its canonical lift. \square

Remark 5.2.6. The Igusa tower $\widehat{\text{Ig}}_{K, p}^{\text{ord}}$ over $\widehat{\mathcal{S}}_{K, p}^{\text{ord}}$, when restricted to $\widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}}$ acquires a reduction of structure group to the subgroup $M_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p) \subset M_{\mathbb{Z}_p}(\mathbb{Z}_p)$, given by its own Igusa tower $\widehat{\text{Ig}}_{K_g^b, p}^{\text{ord}}$. This can be understood explicitly. First, Remark 5.2.1 tells us that, for every algebraically closed point $x_0 \in \widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}}(\kappa)$, the crystalline realization of $\iota : \Lambda \rightarrow V(x_0)$ yields an embedding as a direct summand

$$W_g \xrightarrow[\simeq]{g_p} \Lambda_{\mathbb{Z}_p} \hookrightarrow \mathbf{V}_{\text{cris}, x_0}^{F=1}.$$

The reduction of structure is now given by the subsheaf parameterizing for $x : \text{Spf } R \rightarrow \widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}}$, trivializations

$$R \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_x^{\text{ét}} \times \mathcal{G}_x^{\text{mult}}$$

such that, at every geometric point $s : \text{Spec } \kappa \rightarrow \text{Spf } R$, we obtain a commuting diagram

$$\begin{array}{ccc} W_g & \xlongequal{\quad} & W_g \\ \downarrow & & \downarrow \\ V_{\mathbb{Z}_p}^0 & \xrightarrow[\simeq]{} & \mathbf{V}_{\text{cris}, x \circ s}^{F=1}. \end{array}$$

Lemma 5.2.7. $\mathcal{S}_{K_g^b, \mathbb{F}_p}^{\text{ord}}$ contains a hypersymmetric point (see Definition 3.4.2).

Proof. This just amounts to the observation that there is a rank 2 negative definite lattice $L_{\mathbb{Z}} \subset V_{g, \mathbb{Z}}^b$ such that $L_{\mathbb{Z}_p}$ is isomorphic to the hyperbolic plane. Indeed, $\text{GSpin}(L)$ will then be of the form $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ for an imaginary quadratic extension F/\mathbb{Q} that is split at p , and we can then conclude using Remark 3.4.5. \square

Remark 5.2.8. Let $U_{\mu_g}^- \subset G_{\mathbb{Z}_p}$ be the opposite unipotent associated with μ_g , so that we have a canonical inclusion

$$(5.2.8.1) \quad \text{Lie } U_{\mu_g}^- \subset \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0).$$

This can be made explicit. Namely, let $G_0 = \mathrm{SO}(V_{\mathbb{Z}_p})(\mathbb{Z}_p)$ be the special orthogonal quotient of $G_{\mathbb{Z}_p}$. Then we can also identify $U_{\mu_g}^-$ with the unipotent subgroup of G_0 associated with the isotropic line $V_{\mathbb{Z}_p}^{-1}$. That is, we have

$$\mathrm{Lie} U_{\mu_g}^- = \{(\varphi, \psi) \in \mathrm{Hom}(V_{\mathbb{Z}_p}^0, V_{\mathbb{Z}_p}^{-1}) \times \mathrm{Hom}(V_{\mathbb{Z}_p}^1, V_{\mathbb{Z}_p}^0) : \varphi^\vee + \psi = 0\} \subset \mathrm{End}(V_{\mathbb{Z}_p}).$$

Here, we have used the non-degenerate bilinear form on $V_{\mathbb{Z}_p}$ to identify $V_{\mathbb{Z}_p}^0$ with its own dual, and $V_{\mathbb{Z}_p}^1$ with the dual of $V_{\mathbb{Z}_p}^{-1}$, and hence the dual φ^\vee of φ with a map $\varphi^\vee : V_{\mathbb{Z}_p}^1 \rightarrow V_{\mathbb{Z}_p}^0$. In what follows, we can and will identify $\mathrm{Lie} U_{\mu_g}^-$ with its image in $\mathrm{Hom}(V_{\mathbb{Z}_p}^0, V_{\mathbb{Z}_p}^{-1})$. Fix generators v^\pm of $V_{\mathbb{Z}_p}^{\pm 1}$. Then, as explained in [Mad16, §1], under the left multiplication action of $V_{\mathbb{Z}_p}$ on $H_{\mathbb{Z}_p}$, we have

$$H_{\mathbb{Z}_p}^1 = \ker(v^+) = \mathrm{im}(v^+) ; H_{\mathbb{Z}_p}^0 = \ker(v^-) = \mathrm{im}(v^-).$$

The embedding (5.2.8.1) can now be described as follows: Suppose that we have a map $\varphi : V_{\mathbb{Z}_p}^0 \rightarrow V_{\mathbb{Z}_p}^{-1}$ in $\mathrm{Lie} U_{\mu_g}^-$. There exists a unique $v_\varphi^0 \in V_{\mathbb{Z}_p}^0$ such that, for all $v \in V_{\mathbb{Z}_p}^0$, we have $[v_\varphi^0, v]_Q \cdot v^- = \psi(v)$. Now, one can check that, up to sign, under (5.2.8.1), (φ, ψ) maps to left multiplication by the element $v^- v_\varphi^0$ in the Clifford algebra.

Remark 5.2.9. Now, let $\mathrm{Lie} U_{\mu_g}^{b,-} = \mathrm{Lie} U_{\mu_g}^- \cap \mathrm{Lie} U_{g, \mathbb{Z}_p}^b$. One checks that this is identified with the subspace

$$\mathrm{Hom}(V_{\mathbb{Z}_p}^0/W_g, V_{\mathbb{Z}_p}^1) \subset \mathrm{Hom}(V_{\mathbb{Z}_p}^0, V_{\mathbb{Z}_p}^1).$$

Fix an algebraically closed point $x_0 \in \mathcal{S}_{K_g^b, \mathbb{F}_p}^{\mathrm{ord}}(\kappa)$, and let $\widehat{U}_{x_0}^b$ (resp. \widehat{U}_{x_0}) be the deformation spaces of $\widehat{\mathcal{S}}_{K_g^b, p}^{\mathrm{ord}}$ (resp. $\widehat{\mathcal{S}}_{K, p}^{\mathrm{ord}}$). Using Lemma 5.2.5 and Proposition 1.2.13, we see that we have a diagram

$$\begin{array}{ccc} \widehat{U}_{x_0}^b & \longrightarrow & \widehat{U}_{x_0} \\ \simeq \downarrow & & \downarrow \simeq \\ \widehat{T}_{G_g^b} & \longrightarrow & \widehat{T}_G \end{array}$$

where $\widehat{T}_{G_g^b}$ (resp. \widehat{T}_G) is the formal torus over $W(\kappa)$ with cocharacter group $\mathrm{Lie} U_{\mu_g}^{b,-}$ (resp. $\mathrm{Lie} U_{\mu_g}^-$). This diagram is uniquely determined up to the action of $M_{\mu_g, \mathbb{Z}_p}^b(\mathbb{Z}_p)$.

