KUDLA'S MODULARITY CONJECTURE ON INTEGRAL MODELS OF ORTHOGONAL SHIMURA VARIETIES

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Abstract. We construct a family of special cycle classes on the regular integral model of an orthogonal Shimura variety, and show that these cycle classes appear as Fourier coefficients of a Siegel modular form. Passing to the generic fiber of the Shimura variety recovers a result of Bruinier and Raum, originally conjectured by Kudla.

Contents

1. Introduction

Throughout this paper, we denote by (V, Q) a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$ with $n \geqslant 1$. Associated to V is a Shimura datum $(G, \mathcal{D})$ with reflex field $\mathbb{Q}$, where the reductive group $G = \mathrm{GSpin}(V)$ of spinor similitudes sits in an exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathrm{SO}(V) \to 1.$$

Fixing a $\mathbb{Z}$-lattice $L \subset V$ on which the quadratic form takes integral values determines a compact open subgroup $K \subset G(\mathbb{A}_f)$, and hence a smooth complex orbifold

$$M(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K.$$

By the theory of canonical models of Shimura varieties, these are the complex points of a smooth Deligne-Mumford stack $M \to \mathrm{Spec}(\mathbb{Q})$ of dimension n.

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The Shimura variety M carries special cycles of all codimensions, whose arithmetic properties are the subject of a series of conjectures of Kudla [Kud04]. See also [Kud97], [KR99], [KR00], and [KRY06]. The organizing principle of these conjectures is that the special cycles should behave like coefficients of the theta kernel used to lift automorphic forms from a symplectic group to an orthogonal group.

In particular, the special cycles should themselves be, in a suitable sense, the coefficients of a Siegel modular form. This is a now a theorem of Bruinier and Raum, and the goal of this paper is to extend their modularity result to special cycles on the canonical integral model of M.

1.1. **Modularity on the generic fiber.** For any integer $d \geqslant 1$, let $\mathrm{Sym}_d(\mathbb{Q})$ be the set of symmetric $d \times d$ matrices with rational coefficients.

Let $L^\vee \subset V$ be the dual lattice to L under the bilinear form determined by Q. To each $T \in \mathrm{Sym}_d(\mathbb{Q})$ and each tuple of cosets $\mu = (\mu_1, \dots, \mu_d) \in (L^\vee/L)^d$, Kudla associates a special cycle

$$Z(T, \mu) \to M$$

of pure codimension $\mathrm{rank}(T)$. The Shimura variety M carries a distinguished line bundle ω, called the *tautological bundle* or the *line bundle of weight one modular forms*, and we follow Kudla in using the intersection pairing in the Chow ring to define the *corrected cycle class*

$$C(T, \mu) = \underbrace{c_1(\omega^{-1}) \cdots c_1(\omega^{-1})}_{d - \mathrm{rank}(T)} \cdot Z(T, \mu) \in \mathrm{CH}^d(M) \tag{1.1}$$

in the codimension d Chow group. Here $c_1(\omega^{-1}) \in \mathrm{CH}^1(M)$ is the first Chern class of ω^{-1}. These Chow groups, like all Chow groups appearing in this paper, are taken with $\mathbb{Q}$-coefficients.

The metaplectic double cover of $\mathrm{Sp}_{2d}(\mathbb{Z})$ acts via the Weil representation $\omega_{L,d}$ on the finite dimensional $\mathbb{C}$-vector space $S_{L,d}$ of functions $(L^\vee/L)^d \to \mathbb{C}$. The dual representation has a canonical basis $\{\phi^*_\mu\}_\mu \subset S^*_{L,d}$ indexed by d-tuples μ as above, and so we may form

$$C(T) = \sum_{\mu \in (L^\vee/L)^d} C(T, \mu) \otimes \phi^*_\mu \in \mathrm{CH}^d(M) \otimes_{\mathbb{Q}} S^*_{L,d}.$$

The following conjecture of Kudla was proved by Borcherds [Bor99] in the case of codimension $d = 1$ (and before that by Gross-Kohnen-Zagier [GKZ87] in the very special case where M is a modular curve). The general case was proved by Bruinier and Raum [BWR15], using ideas from the thesis of W. Zhang [Zha09] to reduce to the case $d = 1$.

Theorem A (Bruinier-Raum)**.** *The formal generating series*

$$\sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} C(T) \cdot q^T$$

converges to a Siegel modular form of weight $\frac{n}{2}+1$ *and representation*

$$\omega^*_{L,d} : \mathrm{Sp}_{2d}(\mathbb{Z}) \to \mathrm{GL}(S^*_{L,d}).$$

Convergence and modularity are understood in the following sense: for any $\mathbb{Q}$*-linear functional* $\iota : \mathrm{CH}^d(M) \to \mathbb{C}$*, the formal generating series*

$$\sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} \iota(C(T)) \cdot q^T$$

with coefficients in $S^*_{L,d}$ *is the* q*-expansion of a holomorphic Siegel modular form of the stated weight and representation.*

Strictly speaking, the results of Borcherds and Bruinier-Raum apply to the Chow group of the complex fiber $M(\mathbb{C})$, not the canonical model over $\mathbb{Q}$. The proof for the canonical model is the same, using the fact that all Borcherds products on $M(\mathbb{C})$ are algebraic and defined over the field of rational numbers [HM20]. In any case, Theorem A in the form stated here is a consequence of our main result, Theorem F below.

1.2. **Modularity on the integral model.** Throughout the paper we work with a finite set of primes Σ containing all primes p for which the lattice L_p is not maximal (Definition 2.2.1), and abbreviate

$$\mathbb{Z}[\Sigma^{-1}] = \mathbb{Z}[p^{-1} : p \in \Sigma].$$

In [HM20] one finds the construction of a normal and flat Deligne-Mumford stack

$$\mathcal{M} \to \mathrm{Spec}(\mathbb{Z}[\Sigma^{-1}])$$

with generic fiber M. Soon we will impose stronger assumptions on Σ, to guarantee that $\mathcal{M}$ is regular.

For each $T \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu \in (L^\vee/L)^d$ we define a *naive special cycle*

$$\mathcal{Z}(T,\mu) \to \mathcal{M}$$

whose generic fiber agrees with Kudla's $Z(T,\mu)$. Our definition of this cycle is via a moduli interpretation. The integral model carries a *Kuga-Satake abelian scheme* $\mathcal{A} \to \mathcal{M}$ whose pullback to any $\mathcal{M}$-scheme $S \to \mathcal{M}$ has a distinguished $\mathbb{Z}$-submodule

$$V(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S)$$

of *special endomorphisms*. The space of special endomorphisms is endowed with a positive definite quadratic form, and the S-points of $\mathcal{Z}(T,\mu)$ are in bijection with d-tuples $x = (x_1,\ldots,x_d) \in V(\mathcal{A}_S)^d_{\mathbb{Q}}$ of special quasi-endomorphisms with moment matrix $Q(x) = T$, whose denominators are controlled (in a precise sense) by the tuple $\mu = (\mu_1,\ldots,\mu_d)$. For example, if $\mu_i = 0$ then $x_i \in V(\mathcal{A}_S)$.

We insist on a modular definition of $\mathcal{Z}(T,\mu)$, as opposed to simply taking the Zariski closure of $Z(T,\mu)$ in the integral model, because this is necessary to ensure that the special cycles behave well under intersections and pullbacks to smaller orthogonal Shimura varieties (as in Theorems D and E below).

This insistence comes with a high cost: the naive special cycles need not be flat over $\mathbb{Z}[\Sigma^{-1}]$, and need not be equidimensional. Although those irreducible components of $\mathcal{Z}(T,\mu)$ that are flat over $\mathbb{Z}[\Sigma^{-1}]$ have codimension $\mathrm{rank}(T)$ in M, there will often be irreducible components of the wrong codimension supported in nonzero characteristics.

The intuition behind this phenomenon is easy to explain. At a characteristic p geometric point $s \to \mathcal{M}$ at which $\mathcal{A}_s$ is supersingular, the rank of the space of special endomorphisms $V(\mathcal{A}_s)$ is as large as it can be (namely, $n+2$). This is large enough that if the entries of T are integral and highly divisible, the entire supersingular locus of $\mathcal{M}_{\mathbb{F}_p}$ will be contained in $\mathcal{Z}(T,\mu)$. It is known [HP17] that this supersingular locus has dimension roughly $n/2$, and so the naive cycles $\mathcal{Z}(T,\mu)$ tend to have vertical irreducible components of dimension $> n/2$, regardless of the rank of T. For this reason, one cannot construct cycle classes on $\mathcal{M}$ simply by imitating the construction (1.1).

Hypothesis B. For the remainder of the introduction we assume that Σ satisfies the hypotheses of Proposition 2.2.4, guaranteeing that $\mathcal{M}$ is regular. If, for example, the discriminant of L is odd and squarefree then $\Sigma = \varnothing$ satisfies these hypotheses.

We will construct *corrected* (or perhaps *derived*) cycle classes

$$\mathcal{C}(T,\mu) \in \mathrm{CH}^d(\mathcal{M})$$

for all integers $d \geqslant 1$, all $T \in \mathrm{Sym}_d(\mathbb{Q})$, and all $\mu \in (L^\vee/L)^d$. These cycle classes vanish unless T is positive semi-definite. In §5.5 we prove the following result, showing that our construction is compatible with the classes already constructed in the generic fiber.

Theorem C. *Restricting $\mathcal{C}(T,\mu)$ to the generic fiber recovers* (1.1). *Moreover, if the naive cycle $\mathcal{Z}(T,\mu)$ is equidimensional of codimension* $\mathrm{rank}(T)$ *in $\mathcal{M}$, then*

$$\mathcal{C}(T,\mu) = \underbrace{c_1(\omega^{-1})\cdots c_1(\omega^{-1})}_{d-\mathrm{rank}(T)} \cdot \mathcal{Z}(T,\mu) \in \mathrm{CH}^d(\mathcal{M})$$

for a distinguished line bundle ω on $\mathcal{M}$.

The next two results show that our corrected cycle classes behave well under intersections and pullbacks to smaller Shimura varieties. Analogous formulas in the generic fiber are proved in [YZZ09] and [Kud21].