Remark 5.2.10. If we choose a generator v^1 for $V_{\mathbb{Z}_p}^1$, then we can compatibly identify the character groups of \widehat{T}_G and $\widehat{T}_{G_g^b}$ with $V_{\mathbb{Z}_p}^0$ and its quotient $V_{\mathbb{Z}_p}^0/W_g$, respectively.

5.3. Correspondences between special cycles in characteristic 0. We will now apply the theory of Hecke correspondences from § 2.1 in the particular context of GSpin Shimura varieties.

Remark 5.3.1. Suppose that we have $s, t : \mathrm{Spec} R \rightarrow \mathrm{Sh}_K$ and $f \in \mathrm{QIsog}_G(s, t)$. Then conjugation by f^{-1} induces an isomorphism

$$c(f) \stackrel{\mathrm{defn}}{=} f^{-1} \circ (\cdot) \circ f : \mathrm{End}(\mathcal{A}_s)[p^{-1}] \xrightarrow{\simeq} \mathrm{End}(\mathcal{A}_t)[p^{-1}]$$

carrying $V(s)[p^{-1}]$ onto $V(t)[p^{-1}]$.

Definition 5.3.2. A p -**minimal pair** or simply **minimal pair** $\Xi = (\tilde{\Lambda} \subset \Lambda)$ is a pair of positive definite lattices $(\Lambda, \tilde{\Lambda})$ equipped with an isometric embedding $\tilde{\Lambda} \subset \Lambda$ such that $[\Lambda : \tilde{\Lambda}] = p$.

Definition 5.3.3. Suppose that Ξ is a minimal pair. Define a stack

$$\mathrm{QIsog}_{G, \mu_p, \Xi} \rightarrow {}^\circ Z_K(\Lambda) \times {}^\circ Z_K(\tilde{\Lambda})$$

whose fiber over $((s, \iota), (t, \tilde{\iota})) \in {}^\circ Z_K(\Lambda)(R) \times {}^\circ Z_K(\tilde{\Lambda})(R)$ is given by

$$\mathrm{QIsog}_{G, \mu_p, \Xi}((s, \iota), (t, \tilde{\iota})) = \{f \in \mathrm{QIsog}_{G, \mu_p}(s, t) : c(f) \circ \iota|_{\tilde{\Lambda}} = \tilde{\iota}\}$$

In other words, we are looking for p -quasi-isogenies of type μ_p that ‘shrink’ the isometric embedding of Λ onto a direct summand of the space of special endomorphisms to that of $\tilde{\Lambda}$ onto a direct summand of the space of special

endomorphisms of an isogenous point. Write $s_{\mu_p, \Xi, K} : \text{QIsog}_{G, \mu_p, \Xi} \rightarrow {}^\circ Z_K(\Lambda)$ and $t_{\mu_p, \Xi, K} : \text{QIsog}_{G, \mu_p, \Xi} \rightarrow {}^\circ Z_K(\tilde{\Lambda})$ for the natural maps.

Remark 5.3.4. Suppose that we have $g \in {}^\circ \Upsilon(\Lambda)$. Over $\text{Sh}_{K_g^b}$, the p -adic realization of the tautological embedding of Λ into the space of special endomorphisms gives an embedding

$$\iota_p : \underline{\Lambda}_{\mathbb{Z}_p} \hookrightarrow V_p |_{\text{Sh}_{K_g^b}}$$

as a local direct summand. Using Lemma 4.1.7, we now see that the canonical $K_{g,p}^b$ -torsor $\text{Sh}_{K_{g,p}^b} \rightarrow \text{Sh}_{K_g^b}$ parameterizes $G_{\mathbb{Z}_p}$ -structure preserving trivializations $\underline{H}_{\mathbb{Z}_p} \xrightarrow{\cong} \mathbf{H}_p$ that carry the embedding $W_g \hookrightarrow V_{\mathbb{Z}_p}$ onto ι_p .

Proposition 5.3.5. (1) *The map $s_{\mu_p, \Xi, K}$ is finite étale and induces a bijection on geometric connected components.*

(2) *The map $t_{\mu_p, \Xi, K}$ is an isomorphism.*

Proof. Let Par_{μ_p} be the quadric Grassmannian over \mathbb{Z}_p parameterizing isotropic lines in $V_{\mathbb{Z}_p}$. Fix $g \in {}^\circ \Upsilon(\Lambda)$, consider the map

$$\text{QIsog}_{G, \mu_p, \Xi} \times_{s_{\mu_p, \Xi, K}, {}^\circ Z_K(\Lambda)} \text{Sh}_{K_g^b} \rightarrow \text{Sh}_{K_g^b}$$

By Proposition 2.1.6 and its proof, combined with Remark 4.2.7, we obtain a commutative diagram

$$\begin{array}{ccc} \text{QIsog}_{G, \mu_p, \Xi} \times_{s_{\mu_p, \Xi, K}, {}^\circ Z_K(\Lambda)} \text{Sh}_{K_g^b} & \longrightarrow & \text{Sh}_{K_{g,p}^b} \times^{G_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)} \text{Par}_{\mu_p}(\mathbb{F}_p) \\ & \searrow & \swarrow \\ & \text{Sh}_{K_g^b} & \end{array}$$

Unwinding the definitions and using (1) of Lemma 4.2.9, one finds that the top arrow maps its source isomorphically onto the substack

$$\text{Sh}_{K_{g,p}^b} \times^{G_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)} \text{Par}_{\mu_p}(W_g, U_g)(\mathbb{F}_p) \subset \text{Sh}_{K_{g,p}^b} \times^{G_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)} \text{Par}_{\mu_p}(\mathbb{F}_p),$$

where $U \in \mathbb{P}(W_g)(\mathbb{Z}_p)$ is any hyperplane such that $g_p^{-1} \tilde{\Lambda}_{\mathbb{Z}_p} = U + pW_g$. Moreover, by Lemma 4.2.6, $G_{g, \mathbb{Z}_p}^b(\mathbb{F}_p)$ acts transitively on $\text{Par}_{\mu_p}(W_g, U_g)(\mathbb{F}_p)$, and the stabilizer $Q(\mathbb{F}_p)$ of any point in this set maps surjectively on \mathbb{F}_p^\times via the spinor norm.

To finish, we need to show that the right diagonal arrow induces a bijection on geometric connected components. Since G^b has simply connected derived subgroup, we find from [Del71, Théorème 2.4] that for any level $K' \subset G^b(\mathbb{A}_f)$ the spinor norm induces a bijection

$$\pi_0(\text{Sh}_{K', \overline{\mathbb{Q}}}) \xrightarrow{\cong} \mathbb{A}_f^\times / \mathbb{Q}_{>0} \nu(K').$$

Now, the discussion in the previous paragraph shows that we have

$$\text{Sh}_{K_{g,p}^b} \times^{G_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)} \text{Par}_{\mu_p}(\mathbb{F}_p) = \text{Sh}_{K'}$$

where $K' \subset K_g^b$ is the pre-image of $Q(\mathbb{F}_p) \subset G_{g, \mathbb{Z}_p}^b(\mathbb{F}_p)$. The surjectivity of the spinor norm $Q(\mathbb{F}_p) \rightarrow \mathbb{F}_p^\times$ shows that we have $\nu(K') = \nu(K_g^b)$, and hence that the map on connected components is a bijection as desired.

The proof of assertion (2) is along analogous lines and uses (2) of Lemma 4.2.9. \square

5.4. Ordinary correspondences of minuscule type. Fix a cocharacter $\mu_p : \mathbb{G}_{m, \mathbb{Z}_p} \rightarrow G_{\mathbb{Z}_p}$ as in Remark 5.1.4 with associated splittings (5.1.4.1). Let $M_{\mathbb{Z}_p} \subset G_{\mathbb{Z}_p}$ be the centralizer of μ_p : This is an extension of \mathbb{G}_m by $\text{GSpin}(V_{\mathbb{Z}_p}^0)$. In this subsection, we will study in detail the structure of $\widehat{\text{QIsog}}_{G, \lambda_p}^{\text{ord}}$ for the choice of a non-trivial minuscule cocharacter λ_p of $M_{\mathbb{Z}_p}$. Up to isomorphism as a formal stack over $\widehat{\mathcal{S}}_{K,p}^{\text{ord}} \times \widehat{\mathcal{S}}_{K,p}^{\text{ord}}$, this will not depend on the choice of the pair (μ_p, λ_p) in their $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -conjugacy class.

Definition 5.4.1. Consider the conjugacy class of cocharacters $\lambda_p : \mathbb{G}_{m, \mathbb{Z}_p} \rightarrow M_{\mathbb{Z}_p}$ factoring through $\text{GSpin}(V_{\mathbb{Z}_p}^0)$, acting via weights 0, 1 on the Clifford algebra $C(V_{\mathbb{Z}_p}^0)$ and giving a splitting

$$(5.4.1.1) \quad V_{\mathbb{Z}_p}^0 = V_{\mathbb{Z}_p}^0(-1) \oplus V_{\mathbb{Z}_p}^0(0) \oplus V_{\mathbb{Z}_p}^0(1)$$

where $V_{\mathbb{Z}_p}^0(\pm 1) \subset V_{\mathbb{Z}_p}^0$ are complementary isotropic lines and $V_{\mathbb{Z}_p}^0(0)$ is their mutual orthogonal complement.

Remark 5.4.2. For any cocharacter $\varpi : \mathbb{G}_{m, \mathbb{Z}_p} \rightarrow M_{\mathbb{Z}_p}$, set

$$\widehat{\text{Ig}}_{\varpi}^{\text{ord}} \stackrel{\text{defn}}{=} \widehat{\text{Ig}}_{K, v}^{\text{ord}} \times^{M_{\mathbb{Z}_p}(\mathbb{Z}_p)} \underline{M_{\mathbb{Z}_p}(\mathbb{Z}_p) \varpi(p) M_{\mathbb{Z}_p}(\mathbb{Z}_p) / M_{\mathbb{Z}_p}(\mathbb{Z}_p)}.$$

Then it follows from Proposition 2.2.14 and Corollaries 2.3.5 and 2.2.18 that we have diagrams

$$\begin{array}{ccc} \widehat{\text{QIsog}}_{G, \lambda_p}^{\text{ord}} & \longrightarrow & \widehat{\text{Ig}}_{-\lambda_p}^{\text{ord}} \\ \downarrow s & & \downarrow \\ \widehat{\text{S}}_{K, p}^{\text{ord}} & & \widehat{\text{S}}_{K, p}^{\text{ord}} \end{array} \quad ; \quad \begin{array}{ccc} \widehat{\text{QIsog}}_{G, \lambda_p}^{\text{ord}} & \longrightarrow & \widehat{\text{Ig}}_{\lambda_p}^{\text{ord}} \\ \downarrow t & & \downarrow \\ \widehat{\text{S}}_{K, p}^{\text{ord}} & & \widehat{\text{S}}_{K, p}^{\text{ord}} \end{array}$$

where both top horizontal arrows are finite flat homeomorphisms. Moreover, one finds from Remark 4.2.7 that we have

$$\widehat{\text{Ig}}_{-\lambda_p}^{\text{ord}} \simeq \widehat{\text{Ig}}_{\lambda_p}^{\text{ord}} \simeq \widehat{\text{Ig}}_{K, p}^{\text{ord}} \times^{M_{\mathbb{Z}_p}(\mathbb{Z}_p)} \underline{\text{Par}}_{\lambda_p}(\mathbb{F}_p),$$

where $\text{Par}_{\lambda_p}(\mathbb{F}_p)$ is the set of isotropic lines in $V_{\mathbb{F}_p}^0$.