The following is stated in the text as Proposition 5.2.1.

Theorem D. *For all positive integers d' and d'', symmetric matrices*

$$T' \in \mathrm{Sym}_{d'}(\mathbb{Q}) \quad \textit{and} \quad T'' \in \mathrm{Sym}_{d''}(\mathbb{Q}),$$

and tuples $\mu' \in (L^\vee/L)^{d'}$ *and* $\mu'' \in (L^\vee/L)^{d''}$*, we have the intersection formula*

$$\mathcal{C}(T',\mu') \cdot \mathcal{C}(T'',\mu'') = \sum_{T = \left(\begin{smallmatrix} T' & * \\ * & T'' \end{smallmatrix}\right)} \mathcal{C}(T,\mu)$$

in the codimension $d'+d''$ *Chow group of* $\mathcal{M}$*. On the right hand side* $\mu = (\mu',\mu'')$ *is the concatenation of* μ' *and* μ''*.*

Now fix a positive definite self-dual quadratic lattice Λ, so that the orthogonal direct sum

$$L^\sharp = L \oplus \Lambda$$

has signature $(n + \mathrm{rank}(\Lambda), 2)$. This lattice determines its own Shimura datum, its own regular integral model $\mathcal{M}^\sharp$ over $\mathbb{Z}[\Sigma^{-1}]$, and its own family of corrected special cycle classes

$$\mathcal{C}^\sharp(T^\sharp, \mu^\sharp) \in \mathrm{CH}^d(\mathcal{M}^\sharp)$$

indexed by $T^\sharp \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu^\sharp \in (L^{\sharp,\vee}/L^\sharp)^d$. The isometric embedding $L \to L^\sharp$ determines a finite and unramified morphism $f : \mathcal{M} \to \mathcal{M}^\sharp$, inducing a pullback

$$f^* : \mathrm{CH}^d(\mathcal{M}^\sharp) \to \mathrm{CH}^d(\mathcal{M}).$$

The following theorem is a special case of Proposition 5.6.1.

Theorem E. *There is a decomposition*

$$f^*\mathcal{C}^\sharp(T^\sharp,\mu^\sharp) = \sum_{\substack{S,T \in \mathrm{Sym}_d(\mathbb{Q}) \\ S+T = T^\sharp}} R_\Lambda(S) \cdot \mathcal{C}(T,\mu)$$

of classes in $\mathrm{CH}^d(\mathcal{M})$*, where* $\mu = \mu^\sharp$ *viewed as an element of*

$$(L^\vee/L)^d \cong (L^{\sharp,\vee}/L^\sharp)^d,$$

and $R_\Lambda(S)$ *is the number of tuples* $y \in \Lambda^d$ *with moment matrix* $Q(y) = S$*.*

Our main result, stated in the text as Theorem 6.2.1, is an extension of the Bruinier-Raum theorem (née Kudla's modularity conjecture) to the integral model $\mathcal{M}$.

Theorem F. *The formal generating series*

$$\sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} \mathcal{C}(T) \cdot q^T$$

with coefficients

$$\mathcal{C}(T) \stackrel{\mathrm{def}}{=} \sum_{\mu \in (L^\vee/L)^d} \mathcal{C}(T,\mu) \otimes \phi_\mu^* \in \mathrm{CH}^d(\mathcal{M}) \otimes_{\mathbb{Q}} S_{L,d}^*$$

converges to a holomorphic Siegel modular form of weight $\frac{n}{2}+1$ *and representation* $\omega_{L,d}^*$*.*

Using the finite-dimensionality of the space of holomorphic Siegel modular forms of a fixed weight and representation, one immediately obtains the following corollary of Theorem F.

Corollary G. *As* $T \in \mathrm{Sym}_d(\mathbb{Q})$ *and* $\mu \in (L^\vee/L)^d$ *vary, the cycle classes* $\mathcal{C}(T,\mu)$ *span a finite-dimensional subspace of* $\mathrm{CH}^d(\mathcal{M})$.

In [Mad22], the second author uses methods from derived algebraic geometry to construct derived cycle classes on essentially every Shimura variety to which Kudla's conjectures apply, and shows that they satisfy certain anticipated properties, giving an alternative proof of Theorems C, D, E above as special cases. However, the lack of Borcherds products in this generality has so far prevented progress towards any version of the main Theorem F in settings beyond the one treated in this paper.

1.3. **Outline of the paper.** In §2 we recall the essentials of the theory of integral models of orthogonal Shimura varieties, and the families of special cycles that live on them. Our main reference for this material is [HM20], although many of the results we cite appeared before that in [Mad16], [AGHM17], and [AGHM18].

The first new results appear in §3, in which we investigate some of the finer geometric structure of the special cycles $\mathcal{Z}(T,\mu)$, under the assumption that $\mathrm{rank}(T)$ is small compared to n. We remark that the notion of smallness here depends on the lattice L, not just n, but if L is self-dual then small means $\mathrm{rank}(T) \leqslant (n-4)/3$. What we show is that in this situation the special cycle $\mathcal{Z}(T,\mu)$ is flat over $\mathbb{Z}[\Sigma^{-1}]$, and equidimensional of the expected codimension $\mathrm{rank}(T)$ in $\mathcal{M}$. In particular, when $\mathrm{rank}(T)$ is small one can define a corrected cycle class $\mathcal{C}(T,\mu)$ by imitating (1.1).

The generic fiber $Z(T,\mu)$ is smooth, however even when $\mathrm{rank}(T)$ is small the special cycle $\mathcal{Z}(T,\mu)$ need not be regular, or even locally integral; it will often have irreducible components that cross in positive characteristic. However, we can say enough about the geometry of its irreducible components to prove in §3 the injectivity of the restriction map to the generic fiber

$$\mathrm{CH}^1(\mathcal{Z}(T,\mu)) \to \mathrm{CH}^1(Z(T,\mu)).$$

Now suppose that d is small relative to n. Having shown in §3 that the special cycles $\mathcal{Z}(T,\mu)$ with $T \in \mathrm{Sym}_d(\mathbb{Q})$ are well-behaved, we prove in §4 that the generating series of corrected cycles $\mathcal{C}(T,\mu)$ is modular. This is done by fixing $T \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu \in (L^\vee/L)^d$, and considering the family of special cycles $\mathcal{Z}(T',\mu')$ in which $T' \in \mathrm{Sym}_{d+1}(\mathbb{Q})$ has upper left $d \times d$ block T, and the first d components of μ' are equal to μ. As (T',μ') varies, the resulting special cycles can be viewed as divisors

$$\mathcal{Z}(T,'\mu') \to \mathcal{Z}(T,\mu)$$

on the fixed $\mathcal{Z}(T,\mu)$, and we prove that they form the coefficients of a Jacobi form of index T valued in $\mathrm{CH}^1(\mathcal{Z}(T,u))$. The essential point is that, by the preceding paragraph, it suffices to check this in the generic fiber. In the generic fiber it follows, as in [Zha09], by realizing $Z(T,\mu)$ as a union of orthogonal Shimura varieties and applying the modularity results of Borcherds [Bor99]. By the main result of [BWR15], this Jacobi modularity, for every pair (T,μ), implies that Theorem F holds under our assumption that d is small relative to n.

To remove the assumption that d is small, we must first overcome the lack of equidimensionality of $\mathcal{Z}(T,\mu)$. In §5 we define the corrected classes $\mathcal{C}(T,\mu)$ needed even to state Theorem F in full generality. The construction itself is somewhat formal. It relies on the close relations between Chow groups and K-theory proved in [GS87] for schemes, and extended to stacks in [Gil84] and [Gil09].

Theorem D follows directly from the definition of $\mathcal{C}(T,\mu)$, but Theorems C and E seem to lie much deeper. If the modularity of Theorem F is to hold, the classes $\mathcal{C}(T)$ of that theorem must satisfy the linear invariance property

$$\mathcal{C}(T) = \mathcal{C}({}^tATA)$$

for any $A \in \mathrm{GL}_d(\mathbb{Z})$. While the analogous invariance of the naive special cycles $\mathcal{Z}(T,\mu)$ is obvious, the invariance of the corrected cycle classes encodes subtle information about self-intersections. We prove this invariance in Proposition 5.4.1 by globalizing the arguments used in [How19] to prove the analogous invariance for special cycles on unitary Rapoport-Zink spaces. The linear invariance is then used in an essential way in the proofs of Theorems C and E.

Finally, in §6 we prove Theorem F in full generality. The idea here is simple enough to explain in a few sentences. To prove the modularity of the generating series

$$\phi(\tau) = \sum_T \mathcal{C}(T) q^T$$

with coefficients in $\mathrm{CH}^d(\mathcal{M}) \otimes S^*_{L,d}$, pick an auxiliary positive definite self-dual lattice Λ. As in Theorem E, we may form the quadratic lattice $L^\sharp = L \oplus \Lambda$ of signature $(n^\sharp, 2)$, and the corresponding generating series

$$\phi^\sharp(\tau) = \sum_T \mathcal{C}^\sharp(T) q^T$$

with coefficients in $\mathrm{CH}^d(\mathcal{M}^\sharp) \otimes S^*_{L^\sharp,d}$. One can rephrase the pullback formula of Theorem E as a factorization

$$f^*\phi^\sharp(\tau) = \phi(\tau) \cdot \vartheta_{\Lambda,d}(\tau),$$

where $\vartheta_{\Lambda,d}(\tau)$ is the usual scalar-valued genus d Siegel theta series determined by the lattice Λ. If we choose Λ to have large rank, then d will be much smaller than $n^\sharp$, and so the modularity of the left hand side follows from the results of §3. Combining this with the modularity of $\vartheta_{\Lambda,d}(\tau)$ shows that $\phi(\tau)$ is a meromorphic Siegel modular form with poles supported on the vanishing locus of $\vartheta_{\Lambda,d}(\tau)$. The lattice Λ, being an arbitrary and auxiliary choice, can then be varied to show that $\phi(\tau)$ is actually holomorphic.

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2. The Shimura variety and its special cycles

This section contains little in the way of new results. Our goal is to recall from [HM20] the integral model of the Shimura variety associated to a quadratic space (V, Q) over $\mathbb{Q}$ of signature $(n, 2)$, and the special cycles on that integral model.