Remark 5.4.3. The cocharacter λ_p breaks up $\text{Lie } U_{\mu_p}^-$ and $\text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)$ compatibly into eigenspaces

$$\text{Lie } U_{\mu_p}^- = \bigoplus_{i=-1}^1 \text{Lie } U_{\mu_p}^-(i) ; \quad \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) = \bigoplus_{i=-1}^1 \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)(i),$$

where

$$\text{Lie } U_{\mu_p}^-(i) = \begin{cases} \text{Hom}(V_{\mathbb{Z}_p}^0(1), V_{\mathbb{Z}_p}^{-1}) & \text{if } i = -1; \\ \text{Hom}(V_{\mathbb{Z}_p}^0(0), V_{\mathbb{Z}_p}^{-1}) & \text{if } i = 0; \\ \text{Hom}(V_{\mathbb{Z}_p}^0(-1), V_{\mathbb{Z}_p}^{-1}) & \text{if } i = 1. \end{cases} ,$$

and

$$\text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)(i) = \begin{cases} \text{Hom}(H_{\mathbb{Z}_p}^1(1), H_{\mathbb{Z}_p}^0(0)) & \text{if } i = -1; \\ \text{Hom}(H_{\mathbb{Z}_p}^1(0), H_{\mathbb{Z}_p}^0(0)) \oplus \text{Hom}(H_{\mathbb{Z}_p}^1(1), H_{\mathbb{Z}_p}^0(1)) & \text{if } i = 0; \\ \text{Hom}(H_{\mathbb{Z}_p}^1(0), H_{\mathbb{Z}_p}^0(1)) & \text{if } i = 1. \end{cases}$$

Remark 5.4.4. Recall the notation and setup from Remark 5.2.8. Consider the homomorphism

$$(5.4.4.1) \quad \text{Lie } U_{\mu_p}^- \oplus \text{Lie } U_{\mu_p}^- \xrightarrow{(\varphi_1, \varphi_2) \mapsto v^- v_{\varphi_1}^0 \lambda_p(p) - \lambda_p(p) v^- v_{\varphi_2}^0} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0)$$

If we restrict to each of the non-zero weight spaces for the action of λ_p , then one sees that we obtain

$$\begin{aligned} \text{Lie } U_{\mu_p}^-(-1) \oplus \text{Lie } U_{\mu_p}^-(-1) &\subset \text{Hom}(H_{\mathbb{Z}_p}^1(1), H_{\mathbb{Z}_p}^0(0)) \times \text{Hom}(H_{\mathbb{Z}_p}^1(1), H_{\mathbb{Z}_p}^0(0)) \xrightarrow{(f_1, f_2) \mapsto p f_1 - f_2} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0) \\ \text{Lie } U_{\mu_p}^-(-1) \oplus \text{Lie } U_{\mu_p}^-(-1) &\text{Hom}(H_{\mathbb{Z}_p}^1(0), H_{\mathbb{Z}_p}^0(1)) \oplus \text{Hom}(H_{\mathbb{Z}_p}^1(0), H_{\mathbb{Z}_p}^0(1)) \xrightarrow{((f_1, f_2) \mapsto f_1 - p f_2} \text{Hom}(H_{\mathbb{Z}_p}^1, H_{\mathbb{Z}_p}^0). \end{aligned}$$

For the zero weight space, the restriction of both maps $\varphi \mapsto v^- v_{\varphi}^0 \lambda_p(p)$ and $\varphi \mapsto \lambda_p(p) v^- v_{\varphi}^0$ can be identified with the composition

$$(5.4.4.2) \quad \begin{aligned} \text{Lie } U_{\mu_p}^-(0) &\xrightarrow{\varphi \mapsto v^- v_{\varphi}^0} \text{Hom}(H_{\mathbb{Z}_p}^1(0), H_{\mathbb{Z}_p}^0(0)) \oplus \text{Hom}(H_{\mathbb{Z}_p}^1(1), H_{\mathbb{Z}_p}^0(1)) \\ &\xrightarrow{(\text{id}, p \cdot \text{id})} \text{Hom}(H_{\mathbb{Z}_p}^1(0), H_{\mathbb{Z}_p}^0(0)) \oplus \text{Hom}(H_{\mathbb{Z}_p}^1(1), H_{\mathbb{Z}_p}^0(1)), \end{aligned}$$

which maps onto a direct summand of its target.

Remark 5.4.5. For any algebraically closed field κ , (5.4.4.1) gives a map of formal tori over $W(\kappa)$, $\widehat{T}_G \times \widehat{T}_G \rightarrow \widehat{T}$, and Proposition 2.2.17 now shows that the kernel $\widehat{T}_{G,\lambda_p}$ of this map is the model for the completion of $\widehat{\text{QIsog}}_{G,\lambda_p}^{\text{ord}}$ at any κ -valued point.

The weight decompositions of their cocharacter groups for the action of λ_p yield compatible splittings

$$\widehat{T}_G = \prod_{i=-1}^1 \widehat{T}_G(i) \subset \prod_{i=-1}^1 \widehat{T}(i) \subset \widehat{T}$$

The discussion in Remark 5.4.4 shows that the map $\widehat{T}_G \times \widehat{T}_G \rightarrow \widehat{T}$ respects this decomposition, and its restrictions take the following shape:

$$\begin{aligned} \widehat{T}_G(-1) \times \widehat{T}_G(-1) &\xrightarrow{(x,y) \mapsto x^p y^{-1}} \widehat{T}_G(-1) \subset \widehat{T}(-1); \\ \widehat{T}_G(0) \times \widehat{T}_G(0) &\xrightarrow{(x,y) \mapsto xy^{-1}} \widehat{T}_G(0) \xrightarrow{\varpi} \widehat{T}(0); \\ \widehat{T}_G(1) \times \widehat{T}_G(1) &\xrightarrow{(x,y) \mapsto xy^{-p}} \widehat{T}_G(1) \subset \widehat{T}(1). \end{aligned}$$

Here ϖ is the closed immersion of formal tori arising from the map (5.4.4.2) on cocharacter groups.

Therefore, we obtain a decomposition

$$\widehat{T}_{G,\lambda_p} = \widehat{T}_{G,\lambda_p}(-1) \times \widehat{T}_{G,\lambda_p}(0) \times \widehat{T}_{G,\lambda_p}(1),$$

with

$$(5.4.5.1) \quad \widehat{T}_{G,\lambda_p}(i) = \begin{cases} \{(x, x^p) : x \in \widehat{T}_G(-1)\} & \text{if } i = -1; \\ \{(x, x) : x \in \widehat{T}_G(0)\} & \text{if } i = 0; \\ \{(x^p, x) : x \in \widehat{T}_G(1)\} & \text{if } i = 1. \end{cases}$$

The map s_{λ_p} (resp. t_{λ_p}) from $\widehat{T}_{G,\lambda_p}$ to \widehat{T}_G is therefore given by the identity on $\widehat{T}_{G,\lambda_p}(-1)$ and $\widehat{T}_{G,\lambda_p}(0)$ (resp. $\widehat{T}_{G,\lambda_p}(1)$ and $\widehat{T}_{G,\lambda_p}(0)$) and by the p -power map on $\widehat{T}_{G,\lambda_p}(1)$ (resp. $\widehat{T}_{G,\lambda_p}(-1)$).

Remark 5.4.6. Just as in Remark 5.2.10, via the choice of basis element v^1 for $V_{\mathbb{Z}_p}^1$, we can identify the character group of $\widehat{T}_{G,\lambda_p}$ with $\bigoplus_{i=-1}^1 V_{\mathbb{Z}_p}^0(i) = V_{\mathbb{Z}_p}^0$, and the source and target maps are given on the level of character groups, and for this splitting, by $(p, 1, 1)$ and $(1, 1, p)$, respectively.

5.5. Ordinary correspondences between special cycles. Here we give the ordinary counterpart to the story from §5.3.

Remark 5.5.1. Suppose that we have $s, t : \text{Spf } R \rightarrow \widehat{\mathcal{S}}_{K, \mathbb{F}_p}^{\text{ord}}$ and $f \in \widehat{\text{QIsog}}_G^{\text{ord}}(s, t)$. Then just as in the generic fiber in Remark 5.3.1 conjugation by f^{-1} induces an isomorphism

$$c(f) \stackrel{\text{defn}}{=} f^{-1} \circ (\cdot) \circ f : \text{End}(\mathcal{A}_s)[p^{-1}] \xrightarrow{\simeq} \text{End}(\mathcal{A}_t)[p^{-1}]$$

carrying $V(s)[p^{-1}]$ onto $V(t)[p^{-1}]$.

Definition 5.5.2. Suppose that $\Xi = (\tilde{\Lambda} \subset \Lambda)$ is a minimal pair. Define a formal stack

$$\widehat{\text{QIsog}}_{G,\lambda_p,\Xi}^{\text{ord}} \rightarrow {}^\circ \widehat{\mathcal{Z}}_K^{\text{ord}}(\Lambda) \times {}^\circ \widehat{\mathcal{Z}}_K^{\text{ord}}(\tilde{\Lambda})$$

whose fiber over $((s, \iota), (t, \tilde{\iota})) \in {}^\circ \widehat{\mathcal{Z}}_K^{\text{ord}}(\Lambda)(R) \times {}^\circ \widehat{\mathcal{Z}}_K^{\text{ord}}(\tilde{\Lambda})(R)$ for R p -complete is given by

$$\widehat{\text{QIsog}}_{G,\lambda_p,\Xi}^{\text{ord}}((s, \iota), (t, \tilde{\iota})) = \{f \in \widehat{\text{QIsog}}_{G,\lambda_p}^{\text{ord}}(s, t) : c(f) \circ \iota|_{\tilde{\Lambda}} = \tilde{\iota}\}.$$

Write $s_{\lambda_p,\Xi,K}^{\text{ord}} : \widehat{\text{QIsog}}_{G,\lambda_p,\Xi}^{\text{ord}} \rightarrow {}^\circ \widehat{\mathcal{Z}}_K^{\text{ord}}(\Lambda)$ and $t_{\lambda_p,\Xi,K}^{\text{ord}} : \widehat{\text{QIsog}}_{G,\lambda_p,\Xi}^{\text{ord}} \rightarrow {}^\circ \widehat{\mathcal{Z}}_K^{\text{ord}}(\tilde{\Lambda})$ for the natural maps. For each $g \in {}^\circ \Upsilon(\Lambda)$ (resp. $\tilde{g} \in {}^\circ \Upsilon(\tilde{\Lambda})$), write $s_{\lambda_p,\Xi,K_g^b}^{\text{ord}}$ (resp. $t_{\lambda_p,\Xi,K_g^b}^{\text{ord}}$) for the restriction of $s_{\lambda_p,\Xi,K}^{\text{ord}}$ (resp. $t_{\lambda_p,\Xi,K}^{\text{ord}}$) over the open subspace $\widehat{\mathcal{S}}_{K_g^b,p}^{\text{ord}}$ (resp. $\widehat{\mathcal{S}}_{K_g^b,p}^{\text{ord}}$)

Notation 5.5.3. Suppose that we have $g \in {}^\circ\Upsilon(\Lambda)$. Set $\widetilde{W}_g = g_p^{-1}\widetilde{\Lambda}_{\mathbb{Z}_p} \subset W_g$, and let $U_g \in \mathbb{P}(W_g)(\mathbb{Z}_p)$ be a hyperplane such that $pW_g + U_g = \widetilde{W}_g$. Associated with this is the scheme $\text{Par}_{\lambda_p}^\circ(W_g, U_g)$ from Definition 4.2.3 that parameterizes W_g -generic isotropic lines in $V_{\mathbb{Z}_p}^0$ such that U is the kernel of the pairing of W_g with J .

Remark 5.5.4. Suppose that we have an algebraically closed point

$$z = ((s_0, \iota), (t_0, \tilde{\iota}), f) \in \widehat{\text{QIsog}}_{G, \lambda_p, \Xi}^{\text{ord}}(\kappa)$$

such that $(s_0, \iota) \in \widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}}(\kappa)$ for $g \in {}^\circ\Upsilon(\Lambda)$. We can choose a cocharacter μ_g as in Remark 5.2.4, so that it factors through G_{g, \mathbb{Z}_p}^b , and a trivialization $W(\kappa) \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_{s_0}$ that is a section of $\text{Ig}_{K_g^b, p}^{\text{ord}}$ over s_0 . In particular, this identifies $V_{\mathbb{Z}_p}^0$ with $\mathbf{V}_{\text{cris}, s_0}^{F=1}$ in such a way that $W_g = g_p^{-1}\Lambda_{\mathbb{Z}_p}$ is identified with the \mathbb{Z}_p -submodule generated by the image of ι . Now, conjugation by the crystalline realization of f gives an isomorphism

$$V_{\mathbb{Q}_p}^0 \simeq \mathbf{V}_{\text{cris}, s_0}^{F=1}[p^{-1}] \xrightarrow{\simeq} \mathbf{V}_{\text{cris}, t_0}^{F=1}[p^{-1}].$$

By our hypotheses, the pre-image of the lattice $\mathbf{V}_{\text{cris}, t_0}^{F=1}$ in $V_{\mathbb{Q}_p}^0$ is identified with a lattice of the form $hV_{\mathbb{Z}_p}^0$ with the following properties:

- (1) $h \in M_{\mathbb{Z}_p}(\mathbb{Z}_p)\lambda_p(p)M_{\mathbb{Z}_p}(\mathbb{Z}_p)$;
- (2) $hV_{\mathbb{Z}_p}^0 \cap W_g[p^{-1}] = \widetilde{W}_g$.

By (1) of Lemma 4.2.9, such lattices are in canonical bijection with $\text{Par}_{\lambda_p}^\circ(W_g, U_g)(\mathbb{F}_p)$.