2.1. **The Shimura variety.** As in the introduction, we denote by $G = \mathrm{GSpin}(V)$ the group of spinor similitudes of V. By construction, G is an algebraic subgroup of the group of units of the Clifford algebra $C(V)$. The bilinear form associated to the quadratic form Q is denoted

$$[x, y] = Q(x + y) - Q(x) - Q(y). \tag{2.1}$$

If we define a Hermitian symmetric domain

$$\mathcal{D} = \{z \in V_{\mathbb{C}} : [z, z] = 0,\ [z, \overline{z}] < 0\}/\mathbb{C}^\times \subset \mathbb{P}(V_{\mathbb{C}}),$$

then the pair $(G, \mathcal{D})$ is a Hodge type Shimura datum with reflex field $\mathbb{Q}$.

Any choice of compact open subgroup in $G(\mathbb{A}_f)$ determines a Shimura variety, but we shall only consider subgroups of a particular type. Fix a $\mathbb{Z}$-lattice $L \subset V$ satisfying $Q(L) \subset \mathbb{Z}$, and let $L^\vee$ denote the dual lattice relative to the bilinear form (2.1). For every prime p, abbreviate $L_p = L \otimes \mathbb{Z}_p$, and let $C(L_p) \subset C(V_p)$ be the $\mathbb{Z}_p$-subalgebra generated by $L_p \subset C(V_p)$. The compact open subgroup

$$K_p = G(\mathbb{Q}_p) \cap C(L_p)^\times \tag{2.2}$$

of $G(\mathbb{Q}_p)$ is the largest one that stabilizes the lattice L_p and acts trivially on the discriminant group $L_p^\vee / L_p$. The compact open subgroup

$$K = \prod_p K_p \subset G(\mathbb{A}_f) \tag{2.3}$$

determines a complex orbifold

$$M(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K,$$

whose canonical model $M \to \mathrm{Spec}(\mathbb{Q})$ is a smooth Deligne-Mumford stack of dimension n.

Remark 2.1.1. For a given prime p, one can make the compact open subgroup $K_p \subset G(\mathbb{Q}_p)$ as small as one wants by replacing L by $p^k L$ for some $k \gg 0$. In particular one is free to assume that K is neat, so that M is a scheme rather than a stack. The penalty for doing so appears in the next subsection, when we form the integral model of M over $\mathbb{Z}[\Sigma^{-1}]$. This smaller choice of L will not be maximal at p, and so p must be included in the finite set of bad primes Σ that we invert.

Remark 2.1.2. As the derived subgroup $\mathrm{Spin}(V) \subset G$ is simply connected, we find by [Del71, (2.7.1)] that the space of connected components of $M(\mathbb{C})$ is a torsor under

$$\pi_0(\mathbb{A}^\times/\mathbb{Q}^\times\nu(K)\mathbb{R}_{>0}) \cong \mathbb{A}_f^\times/\mathbb{Q}_{>0}^\times\nu(K),$$

where $\nu : G \to \mathbb{G}_m$ is the spinor norm. It follows that if $\nu(K) = \widehat{\mathbb{Z}}^\times$, then M is geometrically connected. This holds in particular if L contains isotropic vectors $\ell, \ell_* \in L$ with $[\ell, \ell_*] = 1$.

Suppose $V^\sharp$ is a quadratic space of signature $(n^\sharp, 2)$, and let $(G^\sharp, \mathcal{D}^\sharp)$ be the associated Shimura datum. A $\mathbb{Z}$-lattice $L^\sharp \subset V^\sharp$ on which the quadratic form is $\mathbb{Z}$-valued determines a compact open subgroup $K^\sharp \subset G^\sharp(\mathbb{A}_f)$ as in (2.3), and hence a Shimura variety $M^\sharp$ over $\mathbb{Q}$.

An isometric embedding $L \hookrightarrow L^\sharp$ determines an injection of Clifford algebras $C(V) \to C(V^\sharp)$, which then induces a closed immersion of algebraic groups $G \hookrightarrow G^\sharp$ exhibiting G as the pointwise stabilizer of the orthgonal complement of $V \subset V^\sharp$. This embedding of groups induces an embedding of Shimura data

$$(G, \mathcal{D}) \to (G^\sharp, \mathcal{D}^\sharp),$$

As $K \subset K^\sharp \cap G(\mathbb{A}_f)$, the theory of canonical models implies the existence of a finite and unramified morphism

$$M \to M^\sharp \tag{2.4}$$

of Deligne-Mumford stacks, given on $\mathbb{C}$-points by

$$G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K \xrightarrow{(z,g)\mapsto(z,g)} G^\sharp(\mathbb{Q})\backslash\mathcal{D}^\sharp \times G^\sharp(\mathbb{A}_f)/K^\sharp.$$

More generally, for any $g \in G^\sharp(\mathbb{A}_f)$ we may replace L by the quadratic lattice $L_g = V \cap gL^\sharp_{\widehat{\mathbb{Z}}}$ throughout the discussion above. The compact open subgroup associated to this lattice is

$$K_g = gK^\sharp g^{-1} \cap G(\mathbb{A}_f),$$

and the associated Shimura variety M_g admits a finite unramified morphism

$$M_g \to M^\sharp \tag{2.5}$$

given on $\mathbb{C}$-points by

$$G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K_g \xrightarrow{(z,h)\mapsto(z,hg)} G^\sharp(\mathbb{Q})\backslash\mathcal{D}^\sharp \times G^\sharp(\mathbb{A}_f)/K^\sharp.$$

2.2. **Integral models and special cycles.** We will use [HM20, §6] as our primary reference for the theory of integral models of M. See also [KM16], [AGHM17], and [AGHM18].

Definition 2.2.1. For a prime p, we say that L_p is *maximal* if there is no larger $\mathbb{Z}_p$-lattice of V_p on which Q is $\mathbb{Z}_p$-valued. We say that L_p is *hyperspecial* if either

- L_p is self-dual, or

- $p = 2$, $\dim_{\mathbb{Q}}(V)$ is odd, and $[L_2^\vee : L_2]$ is not divisible by 4.

We call L *maximal* or *hyperspecial* if L_p has this property for every p.

Remark 2.2.2. Note that

$$L_p \text{ self-dual} \implies L_p \text{ hyperspecial} \implies L_p \text{ maximal},$$

and that L_p not maximal $\implies p^2$ divides $[L^\vee : L]$.

Remark 2.2.3. A hyperspecial lattice L_p was called an *almost self-dual* lattice in [HM20, Definition 6.1.1]. If L_p is hyperspecial then (2.2) is a hyperspecial subgroup in the usual sense, justifying the terminology. See [HM20, §6.3].

As in the introduction, Σ will always denote a finite set of primes containing all primes p for which L_p is not maximal. The constructions of [HM20, §6] provide us with a normal and flat Deligne-Mumford stack

$$\mathcal{M} \to \mathrm{Spec}(\mathbb{Z}[\Sigma^{-1}])$$

with generic fiber M. Strictly speaking, *loc. cit.* constructs an integral model over the localization $\mathbb{Z}_{(p)}$ for any p at which L_p is maximal; these can be collated into a model over $\mathbb{Z}[\Sigma^{-1}]$ as in [HM20, §9.1].

Proposition 2.2.4. *Assume that Σ satisfies both*

- *$p \in \Sigma$ for all primes p such that p^2 divides $[L^\vee : L]$,*
- *if L_2 is not hyperspecial then $2 \in \Sigma$.*

The stack $\mathcal{M}$ is regular, and for any $p \notin \Sigma$ the localization $\mathcal{M}_{\mathbb{Z}_{(p)}}$ is the canonical integral model of M in the sense of [Mad16, Definition 4.3].

Proof. If $p \notin \Sigma$ then either L_p is hyperspecial, or p is odd and $\mathrm{ord}_p([L^\vee : L]) = 1$. In the former case $\mathcal{M}_{\mathbb{Z}_{(p)}}$ is the smooth canonical integral model of M over $\mathbb{Z}_{(p)}$ constructed [Kis10] and [KM16]. In the latter case $\mathcal{M}_{\mathbb{Z}_{(p)}}$ is the regular canonical integral model constructed in [Mad16, Theorem 7.4]. □

Remark 2.2.5. By Proposition 2.2.4, if $[L^\vee : L]$ is odd and squarefree then $\mathcal{M}$ is regular for any choice of Σ, including $\Sigma = \varnothing$.

Remark 2.2.6. If $p \notin \Sigma$ is an odd prime with $\mathrm{ord}_p([L^\vee : L]) = 2$, the model $\mathcal{M}_{\mathbb{Z}_{(p)}}$ is no longer regular. It does admit a regular resolution constructed by Pappas-Zachos [PZ22], which has a certain canonicity property formulated by Pappas [Pap23], and now proven by Daniels-Youcis [PY24]. It would be interesting to extend the results of this paper to these regular integral models as well.

The integral model $\mathcal{M}$ comes with a *tautological line bundle*

$$\omega \in \mathrm{Pic}(\mathcal{M}), \tag{2.6}$$

called the *line bundle of weight one modular forms* in [HM20, §6.3], whose fiber at a complex point

$$(z, g) \in G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K \cong \mathcal{M}(\mathbb{C})$$

is identified with the isotropic line $\mathbb{C}z \subset V_{\mathbb{C}}$.

Remark 2.2.7. Recall from (2.4) the morphism $M \to M^\sharp$ of canonical models induced by an isometric embedding $L \hookrightarrow L^\sharp$. If Σ also contains all primes p for which $L^\sharp_p$ is not maximal, then $M^\sharp$ has its own flat and normal integral model

$$\mathcal{M}^\sharp \to \mathrm{Spec}(\mathbb{Z}[\Sigma^{-1}])$$

and the morphism above extends uniquely to a finite morphism

$$\mathcal{M} \to \mathcal{M}^\sharp.$$

The tautological bundle on $\omega^\sharp$ on the target pulls back to the tautological bundle ω on the source. See [HM20, Proposition 6.6.1].