Remark 5.5.5. Suppose that we have a point $z \in \widehat{\text{QIsog}}_{G, \lambda_p, \Xi}^{\text{ord}}(\kappa)$ as above with $(t_0, \tilde{\iota}) \in \widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}}(\kappa)$ for $\tilde{g} \in {}^\circ\Upsilon(\tilde{\Lambda})$. Choose the cocharacter $\mu_{\tilde{g}}$ as in Remark 5.2.4, so that it factors through $G_{\tilde{g}, \mathbb{Z}_p}^b$. Set

$$\widetilde{W}_{\tilde{g}} = \tilde{g}_p^{-1}\widetilde{\Lambda}_{\mathbb{Z}_p} \subset V_{\mathbb{Z}_p}^0.$$

Choose also a trivialization $W(\kappa) \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_{t_0}$ that is a section of $\text{Ig}_{K_g^b, p}^{\text{ord}}$ over t_0 . Conjugation by the crystalline realization of f now gives us an isomorphism $\mathbf{V}_{\text{cris}, s_0}^{F=1}[p^{-1}] \xrightarrow{\simeq} V_{\mathbb{Q}_p}^0$ such that the image of $\mathbf{V}_{\text{cris}, s_0}^{F=1}$ is of the form $\tilde{h}V_{\mathbb{Z}_p}^0$ with the following properties:

- (1) $\tilde{h} \in M_{\mathbb{Z}_p}(\mathbb{Z}_p)\lambda_p(p)^{-1}M_{\mathbb{Z}_p}(\mathbb{Z}_p)$;
- (2) $\tilde{h}V_{\mathbb{Z}_p}^0 \cap \widetilde{W}_{\tilde{g}}[p^{-1}] = W_{\tilde{g}} \stackrel{\text{defn}}{=} \tilde{g}_p^{-1}\Lambda_{\mathbb{Z}_p}$.

By (2) of Lemma 4.2.9, there is a unique such lattice.

Notation 5.5.6. Set $\mathbf{X}_g^b \stackrel{\text{defn}}{=} \text{Par}_{\lambda_g}^\circ(W_g, U_g)$ and

$$\widehat{\text{Ig}}_{\mathbf{X}_g^b}^{\text{ord}} \stackrel{\text{defn}}{=} \widehat{\text{Ig}}_{K_g^b, p}^{\text{ord}} \times^{M_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)} \mathbf{X}_g^b(\mathbb{F}_p).$$

Proposition 5.5.7. Fix $g \in {}^\circ\Upsilon(\Lambda)$ and $\tilde{g} \in {}^\circ\Upsilon(\tilde{\Lambda})$. Then:

- (1) The map $s_{\lambda_p, \Xi, K_g^b}^{\text{ord}}$ factors as

$$\widehat{\text{QIsog}}_{G, \lambda_p, \Xi, g}^{\text{ord}} \rightarrow \widehat{\text{Ig}}_{\mathbf{X}_g^b}^{\text{ord}} \rightarrow \widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}}$$

where the first map is a finite flat homeomorphism.

- (2) The map $t_{\lambda_p, \Xi, K_g^b}^{\text{ord}}$ is an isomorphism

Proof. By Remark 5.4.2, we have a commuting diagram

$$\begin{array}{ccc} \widehat{\text{QIsog}}_{G, \lambda_p, \Xi}^{\text{ord}} & \longrightarrow & \widehat{\text{Ig}}_{K_g^b, p}^{\text{ord}} \times^{M_{g, \mathbb{Z}_p}^b(\mathbb{Z}_p)} \text{Par}_{\lambda_g}(\mathbb{F}_p) \\ & \searrow s & \swarrow \\ & & \widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}} \end{array}$$

Here $\text{Par}_{\lambda_g}(\mathbb{F}_p)$ is the set of isotropic lines in $V_{\mathbb{F}_p}^0$. Remark 5.5.4 shows that, on κ -points for κ algebraically closed, the top arrow maps isomorphically onto the κ -points of

$$\widehat{\text{Ig}}_{K_g^b, p}^{\text{ord}} \times^{M_{g, Z_p}^b(\mathbb{Z}_p)} \text{Par}_{\lambda_g}^{\circ}(W_g, U_g)(\mathbb{F}_p) = \widehat{\text{Ig}}_{X_g^b}^{\text{ord}}.$$

Since $\widehat{\text{Ig}}_{X_g^b}^{\text{ord}}$ is étale over the base, this shows that the top horizontal map actually factors through a map

$$\widehat{\text{QIsog}}_{G, \lambda_p, \Xi, g}^{\text{ord}} \rightarrow \widehat{\text{Ig}}_{X_g^b}^{\text{ord}}.$$

that is an isomorphism on algebraically closed points. Similarly, Remark 5.5.5 shows that the map $t_{\lambda_p, \Xi, K_g^b}^{\text{ord}}$ is a bijection on κ -points for κ algebraically closed.

To finish the proof, we can now work on the level of deformation rings. More precisely, suppose that we are in the situation of Remark 5.5.4. If \widehat{U}_z (resp. $\widehat{U}_{s_0}^b, \widehat{U}_{t_0}^b$) is the completion of $\widehat{\text{QIsog}}_{G, \lambda_p, \Xi}^{\text{ord}}$ at z (resp. $\widehat{\mathcal{S}}_{K_g^b, p}^{\text{ord}}$ at (s_0, ι) , resp. $\widehat{\mathcal{Z}}_K^{\text{ord}}(\tilde{\Lambda})$ at $(t_0, \tilde{\iota})$), we want to show that the source map $\widehat{U}_z \rightarrow \widehat{U}_{s_0}^b$ is a finite flat homeomorphism and that the target map $\widehat{U}_z \rightarrow \widehat{U}_{t_0}^b$ is an isomorphism.

By Remark 5.5.4 again, we can work with the cocharacter μ_g factoring through G_{g, Z_p}^b and choose the trivializations

$$\mathcal{G}_{t_0} \xleftarrow{\simeq} W(\kappa) \otimes_{\mathbb{Z}_p} \mathcal{G}_0 \xrightarrow{\simeq} \mathcal{G}_{s_0}$$

such that the crystalline realization of f from $\mathbf{V}_{\text{cris}, s_0}[p^{-1}]$ to $\mathbf{V}_{\text{cris}, t_0}[p^{-1}]$ is carried via these isomorphisms to the map

$$W(\kappa) \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}[p^{-1}] \xrightarrow{1 \otimes \lambda_g(p)} W(\kappa) \otimes_{\mathbb{Z}_p} V_{\mathbb{Z}_p}[p^{-1}],$$

where λ_g is a cocharacter of $M_{\mathbb{Z}_p}$ that is conjugate to λ_p and whose -1 -eigenspace maps to an isotropic line in $\text{Par}_{\lambda_p}^{\circ}(W_g, U_g)(\mathbb{Z}_p)$. That is, if $V_{\mathbb{Z}_p}^0 = \bigoplus_{i=-1}^1 V_{\mathbb{Z}_p}^0(i)$ is the weight space decomposition for λ_g , then W_g pairs non-degenerately with $V_{\mathbb{Z}_p}^0(-1)$ (equivalently, its projection on $V_{\mathbb{Z}_p}^0(1)$ is surjective), and $V_{\mathbb{Z}_p}^0(-1)$ maps onto a direct summand of $V_{\mathbb{Z}_p}^0/W_g$. Furthermore, $\tilde{W}_g = \lambda_g(p)V_{\mathbb{Z}_p}^0 \cap W_g[p^{-1}]$.

Combining this with Remarks 5.2.10 and 5.4.6, one sees that the map

$$\widehat{U}_z \rightarrow \widehat{U}_{s_0}^b \times \widehat{U}_{t_0}^b$$

is isomorphic to a map of formal diagonalizable groups

$$\widehat{T}_{\lambda_g} \rightarrow \widehat{T}_1^b \times \widehat{T}_2^b$$

over $W(\kappa)$, where \widehat{T}_1^b (resp. \widehat{T}_2^b) has character group $V_{\mathbb{Z}_p}^0/W_g$ (resp. $V_{\mathbb{Z}_p}^0/\lambda_g(p)^{-1}\tilde{W}_g$) and the character group of \widehat{T}_{λ_g} is the cokernel of the map

$$\begin{aligned} V_{\mathbb{Z}_p}^0 &= \bigoplus_{i=-1}^1 V_{\mathbb{Z}_p}^0(i) \rightarrow V_{\mathbb{Z}_p}^0/W_g \oplus V_{\mathbb{Z}_p}^0/\lambda_g(p)^{-1}\tilde{W}_g \\ (z_{-1}, z_0, z_1) &\mapsto ((z_{-1}, z_0, pz_1) + W_g, (pz_{-1}, z_0, z_1) + \lambda_g(p)^{-1}\tilde{W}_g). \end{aligned}$$

I now claim which shows that $V_{\mathbb{Z}_p}^0/W_g$ maps injectively onto an index p subgroup of M while $V_{\mathbb{Z}_p}^0/\lambda_g(p)^{-1}\tilde{W}_g$ maps isomorphically onto M . This will imply that the map $\widehat{T}_{\lambda_g} \rightarrow \widehat{T}_1^b$ is a finite flat homeomorphism and that $\widehat{T}_{\lambda_g} \rightarrow \widehat{T}_2^b$ is an isomorphism, and so will complete the proof of the proposition.

To prove the claim, note that the elements of the kernel of $V_{\mathbb{Z}_p}^0/W_g \rightarrow M$ have representatives of the form (z_{-1}, z_0, pz_1) where (pz_{-1}, z_0, z_1) belongs to $\lambda_g(p)^{-1}\tilde{W}_g$. But then

$$(z_{-1}, z_0, pz_1) = \lambda_g(p) \cdot (pz_{-1}, z_0, z_1) \in \tilde{W}_g \subset W_g.$$

This shows the injectivity of $V_{\mathbb{Z}_p}^0/W_g \rightarrow M$. Similarly, the kernel of $V_{\mathbb{Z}_p}^0/\lambda_g(p)^{-1}\tilde{W}_g \rightarrow M$ consists of elements represented by (pz_{-1}, z_0, z_1) with $(z_{-1}, z_0, pz_1) \in W_g$. But then

$$(pz_{-1}, z_0, z_1) \in V_{\mathbb{Z}_p}^0 \cap \lambda_g(p)^{-1}\tilde{W}_g = \lambda_p(g)^{-1}\tilde{W}_g.$$

Thus this map is also injective.

Now, since W_g projects surjectively onto $V_{\mathbb{Z}_p}^0(1)$, every element of $V_{\mathbb{Z}_p}^0/W_g$ can be represented by a tuple of the form $(z_{-1}, z_0, 0)$ and so has the same image in M as the element $-(pz_{-1}, z_0, 0) + \lambda_g(p)^{-1}\tilde{W}_g \in V_{\mathbb{Z}_p}^0/\lambda_g(p)^{-1}\tilde{W}_g$. This shows that $V_{\mathbb{Z}_p}^0/\lambda_g(p)^{-1}\tilde{W}_g$ surjects onto M , and also shows that the image of $V_{\mathbb{Z}_p}^0/W_g$ has index p in M . \square

5.6. Unique specialization of connected components. Here we will prove the main technical result of this paper, Proposition 5.6.5, and so complete the proof of Theorem B.

Remark 5.6.1. Let $\Xi = (\Lambda \subset \tilde{\Lambda})$ be a minimal pair, and write

$$\pi_K^{\text{ord}} = s_{\lambda_0, \Xi, K}^{\text{ord}} \circ (t_{\lambda_0, \Xi, K}^{\text{ord}})^{-1} : {}^\circ\widehat{\mathcal{Z}}_K^{\text{ord}}(\tilde{\Lambda})_{\mathbb{F}_p} \rightarrow {}^\circ\widehat{\mathcal{Z}}_K^{\text{ord}}(\Lambda); \quad \pi_K = s_{\mu, \Xi, K} \circ (t_{\mu, \Xi, K})^{-1} : {}^\circ\mathcal{Z}_K(\tilde{\Lambda}) \rightarrow {}^\circ\mathcal{Z}_K(\Lambda).$$

We obtain a diagram:

$$(5.6.1.1) \quad \begin{array}{ccc} \pi_0({}^\circ\mathcal{Z}_K(\tilde{\Lambda})_{\mathbb{F}_p}) & \xrightarrow{\simeq} & \pi_0({}^\circ\mathcal{Z}_K^{\text{ord}}(\tilde{\Lambda})_{\mathbb{F}_p}) \rightarrow \pi_0({}^\circ\mathcal{Z}_K(\tilde{\Lambda})_{\mathbb{Q}}); \\ & & \downarrow \pi_{K,*}^{\text{ord}} \qquad \qquad \simeq \downarrow \pi_{K,*} \\ \pi_0({}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}) & \xrightarrow{\simeq} & \pi_0({}^\circ\mathcal{Z}_K^{\text{ord}}(\Lambda)_{\mathbb{F}_p}) \rightarrow \pi_0({}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Q}}). \end{array}$$

The horizontal maps are the natural ones, and the ones on the left are bijections because the ordinary locus is dense in ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$; see (3) of Proposition 5.1.12. The map $\pi_{K,*}$ is a bijection by Proposition 5.3.5. One can verify that the square on the right commutes using Remarks 2.3.4 and 2.3.3.