The integral model $\mathcal{M}$ also comes with a *Kuga-Satake abelian scheme* $\mathcal{A} \to \mathcal{M}$. For every scheme $S \to \mathcal{M}$ there is a canonical (*e.g.* functorial in S) subspace

$$V(\mathcal{A}_S)_{\mathbb{Q}} \subset \mathrm{End}(\mathcal{A}_S)_{\mathbb{Q}}$$

of *special quasi-endomorphisms*, carrying a positive definite quadratic form defined by $Q(x) = x \circ x$ as elements of $\mathbb{Q} \subset \mathrm{End}(\mathcal{A}_S)_{\mathbb{Q}}$. More generally, for every coset $\mu \in L^\vee/L$ there is a subset

$$V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S)_{\mathbb{Q}}$$

of *special quasi-endomorphisms with denominator* μ. When $\mu = 0$ this agrees with

$$V(\mathcal{A}_S) \stackrel{\mathrm{def}}{=} V(\mathcal{A}_S)_{\mathbb{Q}} \cap \mathrm{End}(\mathcal{A}_S),$$

and the subsets indexed by distinct cosets are disjoint. Again, we refer the reader to [HM20, §6] for details.

Remark 2.2.8. For any $x \in V_\mu(\mathcal{A}_S)$, we have

$$Q(x) \equiv Q(\tilde{\mu}) \pmod{\mathbb{Z}},$$

where $\tilde{\mu} \in L^\vee$ is any lift of μ. See [AGHM18, Proposition 4.5.4].

Remark 2.2.9. Suppose $s \in \mathcal{M}(\mathbb{C})$ is a complex point corresponding to a pair

$$(z, g) \in G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K \cong \mathcal{M}(\mathbb{C}).$$

Recalling that $z \in V_{\mathbb{C}}$ is a nonzero isotropic vector, there is a canonical identification

$$V(\mathcal{A}_s)_{\mathbb{Q}} = \{x \in V : [x, z] = 0\},$$

respecting quadratic forms and satisfying

$$V_\mu(\mathcal{A}_s) = \{x \in V : x \in g \cdot (\mu + L_{\widehat{\mathbb{Z}}})\}.$$

Here we are regarding $\mu + L_{\widehat{\mathbb{Z}}} \subset V_{\mathbb{A}_f}$.

Definition 2.2.10. For $t \in \mathbb{Q}$ and $\mu \in L^\vee/L$, the *special divisor*

$$\mathcal{Z}(t, \mu) \to \mathcal{M} \tag{2.7}$$

is the finite, unramified, and relatively representable $\mathcal{M}$-stack whose functor of points assigns to any scheme $S \to \mathcal{M}$ the set

$$\mathcal{Z}(t,\mu)(S) = \{x \in V_\mu(\mathcal{A}_S) : Q(x) = t\}.$$

The definition of special divisors can be generalized as follows.

Definition 2.2.11. Given an integer $d \geqslant 1$, a matrix $T \in \mathrm{Sym}_d(\mathbb{Q})$, and a tuple of cosets $\mu = (\mu_1, \ldots, \mu_d) \in (L^\vee/L)^d$, the *special cycle*

$$\mathcal{Z}(T,\mu) \to \mathcal{M} \tag{2.8}$$

is the finite, unramified, and relatively representable $\mathcal{M}$-stack whose functor of points assigns to a scheme $S \to \mathcal{M}$ the set of all tuples

$$x = (x_1, \ldots, x_d) \in V_{\mu_1}(\mathcal{A}_S) \times \cdots \times V_{\mu_d}(\mathcal{A}_S)$$

whose moment matrix

$$Q(x) \stackrel{\mathrm{def}}{=} \left(\frac{[x_i, x_j]}{2} \right) \in \mathrm{Sym}_d(\mathbb{Q}) \tag{2.9}$$

satisfies $Q(x) = T$.

Remark 2.2.12. As $V(\mathcal{A}_S)_\mathbb{Q}$ is a positive definite quadratic space, the special cycle $\mathcal{Z}(T,\mu)$ is empty unless T is positive semi-definite.

2.3. **Special cycles as Shimura varieties.** Given a special cycle (2.8), we explain how to write its generic fiber

$$Z(T,\mu) = \mathcal{Z}(T,\mu)_\mathbb{Q}$$

as a disjoint union of Shimura varieties. We may assume that $T \in \mathrm{Sym}_d(\mathbb{Q})$ is positive semi-definite, for otherwise $Z(T,\mu) = \varnothing$ by Remark 2.2.12.

Endow the space of column vectors $\mathbb{Q}^d$ with the (possibly degenerate) quadratic form $Q(w) = {}^t wTw$, let $\mathrm{rad}(Q) \subset \mathbb{Q}^d$ be its radical, and define a positive definite quadratic space

$$W = \mathbb{Q}^d/\mathrm{rad}(Q) \tag{2.10}$$

of dimension $\mathrm{rank}(T)$. Let $e_1, \ldots, e_d \in W$ be the images of the standard basis vectors in $\mathbb{Q}^d$. Using the notation (2.9), the tuple $e = (e_1, \ldots, e_d)$ has moment matrix $Q(e) = T$.

If S is any scheme, a morphism $S \to \mathcal{Z}(T,\mu)$ determines a tuple

$$x = (x_1, \ldots, x_d) \in V(\mathcal{A}_S)_\mathbb{Q}^d$$

with $Q(x) = T$, and hence an isometric embedding

$$W \xrightarrow{e_i \mapsto x_i} V(\mathcal{A}_S)_\mathbb{Q}. \tag{2.11}$$

Lemma 2.3.1. *If $Z(T,\mu)$ is non-empty then there exists an isometric embedding $W \hookrightarrow V$.*

Proof. Using Remark 2.2.9, a complex point $s \in Z(T,\mu)(\mathbb{C})$ determines an isometric embedding

$$W \xrightarrow{(2.11)} V(\mathcal{A}_s)_\mathbb{Q} \subset V.$$

$\square$

By Lemma 2.3.1 we may assume there exists an isometric embedding $W \hookrightarrow V$, which we now fix. Any two embeddings lie in the same $G(\mathbb{Q})$-orbit, so the particular choice is unimportant. Let $V^\flat \subset V$ be the orthogonal complement of W, so that $V^\flat$ has signature $(n^\flat, 2)$ with $n^\flat = n - \mathrm{rank}(T)$, and

$$V = V^\flat \oplus W.$$

Applying the constructions of §2.1 to $V^\flat$, we obtain a reductive group $G^\flat = \mathrm{GSpin}(V^\flat)$ and an embedding of Shimura data

$$(G^\flat, \mathcal{D}^\flat) \hookrightarrow (G, \mathcal{D}).$$

The vectors $e_1, \ldots, e_d \in W \subset V$ determine a subset

$$\Xi(T, \mu) \stackrel{\text{def}}{=} \{g \in G(\mathbb{A}_f) : e_i \in g \cdot (\mu_i + L_{\widehat{\mathbb{Z}}}) \text{ for all } 1 \leqslant i \leqslant d\},$$

in which we regard $\mu + L_{\widehat{\mathbb{Z}}} \subset V_{\mathbb{A}_f}$, and each $g \in \Xi(T, \mu)$ determines $\mathbb{Z}$-lattices

$$L_g^\flat = V^\flat \cap gL_{\widehat{\mathbb{Z}}}, \qquad \Lambda_g = W \cap gL_{\widehat{\mathbb{Z}}}. \tag{2.12}$$

As the quadratic form on $V^\flat$ is $\mathbb{Z}$-valued on $L_g^\flat$, the constructions of §2.1 associate to it a Shimura datum $(G^\flat, \mathcal{D}^\flat)$, Shimura variety $M_g^\flat$ over $\mathbb{Q}$, and a finite unramified morphism $M_g^\flat \to M$ as in (2.5).

Proposition 2.3.2. *The set $\Xi(T, \mu)$ is stable under left multiplication by $G^\flat(\mathbb{A}_f) \subset G(\mathbb{A}_f)$ and right multiplication by the compact open subgroup $K \subset G(\mathbb{A}_f)$ of (2.3). The Shimura variety $M_g^\flat$ depends only on the double coset $G^\flat(\mathbb{Q})gK$, and there is an isomorphism of M-stacks*

$$\bigsqcup_{g \in G^\flat(\mathbb{Q}) \backslash \Xi(T,\mu)/K} M_g^\flat \cong Z(T, \mu). \tag{2.13}$$

Proof. Only the decomposition (2.13) is nontrivial. For that we use Remark 2.2.9 to identify points of $Z(T, \mu)(\mathbb{C})$ with

$$G(\mathbb{Q}) \backslash \left\{ (z, x, g) \in \mathcal{D} \times V^d \times G(\mathbb{A}_f) : \begin{array}{c} Q(x) = T \\ [z, x_i] = 0, \ \forall\, 1 \leqslant i \leqslant d \\ x_i \in g \cdot (\mu_i + L_{\widehat{\mathbb{Z}}}), \ \forall\, 1 \leqslant i \leqslant d \end{array} \right\} / K.$$

The key point is that the group $G(\mathbb{Q})$ acts transitively on the set of tuples $x \in V^d$ satisfying $Q(x) = T$. Thus any element of the double quotient above is represented by a triple of the form (z, e, g), where $e = (e_1, \ldots, e_d) \in W^d \subset V^d$. As the stabilizer of e is precisely $G^\flat(\mathbb{Q}) \subset G(\mathbb{Q})$, and the condition $[z, e_i] = 0$ for $1 \leqslant i \leqslant d$ is equivalent to $z \in \mathcal{D}^\flat \subset \mathcal{D}$, we may rewrite the double quotient above as

$$Z(T, \mu)(\mathbb{C}) \cong G^\flat(\mathbb{Q}) \backslash \mathcal{D}^\flat \times \Xi(T, \mu)/K.$$

Over the complex fiber, the decomposition (2.13) follows easily from this. In particular, for every g, we have maps $M_{g,\mathbb{C}} \to Z(T, \mu)_{\mathbb{C}}$ of finite unramified stacks over $M_{\mathbb{C}}$. To finish, it is enough to know that these maps descend over $\mathbb{Q}$: This will give the map underlying the isomorphism (2.13), and that it is an isomorphism can be checked over $\mathbb{C}$. The desired descent to $\mathbb{Q}$ is in fact a consequence of the theory of canonical models for Shimura varieties and uses the moduli interpretation of the cycle $Z(T, \mu)$; see the argument in [Mad16, Proposition 6.5]. □

2.4. **Basic properties of special cycles.** First we explain in what sense the morphisms (2.7), which are not closed immersions, deserve to be called special divisors.

Definition 2.4.1. Suppose $D \to X$ is any finite, unramified, and relatively representable morphism of Deligne-Mumford stacks. By [Sta22, Tag 04HG] there is an étale cover $U \to X$ by a scheme such that the pullback $D_U \to U$ is a finite disjoint union

$$D_U = \bigsqcup_i D_U^i$$

with each map $D_U^i \to U$ a closed immersion. If each of these closed immersions is an effective Cartier divisor on U in the usual sense (the corresponding ideal sheaves are invertible), then we call $D \to X$ a *generalized Cartier divisor.*

Remark 2.4.2. Any generalized Cartier divisor $D \to X$ determines an effective Cartier divisor (in the usual sense) on X. Indeed, if we choose an étale cover $U \to X$ as in Definition 2.4.1, then $\boldsymbol{D}_U = \sum_i D_U^i$ is an effective Cartier divisor on U. The descent data for D_U relative to $U \to X$ induces descent data for $\boldsymbol{D}_U$, which then determines an effective Cartier divisor $\boldsymbol{D} \hookrightarrow X$.

Proposition 2.4.3. *Fix $t \in \mathbb{Q}$ and $\mu \in L^\vee/L$.*

1. *If $t > 0$ then $\mathcal{Z}(t, \mu) \to \mathcal{M}$ is a generalized Cartier divisor.*
2. *If $t < 0$ then $\mathcal{Z}(t, \mu) = \varnothing$.*
3. *If $t = 0$ then*

$$\mathcal{Z}(0, \mu) = \begin{cases} \mathcal{M} & \text{if } \mu = 0 \\ \varnothing & \text{if } \mu \neq 0. \end{cases}$$

Proof. The first assertion is [HM20, Proposition 6.5.2], while the second and third follow immediately from the definitions (and Remark 2.2.12).

For future reference, we recall the main ingredient of the proof of (1). Suppose that

$$\begin{CD} S @>>> \mathcal{Z}(t,\mu) \\ @VVV @VVV \\ \widetilde{S} @>>> \mathcal{M} \end{CD}$$

is a commutative diagram of stacks in which $S \to \widetilde{S}$ is a closed immersion of schemes defined by an ideal sheaf $J \subset \mathcal{O}_{\widetilde{S}}$ with $J^2 = 0$. The top horizontal arrow corresponds to a special quasi-endomorphism $x \in V_\mu(\mathcal{A}_S)$, and we want to know when x lies in the image of the (injective) restriction map

$$V_\mu(\mathcal{A}_{\widetilde{S}}) \to V_\mu(\mathcal{A}_S). \tag{2.14}$$

Equivalently, when there is a (necessarily unique) dotted arrow

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{Z}(t,\mu) \\ \downarrow & \nearrow & \downarrow \\ \widetilde{S} & \longrightarrow & \mathcal{M} \end{array}$$

making the diagram commute. In this situation, [HM20, Proposition 6.5.1] provides us with a canonical section

$$\mathrm{obst}_x \in H^0\big(\widetilde{S}, \omega|_{\widetilde{S}}^{-1}\big),$$

called the *obstruction to deforming* x, with the property that x lies in the image of (2.14) if and only if $\mathrm{obst}_x = 0$.

Using this and Nakayama's lemma, one shows that at any geometric point $z \to \mathcal{Z}(t,\mu)$, the kernel of the natural surjection

$$\mathcal{O}^{\mathrm{et}}_{\mathcal{M},z} \to \mathcal{O}^{\mathrm{et}}_{\mathcal{Z}(t,\mu),z}$$

is a principal ideal, and (1) follows from this. Again, see [HM20, Proposition 6.5.2] for details. □

Proposition 2.4.4. *Fix a special cycle* (2.8)*. Every irreducible component* $\mathcal{Z} \subset \mathcal{Z}(T,\mu)$ *satisfies*

$$\dim(\mathcal{Z}) \geqslant \dim(\mathcal{M}) - \mathrm{rank}(T).$$

If equality holds, then $\mathcal{Z}(T,\mu)$ *is a local complete intersection over* $\mathbb{Z}[\Sigma^{-1}]$ *at every point of* $\mathcal{Z}$.

Proof. For any geometric point $z \to \mathcal{Z}(T,\mu)$, the kernel of the natural surjection

$$\mathcal{O}^{\mathrm{et}}_{\mathcal{M},z} \to \mathcal{O}^{\mathrm{et}}_{\mathcal{Z}(T,\mu),z}$$

is generated by $d = \mathrm{rank}(T)$ elements. This follows from Nakayama's lemma and the deformation theory used in the proof of Proposition 2.4.3; see also [Mad16, Corollary 5.17]. From this, the asserted inequality is immediate. Moreover, it is clear that $\mathcal{Z}(T,\mu)$ is a local complete intersection at z whenever

$$\dim(\mathcal{O}^{\mathrm{et}}_{\mathcal{Z}(T,\mu),z}) = \dim(\mathcal{O}^{\mathrm{et}}_{\mathcal{M},z}) - d = \dim(\mathcal{M}) - \mathrm{rank}(T).$$ □

Proposition 2.4.5. *For any special cycle* (2.8) *and any* $A \in \mathrm{GL}_d(\mathbb{Z})$*, there is an isomorphism of* $\mathcal{M}$*-stacks*

$$\mathcal{Z}(T,\mu) \cong \mathcal{Z}({}^tATA, \mu A).$$

Proof. Given a scheme $S \to \mathcal{M}$, the isomorphism sends a tuple

$$(x_1, \ldots, x_d) \in V(\mathcal{A}_S)^d_{\mathbb{Q}}$$

to the tuple $(x_1, \ldots, x_d) \cdot A \in V(\mathcal{A}_S)^d_{\mathbb{Q}}$. □

Proposition 2.4.6. *Given positive integers d' and d'', symmetric matrices*

$$T' \in \mathrm{Sym}_{d'}(\mathbb{Q}) \quad \text{and} \quad T'' \in \mathrm{Sym}_{d''}(\mathbb{Q}),$$

and tuples $\mu' \in (L^\vee/L)^{d'}$ and $\mu'' \in (L^\vee/L)^{d''}$, there is a canonical isomorphism of $\mathcal{M}$-stacks

$$\mathcal{Z}(T',\mu') \times_{\mathcal{M}} \mathcal{Z}(T'',\mu'') \cong \bigsqcup_{T=\left(\begin{smallmatrix} T' & * \\ * & T'' \end{smallmatrix}\right)} \mathcal{Z}(T,\mu)$$

where the disjoint union is over all $T \in \mathrm{Sym}_{d'+d''}(\mathbb{Q})$ of the indicated form, and $\mu = (\mu',\mu'') \in (L^\vee/L)^{d'+d''}$ is the concatenation of the tuples μ' and μ''.

Proof. For any $\mathcal{M}$-scheme S, the S-valued points on both sides can be identified with the set of tuples

$$(x',x'') \in \prod_{i=1}^{d'} V_{\mu'_i}(\mathcal{A}_S) \times \prod_{j=1}^{d''} V_{\mu''_j}(\mathcal{A}_S)$$

such that $Q(x') = T'$ and $Q(x'') = T''$. □

Suppose we have an isometric embedding $L \hookrightarrow L^\sharp$ as in the discussion leading to (2.4). As in Remark 2.2.7, we assume that Σ contains all primes p for which $L^\sharp_p$ is not maximal, so that there is a morphism of integral models

$$\mathcal{M} \to \mathcal{M}^\sharp$$

over $\mathbb{Z}[\Sigma^{-1}]$. The target of this morphism has its own special cycles

$$\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp) \to \mathcal{M}^\sharp$$

indexed by $T^\sharp \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu^\sharp \in (L^{\sharp,\vee}/L^\sharp)^d$, and we wish to describe their pullbacks to $\mathcal{M}$.

Denoting by $\Lambda \subset L^\sharp$ the set of vectors orthogonal to L, there are inclusions of lattices

$$L \oplus \Lambda \subset L^\sharp \subset L^{\sharp,\vee} \subset L^\vee \oplus \Lambda^\vee.$$

Given cosets

$$\mu \in L^\vee/L, \qquad \nu \in \Lambda^\vee/\Lambda, \qquad \mu^\sharp \in L^{\sharp,\vee}/L^\sharp,$$

we write $\mu + \nu = \mu^\sharp$ to indicate that the natural map

$$(L^\vee \oplus \Lambda^\vee)/(L \oplus \Lambda) \to (L^\vee \oplus \Lambda^\vee)/L^\sharp$$

sends

$$\mu + \nu \mapsto \mu^\sharp \in L^{\sharp,\vee}/L^\sharp \subset (L^\vee \oplus \Lambda^\vee)/L^\sharp.$$

Proposition 2.4.7. *There is an isomorphism of $\mathcal{M}$-stacks*

$$\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp) \times_{\mathcal{M}^\sharp} \mathcal{M} \cong \bigsqcup_{\substack{T \in \mathrm{Sym}_d(\mathbb{Q}) \\ \mu \in (L^\vee/L)^d}} \ \bigsqcup_{\substack{\nu \in (\Lambda^\vee/\Lambda)^d \\ \mu+\nu=\mu^\sharp}} \ \bigsqcup_{\substack{y \in \nu + \Lambda^d \\ T+Q(y)=T^\sharp}} \mathcal{Z}(T,\mu),$$

where $\mu + \nu = \mu^\sharp$ is understood as above, but componentwise (that is to say, $\mu_i + \nu_i = \mu^\sharp_i$ for every $1 \leqslant i \leqslant d$).