Notation 5.6.2. Let $P(\Lambda)$ be the assertion that the lower right horizontal arrow is bijective in the diagram above for all prime-to- p levels K^p , and let $Q(\Xi)$ be the assertion that $\pi_{K,*}^{\text{ord}}$ is bijective for all prime-to- p levels K^p .

Proposition 5.6.3. $P(\Lambda)$ is true if Λ is maximal at p .

Proof. The maximality at p here means (by definition) that, for all $\Lambda' \in \mathbf{L}(\Lambda)$, $[\Lambda' : \Lambda]$ is prime to p . Therefore, by Remark 5.1.11, we see that ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$ is an open and closed substack of $\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$, and in particular is finite over \mathcal{S}_K . Moreover, Proposition 5.1.12 tells us that ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}}$ and ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}$ are normal. In particular, the irreducible and connected components of the latter stack agree.

By Remark 5.1.13 and the finiteness of ${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}} \rightarrow \mathcal{S}_{K,(p)}$, we have a decomposition into normal stacks

$${}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Z}_{(p)}} \simeq \bigsqcup_{g \in G^b(\mathbb{Q}) \setminus {}^\circ\Xi_K(\Lambda)/K} \mathcal{S}_{K_g^b,(p)}$$

finite over $\mathcal{S}_{K,(p)}$.

Therefore [Mad19, Corollary 4.1.11] shows that every connected component of its special fiber is the specialization of a unique connected component of its generic fiber. \square

Proposition 5.6.4. $P(\Lambda)$ implies $Q(\Xi)$ for all minimal pairs of the form $\Xi = (\tilde{\Lambda} \subset \Lambda)$.

Proof. By Proposition 5.5.7, this reduces to knowing that, for all $g \in {}^\circ\Xi_K(\Lambda)$, the map

$$\pi_0\left(\widehat{\text{Ig}}_{X_g^b, \mathbb{F}_p}^{\text{ord}}\right) \rightarrow \pi_0(\mathcal{S}_{K_g^b, \mathbb{F}_p}^{\text{ord}})$$

is a bijection. We will check this using Proposition 3.4.1. Assertions (1)-(3) are taken care of by Lemma 4.2.6. Assertion (4) holds because we are assuming $P(\Lambda)$ (and by (3) of Proposition 5.1.12). Our group G is already isotropic over \mathbb{Q} , so Assertion (5) is okay. The last Assertion (6) is also valid by Lemma 5.2.7. \square

Proposition 5.6.5. For any Λ , we have a bijection

$$\pi_0({}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{F}_p}) \xrightarrow{\simeq} \pi_0({}^\circ\mathcal{Z}_K(\Lambda)_{\mathbb{Q}}).$$

Proof. Using induction on the p -adic valuation of $[\Lambda^\vee : \Lambda]$, this follows from the diagram (5.6.1.1) and Propositions 5.6.3 and 5.6.4. \square

5.7. Application to the moduli of K3 surfaces. When Λ is a rank 1 lattice spanned by a vector v with $Q(v) = m$, write $\mathcal{Z}_K(m)$ for $\mathcal{Z}_K(\Lambda)$, and similarly for all the sub-loci of the special cycle. For an integer $d \geq 1$, let $\mathcal{M}_{2d,(p)}^\circ$ be the moduli stack of primitively polarized K3 surfaces over $\mathbb{Z}_{(p)}$ of degree $2d$ (see [Mad15, §3]).

Proof of Theorem A. Let N be the self-dual quadratic \mathbb{Z} -lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$, where U is the hyperbolic plane. Choose a hyperbolic basis e, f for the first copy of U . Set

$$L_d = \langle e - df \rangle^\perp \subset N.$$

This is a quadratic space of signature $(19, 2)$. We can choose our quadratic space V , and the \mathbb{Z} -lattice $V_{\mathbb{Z}}$ such that V has signature $(20, 2)$, $V_{\mathbb{Z}(p)}$ is self-dual, and such that there exists an isometric embedding as a direct summand

$$L_d \hookrightarrow V_{\mathbb{Z}}.$$

Associated with the lattice $V_{\mathbb{Z}}$ and a suitable neat level subgroup $K^p \subset \mathrm{GSpin}(V)(\mathbb{A}_f^p)$, we have the integral model $\mathcal{S}_{K_p K^p, (p)}$ over $\mathbb{Z}_{(p)}$.

Let $\mathcal{M}_{2d,(p)}^{\mathrm{sm}}$ be the open smooth locus of $\mathcal{M}_{2d,(p)}$: This is a fiber-by-fiber dense subspace. In particular, it suffices to show that $\mathcal{M}_{2d,\overline{\mathbb{F}}_p}^{\mathrm{sm}}$ is irreducible.

By the theory of [Mad15, §5], extended to the case $p = 2$ in [KM16, Prop. A. 12] (see also the erratum at [Mad20]), there is a finite étale cover $\tilde{\mathcal{M}}_{2d,K}^{\mathrm{sm}}$ of $\mathcal{M}_{2d,\mathbb{Z}(p)}^{\mathrm{sm}}$, and an étale period map

$$\tilde{\mathcal{M}}_{2d,K}^{\mathrm{sm}} \rightarrow \mathcal{Z}_K(2d)_{\mathbb{Z}(p)}$$

that is in fact an open immersion, since it is one in the generic fiber; see [Mad15, Cor. 5.15]. This map lands inside ${}^\circ\mathcal{Z}_K(2d)_{\mathbb{Z}(p)}$, and in fact inside the *primitive* locus $\mathcal{Z}_K^{\mathrm{pr}}(2d)_{\mathbb{Z}(p)}$ where the de Rham realization of the special endomorphism of degree $2d$ generates a direct summand of \mathbf{V}_{dR} .¹³

Combined with Proposition 5.6.5, this shows that every irreducible component of $\tilde{\mathcal{M}}_{2d,K,\overline{\mathbb{F}}_p}^{\mathrm{sm}}$ is the specialization of a unique irreducible component of $\tilde{\mathcal{M}}_{2d,K,\overline{\mathbb{Q}}}^{\mathrm{sm}}$. From this, we deduce the same assertion for the fibers of $\mathcal{M}_{2d,(p)}^{\mathrm{sm}}$. However, it is well-known that this moduli stack is irreducible over \mathbb{C} . For instance, this follows from the Torelli theorem; see the proof of [Mad15, Prop. 5.3]. \square

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¹³It is the verification of this primitivity when $p = 2$ that is the main focus of the erratum [Mad20].

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