Proof. By [HM20, Proposition 6.6.2], for any scheme $S \to \mathcal{M}$ there is a canonical isometric embedding

$$V(\mathcal{A}_S) \oplus \Lambda \hookrightarrow V(\mathcal{A}^\sharp_S).$$

Here $\mathcal{A} \to \mathcal{M}$ and $\mathcal{A}^\sharp \to \mathcal{M}^\sharp$ are the Kuga-Satake abelian schemes, and the $\oplus$ on the left is the orthogonal direct sum. This embedding determines a $\mathbb{Q}$-linear isometry

$$V(\mathcal{A}^\sharp_S)_\mathbb{Q} \cong V(\mathcal{A}_S)_\mathbb{Q} \oplus \Lambda_\mathbb{Q},$$

which restricts to a bijection

$$V_{\mu^\sharp}(\mathcal{A}^\sharp_S) \cong \bigsqcup_{\substack{\mu \in L^\vee/L \\ \nu \in \Lambda^\vee/\Lambda \\ \mu+\nu=\mu^\sharp}} V_\mu(\mathcal{A}_S) \times (\nu + \Lambda)$$

for every $\mu^\sharp \in L^{\sharp,\vee}/L^\sharp$. The proposition follows easily from this and the definition of special cycles. ☐

3. Special cycles of low codimension

Keep $L \subset V$ and $\mathcal{M} \to \mathrm{Spec}(\mathbb{Z}[\Sigma^{-1}])$ as in §2.1 and§2.2. Given a positive semi-definite $T \in \mathrm{Sym}_d(\mathbb{Q})$ and a $\mu \in (L^\vee/L)^d$, our goal is prove that if $\mathrm{rank}(T)$ is small relative to $n = \dim(V)$, then the special cycle $\mathcal{Z}(T,\mu)$ is equidimensional and flat over $\mathbb{Z}[\Sigma^{-1}]$.

We also show that divisor classes on $\mathcal{Z}(T,\mu)$ are determined by their restriction to the generic fiber. In §4, this property will allow us to deduce modularity results for cycles on $\mathcal{M}$ from known modularity results on its generic fiber.

3.1. **Connectedness in low codimension.** Our notion of smallness of $\mathrm{rank}(T)$ is always relative to the fixed lattice L, and depends on the following integer $r(L)$ associated to it.

Definition 3.1.1. Denote by $r(L)$ the smallest integer $r \geqslant 0$ such that L is isometric to a $\mathbb{Z}$-module direct summand of a self-dual quadratic $\mathbb{Z}$-module $L^\sharp$ of signature $(n+r,2)$. The existence of such an $L^\sharp$ follows from Proposition B.2.2.

Remark 3.1.2. For any $g \in G(\mathbb{A}_f)$ we have

$$r(L) = r(L_g),$$

where $L_g = V \cap gL_{\widehat{\mathbb{Z}}}$. This is immediate from the fact that if L embeds isometrically as a $\mathbb{Z}$-module direct summand of $L^\sharp$, then L_g embeds isometrically as a $\mathbb{Z}$-module direct summand of $L^\sharp_g = V^\sharp \cap gL^\sharp_{\widehat{\mathbb{Z}}}$.

Proposition 3.1.3. *Suppose $T \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu \in (L^\vee/L)^d$. If*

$$\mathrm{rank}(T) \leqslant \frac{n - 2r(L) - 4}{3},$$

then every connected component of the generic fiber $Z(T,\mu) = \mathcal{Z}(T,\mu)_{\mathbb{Q}}$ *is geometrically connected.*

Proof. It suffices to show that each $\mathbb{Q}$-stack $M_g^\flat$ appearing in (2.13) is geometrically connected. Set $n^\flat = n - \mathrm{rank}(T)$, and recall from (2.12) the quadratic lattice $L_g^\flat \subset V^\flat$ of signature $(n^\flat, 2)$ used to define $M_g^\flat$. Using Remark 2.1.2, we are reduced to proving the existence of isotropic vectors $\ell, \ell_* \in L_g^\flat$ with $[\ell, \ell_*] = 1$.

In general, if N is a quadratic $\mathbb{Z}$-module with $N_{\mathbb{Q}}$ non-degenerate, let $\gamma(N)$ the minimal number of elements needed to generate the finite abelian group $N^\vee/N$. This quantity only depends on the $\widehat{\mathbb{Z}}$-quadratic space $N_{\widehat{\mathbb{Z}}}$. Moreover, if we realize $N \subset N^\sharp$ as a $\mathbb{Z}$-module direct summand of a self-dual quadratic $\mathbb{Z}$-module as in Definition 3.1.1, there is a canonical surjection

$$N^\sharp \cong N^{\sharp,\vee} \to N^\vee$$

whose restriction to N is just the inclusion $N \to N^\vee$. The induced surjection $N^\sharp/N \to N^\vee/N$ shows that

$$\gamma(N) \leqslant \mathrm{rank}_{\mathbb{Z}}(N^\sharp) - \mathrm{rank}_{\mathbb{Z}}(N).$$

As in Remark 3.1.2, set $L_g = V \cap gL_{\widehat{\mathbb{Z}}}$ and abbreviate

$$r = r(L) = r(L_g).$$

Fix an embedding $L_g \to L^\sharp$ as a $\mathbb{Z}$-module direct summand of a self-dual quadratic lattice of signature $(n+r, 2)$. As the submodule $L_g^\flat \subset L_g$ of (2.12) is a $\mathbb{Z}$-module direct summand, the paragraph above implies

$$\gamma(L_g^\flat) \leqslant \mathrm{rank}_{\mathbb{Z}}(L^\sharp) - \mathrm{rank}_{\mathbb{Z}}(L_g^\flat) = \mathrm{rank}(T) + r.$$

This implies the first inequality in

$$2 \cdot \gamma(L_g^\flat) + 6 \leqslant 2 \cdot \mathrm{rank}(T) + 2r + 6 \leqslant n - \mathrm{rank}(T) + 2 = \mathrm{rank}_{\mathbb{Z}}(L_g^\flat)$$

(the second is by the hypotheses of the proposition), and so Proposition B.1.2 implies the existence of the desired isotropic vectors $\ell, \ell_* \in L_g^\flat$. □

3.2. **Geometric properties in low codimension: the self-dual case.** In this subsection, we assume that L is self-dual. In particular, L is hyperspecial (Definition 2.2.1), and the integral model $\mathcal{M}$ is a smooth $\mathbb{Z}[\Sigma^{-1}]$-stack by the proof of Proposition 2.2.4.

Let Λ be a positive definite quadratic $\mathbb{Z}$-module. Set

$$\mathsf{L}(\Lambda) = \{\mathbb{Z}\text{-lattices } \Lambda' \subset \Lambda_{\mathbb{Q}} : \ \Lambda \subset \Lambda' \subset (\Lambda')^\vee \subset \Lambda^\vee\},$$

and for each $\Lambda' \in \mathsf{L}(\Lambda)$, write $\mathcal{Z}(\Lambda')$ for the finite unramified stack over $\mathcal{M}$ with functor of points

$$\mathcal{Z}(\Lambda')(S) = \{\text{isometric embeddings } \iota : \Lambda' \hookrightarrow V(\mathcal{A}_S)\}$$

for any scheme $S \to \mathcal{M}$.

Remark 3.2.1. The above stacks are actually special cycles under a different name. In what follows we fix a basis $e_1, \ldots, e_d \in \Lambda$, and let $T = Q(e) \in \mathrm{Sym}_d(\mathbb{Q})$ be the moment matrix of $e = (e_1, \ldots, e_d)$. There is a canonical isometry $\Lambda_{\mathbb{Q}} \cong W$, where the right hand side is the quadratic space (2.10) determined by T, and a canonical isomorphism of $\mathcal{M}$-stacks

$$\mathcal{Z}(\Lambda) \cong \mathcal{Z}(T, 0)$$

where $0 = (0, \ldots, 0) \in (L^\vee/L)^d$. Indeed, an S-valued point of the left hand side is an isometric embedding $\iota : \Lambda \to V(\mathcal{A}_S)$, and the tuple $x = (\iota(e_1), \ldots, \iota(e_d)) \in V(\mathcal{A}_S)$, defines an S-point of the right hand side.

For each $\Lambda' \in \mathsf{L}(\Lambda)$, the natural map

$$\mathcal{Z}(\Lambda') \xrightarrow{\iota \mapsto \iota|_\Lambda} \mathcal{Z}(\Lambda)$$

is a closed immersion. Henceforth we regard $\mathcal{Z}(\Lambda')$ as a closed substack of $\mathcal{Z}(\Lambda)$, so that $\mathcal{Z}(\Lambda') \subset \mathcal{Z}(\Lambda'')$ whenever $\Lambda'' \subset \Lambda'$ is an inclusion of lattices in $\mathsf{L}(\Lambda)$. The open substack of $\mathcal{Z}(\Lambda')$ defined by

$$^\circ\mathcal{Z}(\Lambda') = \mathcal{Z}(\Lambda') \smallsetminus \bigcup_{\Lambda' \subsetneq \Lambda''} \mathcal{Z}(\Lambda'')$$

is then a locally closed substack of $\mathcal{Z}(\Lambda)$.

By construction, we have the equality of sets

$$\mathcal{Z}(\Lambda)(k) = \bigsqcup_{\Lambda' \in \mathsf{L}(\Lambda)} {}^\circ\mathcal{Z}(\Lambda')(k) \tag{3.1}$$

for any algebraically closed field k with $\mathrm{char}(k) \notin \Sigma$. In fact, given a point $s \in \mathcal{Z}(\Lambda)(k)$ corresponding to an isometric embedding $\iota : \Lambda \to V(\mathcal{A}_s)$, we have $s \in {}^\circ\mathcal{Z}(\Lambda')(k)$ if and only if

$$\Lambda' = V(\mathcal{A}_s) \cap \iota(\Lambda)_{\mathbb{Q}}.$$

This last equality says simply that $\Lambda' \subset \Lambda_{\mathbb{Q}}$ is the largest lattice such that ι extends to $\iota : \Lambda' \to V(\mathcal{A}_s)$.

Remark 3.2.2. In the notation of §A.1, the decomposition (3.1) amounts to saying that the topological space $|\mathcal{Z}(\Lambda)|$ is the disjoint union of its locally closed subsets $|{}^\circ\mathcal{Z}(\Lambda')|$.

In the generic fiber we have the following strengthening of (3.1).

Lemma 3.2.3. *For every $\Lambda' \in \mathsf{L}(\Lambda)$ the morphism*

$$^\circ\mathcal{Z}(\Lambda')_{\mathbb{Q}} \to \mathcal{Z}(\Lambda)_{\mathbb{Q}}$$

is an open and closed immersion, and there is an isomorphism of $\mathbb{Q}$-stacks

$$\mathcal{Z}(\Lambda)_{\mathbb{Q}} \cong \bigsqcup_{\Lambda' \in \mathsf{L}(\Lambda)} {}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Q}}$$

inducing the bijection (3.1) *on geometric points of characteristic* 0.

Proof. Proposition 2.3.2 and Remark 3.2.1 give us a decomposition

$$\mathcal{Z}(\Lambda)_{\mathbb{Q}} \cong Z(T,0) \cong \bigsqcup_{g\in G^\flat(\mathbb{Q})\backslash \Xi(T,0)/K} M_g^\flat,$$

which depends on a choice[1] of isometric embedding $\Lambda_{\mathbb{Q}} \hookrightarrow V$. Under this bijection, the locally closed substack ${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Q}}$ is identified with the disjoint union of those $M_g^\flat$ for which g satisfies $\Lambda' = \Lambda_{\mathbb{Q}} \cap gL_{\widehat{\mathbb{Z}}}$. Here the intersection is taken inside $V_{\widehat{\mathbb{Z}}}$. The lemma follows immediately. □

The key geometric result is the following.

Proposition 3.2.4. *Fix a* $\Lambda' \in \mathsf{L}(\Lambda)$, *and assume* $\mathrm{rank}_{\mathbb{Z}}(\Lambda) \leqslant (n-4)/2$.

(1) *The* $\mathbb{Z}[\Sigma^{-1}]$*-stack* ${}^\circ\mathcal{Z}(\Lambda')$ *is normal and flat, and equidimensional of dimension*

$$n - \mathrm{rank}_{\mathbb{Z}}(\Lambda) + 1 = \dim(\mathcal{M}) - \mathrm{rank}_{\mathbb{Z}}(\Lambda).$$

(2) *For any prime* $p \notin \Sigma$, *the special fiber* ${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}$ *is geometrically normal and equidimensional of dimension* $n - \mathrm{rank}_{\mathbb{Z}}(\Lambda)$.

(3) *For any prime* $p \notin \Sigma$, *the natural maps*

$$\pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p^{\mathrm{alg}}}\big) \to \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Z}_{(p)}^{\mathrm{alg}}}\big) \leftarrow \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Q}^{\mathrm{alg}}}\big)$$

are bijections, where $\mathbb{Z}_{(p)}^{\mathrm{alg}}$ *is the integral closure of* $\mathbb{Z}_{(p)}$ *in* $\mathbb{Q}^{\mathrm{alg}}$.

Proof. We will use results from [HM20, §7.1] to which the reader is encouraged to refer for details. The key point is that, under our hypotheses, there exists an open substack (see Proposition 7.1.2 of *loc. cit.*)

$$\mathcal{Z}^{\mathrm{pr}}(\Lambda') \subset {}^\circ\mathcal{Z}(\Lambda')$$

with the following properties:

(1) It has the same generic fiber as ${}^\circ\mathcal{Z}(\Lambda')$.
(2) For any prime $p \notin \Sigma$, the special fiber $\mathcal{Z}^{\mathrm{pr}}(\Lambda')_{\mathbb{F}_p}$ is smooth outside of a codimension 2 substack.

Moreover, Lemma 7.1.5 of *loc. cit.* shows that the complement of $\mathcal{Z}^{\mathrm{pr}}(\Lambda')_{\mathbb{F}_p}$ in ${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}$ has codimension at least 2. The statement there assumes that Λ is maximal, but this is only used to ensure that Λ' maps to a direct summand of $V(\mathcal{A}_s)$ for every geometric point $s \to {}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}$. For us, this holds essentially by definition of ${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}$; see the comments after (3.1).

Combining the above with the argument of Proposition 7.1.6 of *loc. cit.* proves assertions (1) and (2). Assertion (3) follows from [Mad25, Theorem B]. □

Proposition 3.2.4 has two consequences, which are of fundamental importance to our arguments. The first requires the following technical lemma of commutative algebra.

[1]If no such embedding exists then $\mathcal{Z}(\Lambda)_{\mathbb{Q}} = \varnothing$ by Lemma 2.3.1, and there is nothing to prove.

Lemma 3.2.5. *Suppose R is a Cohen-Macaulay local ring over $\mathbb{Z}[\Sigma^{-1}]$. The following are equivalent:*

(1) R is flat over $\mathbb{Z}[\Sigma^{-1}]$.
(2) For every minimal prime $P \subset R$, R/P is flat over $\mathbb{Z}[\Sigma^{-1}]$.

Proof. This is easily deduced from the following two facts. First, a $\mathbb{Z}[\Sigma^{-1}]$-algebra S is flat if and only if every prime $p \notin \Sigma$ is a non-zero divisor in S. Second, since R is Cohen-Macaulay, its zero-divisors are precisely those contained in some minimal prime of R. □

Proposition 3.2.6. *If* $\mathrm{rank}_{\mathbb{Z}}(\Lambda) \leqslant (n-4)/2$, *then* $\mathcal{Z}(\Lambda)$ *is a flat, reduced, local complete intersection over* $\mathbb{Z}[\Sigma^{-1}]$*, and is equidimensional of dimension* $\dim(\mathcal{M}) - \mathrm{rank}_{\mathbb{Z}}(\Lambda)$.

Proof. Fix an irreducible component

$$\mathcal{Z} \subset \mathcal{Z}(\Lambda) = \mathcal{Z}(T, 0)$$

and a geometric generic point $s \to \mathcal{Z}$.[2]

By (3.1), there is a unique $\Lambda' \in \mathsf{L}(\Lambda)$ such that $s \to {}^{\circ}\mathcal{Z}(\Lambda')$. By claim (1) of Proposition 3.2.4, the stack ${}^{\circ}\mathcal{Z}(\Lambda')$ is equidimensional of dimension $\dim(\mathcal{M}) - \mathrm{rank}_{\mathbb{Z}}(\Lambda)$, and so the same is true of its Zariski closure $\overline{{}^{\circ}\mathcal{Z}(\Lambda')} \subset \mathcal{Z}(\Lambda)$. The inclusion $\mathcal{Z} \subset \overline{{}^{\circ}\mathcal{Z}(\Lambda')}$ therefore implies

$$\dim(\mathcal{Z}) \leqslant \dim(\mathcal{M}) - \mathrm{rank}_{\mathbb{Z}}(\Lambda).$$

As the other inequality follows from Proposition 2.4.4, we have proved both that $\mathcal{Z}$ has the expected dimension, and that it is equal to an irreducible component of $\overline{{}^{\circ}\mathcal{Z}(\Lambda')}$. This latter stack is flat over $\mathbb{Z}[\Sigma^{-1}]$ by claim (1) of Proposition 3.2.4, and hence so is $\mathcal{Z}$.

It now follows from Proposition 2.4.4 that $\mathcal{Z}(\Lambda)$ is a local complete intersection over $\mathbb{Z}[\Sigma^{-1}]$. In particular $\mathcal{Z}(\Lambda)$ is Cohen-Macaulay, and hence flat by Lemma 3.2.5 and the flatness of its irreducible components proved above. To now see that it is reduced, it is enough to know that it is generically smooth, which follows from the complex uniformization in Proposition 2.3.2. □

Proposition 3.2.7. *If* $\mathrm{rank}_{\mathbb{Z}}(\Lambda) \leqslant (n-4)/3$, *then restriction*

$$\mathrm{CH}^1(\mathcal{Z}(\Lambda)) \to \mathrm{CH}^1(\mathcal{Z}(\Lambda)_{\mathbb{Q}})$$

to the generic fiber is injective.

Proof. This amounts to proving the triviality of the subspace

$$\mathrm{CH}^1_{\mathrm{vert}}(\mathcal{Z}(\Lambda)) \subset \mathrm{CH}^1(\mathcal{Z}(\Lambda)) \tag{3.2}$$

[2]In Appendix A.1, we explain the notion of a 'generic point' $\xi \to M$ of an irreducible component Z of a Deligne-Mumford stack M, where ξ is a *punctual stack*. Such a punctual stack admits a finite étale cover Spec $L \to \xi$ by the spectrum of a field L, and a geometric generic point is one obtained by taking a separably closed extension of such a field L.

spanned by the irreducible components of $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ as $p \notin \Sigma$ varies. For this we will use the following lemma, which provides a parametrization of those components.

Lemma 3.2.8. *For a Deligne-Mumford stack $\mathcal{X}$, denote by $\pi_{\mathrm{irr}}(\mathcal{X})$ its set of irreducible components. For any prime $p \notin \Sigma$ there are canonical bijections*

$$\begin{array}{ccc} \bigsqcup_{\Lambda' \in \mathsf{L}(\Lambda)} \pi_{\mathrm{irr}}\big({}^\circ\mathcal{Z}(\Lambda')\big) & \longrightarrow & \bigsqcup_{\Lambda' \in \mathsf{L}(\Lambda)} \pi_{\mathrm{irr}}\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}\big) \\ \downarrow & & \downarrow \\ \pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)) & & \pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)_{\mathbb{F}_p}) \end{array} \tag{3.3}$$

characterized as follows:

1. *The horizontal arrow takes an irreducible component ${}^\circ\mathcal{Z} \subset {}^\circ\mathcal{Z}(\Lambda')$ to its reduction ${}^\circ\mathcal{Z}_{\mathbb{F}_p} \subset {}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}$.*
2. *The vertical arrow on the left takes an irreducible component of the locally closed substack ${}^\circ\mathcal{Z}(\Lambda') \subset \mathcal{Z}(\Lambda)$ to its Zariski closure.*
3. *The vertical arrow on the right takes an irreducible component of the locally closed substack ${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p} \subset \mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ to its Zariski closure.*

Moreover, given distinct irreducible components

$${}^\circ\mathcal{Z}_1, {}^\circ\mathcal{Z}_2 \subset {}^\circ\mathcal{Z}(\Lambda'),$$

the intersection of ${}^\circ\mathcal{Z}_1$ with the Zariski closure of ${}^\circ\mathcal{Z}_2$ in $\mathcal{Z}(\Lambda)$ is empty.

Proof. We first show that for any $\Lambda' \in \mathsf{L}(\Lambda)$, all arrows in

$$\begin{array}{ccccc} \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p^{\mathrm{alg}}}\big) & \longrightarrow & \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Z}_{(p)}^{\mathrm{alg}}}\big) & \longleftarrow & \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Q}^{\mathrm{alg}}}\big) \\ \downarrow a & & \downarrow b & & \downarrow c \\ \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}\big) & \xrightarrow{d} & \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Z}_{(p)}}\big) & \xleftarrow{e} & \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Q}}\big) \end{array}$$

are bijective. Claim (3) of Proposition 3.2.4 shows that both horizontal arrows in the top row are bijective. Proposition 3.1.3 and our hypothesis on $\mathrm{rank}(\Lambda) = \mathrm{rank}(T)$ guarantee that every connected component of $\mathcal{Z}(\Lambda)_{\mathbb{Q}} = Z(T, 0)$ is geometrically connected (note that $r(L) = 0$ by our assumption that L is self-dual), and so the same is true of ${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Q}}$ by Lemma 3.2.3. This shows that the arrow labeled c is bijective. The morphism

$${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Z}_{(p)}} \to \mathrm{Spec}(\mathbb{Z}_{(p)})$$

is flat with reduced special fiber by claims (1) and (2) of Proposition 3.2.4, and so [Sta22, Tag 055J] implies that the arrow labeled e is injective. The arrow labeled b is surjective because it is induced by a surjective morphism of stacks. It follows that all arrows in the square on the right are bijections, as is the composition $d \circ a$. This implies the injectivity of a, and surjectivity follows by the same reasoning as for b. The arrow labeled d is bijective because, at this point, we know the bijectivity of all the other arrows.

Now we turn to the diagram (3.3). By Proposition 3.2.4 each ${}^\circ\mathcal{Z}(\Lambda')$ is normal and flat over $\mathbb{Z}[\Sigma^{-1}]$, and so there are canonical identifications

$$\pi_{\mathrm{irr}}\big({}^\circ\mathcal{Z}(\Lambda')\big) = \pi_{\mathrm{irr}}\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Z}_{(p)}}\big) = \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{Z}_{(p)}}\big).$$

Similarly, the normality of ${}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}$ implies

$$\pi_{\mathrm{irr}}\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}\big) = \pi_0\big({}^\circ\mathcal{Z}(\Lambda')_{\mathbb{F}_p}\big).$$

Combining this with the paragraph above yield the top horizontal bijection in (3.3). The vertical bijections in (3.3) are formal consequences of (3.1) and our dimension calculations; see the proof of Proposition 3.2.6.

The final claim follows from the normality of ${}^\circ\mathcal{Z}(\Lambda')$. Suppose $s \to {}^\circ\mathcal{Z}_1$ is a geometric point also contained in the Zariski closure of ${}^\circ\mathcal{Z}_2$ in $\mathcal{Z}(\Lambda)$. This is the same as the Zariski closure of ${}^\circ\mathcal{Z}_2$ in $\mathcal{Z}(\Lambda')$, and hence any open subset of $\mathcal{Z}(\Lambda')$ containing s must intersect ${}^\circ\mathcal{Z}_2$. One such open subset is ${}^\circ\mathcal{Z}_1$ itself, and so ${}^\circ\mathcal{Z}_1 \cap {}^\circ\mathcal{Z}_2 \neq \varnothing$. This is impossible, as these are distinct *connected* components of ${}^\circ\mathcal{Z}(\Lambda')$. □

Lemma 3.2.8 determines a canonical bijection

$$\pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)) \to \pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)_{\mathbb{F}_p}), \tag{3.4}$$

but this does *not* send an irreducible component $\mathcal{Z} \subset \mathcal{Z}(\Lambda)$ to its reduction $\mathcal{Z}_{\mathbb{F}_p} \subset \mathcal{Z}(\Lambda)_{\mathbb{F}_p}$. Indeed, this reduction need not be irreducible (or reduced), and an irreducible component of $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ may be contained in $\mathcal{Z}_{\mathbb{F}_p}$ for more than one $\mathcal{Z}$. Instead, the bijection sends $\mathcal{Z}$ to a distinguished irreducible component of $\mathcal{Z}_{\mathbb{F}_p}$.

Remark 3.2.9. Although we will not need to do so, one can show that this distinguished component can be characterized in the following way. If we pull back the Kuga-Satake abelian scheme to a generic geometric point $\eta \to \mathcal{Z}$ then there is a tautological isometric embedding $\Lambda \subset V(\mathcal{A}_\eta)$, and a largest $\Lambda' \subset \Lambda_{\mathbb{Q}}$ for which this extends to $\Lambda' \subset V(\mathcal{A}_\eta)$. It follows that if $s \to \mathcal{Z}_{\mathbb{F}_p}$ is a geometric generic point of an irreducible component, then also $\Lambda' \subset V(\mathcal{A}_s)$. The distinguished irreducible component is the unique one for which this last inclusion cannot be extended to any larger lattice in $\Lambda_{\mathbb{Q}}$.

We now return to the proof of Proposition 3.2.7. Fix a prime $p \notin \Sigma$, a $\Lambda' \in \mathsf{L}(\Lambda)$, and an irreducible component ${}^\circ\mathcal{Z} \subset {}^\circ\mathcal{Z}(\Lambda')$. Using Lemma 3.2.8, we see that the Zariski closure of ${}^\circ\mathcal{Z}$ in $\mathcal{Z}(\Lambda)$ is an irreducible component

$$I(\Lambda', {}^\circ\mathcal{Z}) \in \pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)), \tag{3.5}$$

while the Zariski closure of ${}^\circ\mathcal{Z}_{\mathbb{F}_p}$ in $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ is an irreducible component

$$I_p(\Lambda', {}^\circ\mathcal{Z}) \in \pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)_{\mathbb{F}_p})$$

contained in (3.5). Theses two components correspond under the bijection (3.4), and all irreducible components of $\mathcal{Z}(\Lambda)$ and $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ are of this form.

To prove the triviality of the subspace (3.2), we must therefore prove the triviality of all cycle classes

$$I_p(\Lambda', {}^\circ\mathcal{Z}) \in \mathrm{CH}^1(\mathcal{Z}(\Lambda)). \tag{3.6}$$

This will be by induction on the size of $\mathsf{L}(\Lambda)$.

The base case is when $\mathsf{L}(\Lambda) = \{\Lambda\}$, which happens exactly when Λ is maximal. In this case ${}^\circ\mathcal{Z}(\Lambda) = \mathcal{Z}(\Lambda)$, and so every irreducible component ${}^\circ\mathcal{Z} \subset {}^\circ\mathcal{Z}(\Lambda)$ is already Zariski closed in $\mathcal{Z}(\Lambda)$. It follows that

$$I_p(\Lambda', {}^\circ\mathcal{Z}) = {}^\circ\mathcal{Z}_{\mathbb{F}_p} = I(\Lambda', {}^\circ\mathcal{Z})_{\mathbb{F}_p},$$

which is trivial in $\mathrm{CH}^1(\mathcal{Z}(\Lambda))$. Indeed, it is the Weil divisor of the rational function on $\mathcal{Z}(\Lambda)$ that is p on $I(\Lambda', {}^\circ\mathcal{Z})$ and 1 on all other irreducible components.

We now turn to the inductive step. For any $\Lambda' \in \mathsf{L}(\Lambda)$, the inclusion $\mathcal{Z}(\Lambda') \subset \mathcal{Z}(\Lambda)$ induces a pushforward (Proposition A.1.3)

$$\mathrm{CH}^1(\mathcal{Z}(\Lambda')) \to \mathrm{CH}^1(\mathcal{Z}(\Lambda)).$$

For an irreducible component ${}^\circ\mathcal{Z} \subset {}^\circ\mathcal{Z}(\Lambda')$ we have

$$I_p(\Lambda', {}^\circ\mathcal{Z}) \in \pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda')_{\mathbb{F}_p}) \subset \pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)_{\mathbb{F}_p})$$

by construction, and (3.6) is the pushforward of the corresponding class

$$I_p(\Lambda', {}^\circ\mathcal{Z}) \in \mathrm{CH}^1(\mathcal{Z}(\Lambda')).$$

If $\Lambda \subsetneq \Lambda'$ then this last class is trivial by the induction hypothesis, and hence so is (3.6).

It now suffices to show that every $I_p(\Lambda, {}^\circ\mathcal{Z})$ is rationally equivalent to 0 on $\mathcal{Z}(\Lambda)$. Consider the corresponding irreducible component $I(\Lambda, {}^\circ\mathcal{Z})$ of $\mathcal{Z}(\Lambda)$. By the parametrization of the irreducible components of $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$, there is an equality

$$I(\Lambda, {}^\circ\mathcal{Z})_{\mathbb{F}_p} = \sum_{\substack{\Lambda' \in \mathsf{L}(\Lambda) \\ {}^\circ\mathcal{Z}' \in \pi_{\mathrm{irr}}({}^\circ\mathcal{Z}(\Lambda'))}} m(\Lambda', {}^\circ\mathcal{Z}') \cdot I_p(\Lambda', {}^\circ\mathcal{Z}') \in \mathcal{Z}^1(\mathcal{Z}(\Lambda)) \tag{3.7}$$

for some multiplicities $m(\Lambda', {}^\circ\mathcal{Z}') \in \mathbb{Z}$. More precisely, $I(\Lambda, {}^\circ\mathcal{Z})_{\mathbb{F}_p}$ is an effective Cartier divisor on $\mathcal{Z}(\Lambda)$, and the multiplicities are given by the length of its étale local rings at each of its generic points, each of which of course is the generic point of an irreducible component in $\pi_{\mathrm{irr}}(\mathcal{Z}(\Lambda)_{\mathbb{F}_p})$. Note in particular that this means that the multiplicity $m(\Lambda, {}^\circ\mathcal{Z})$ is *non-zero.*

First we consider those terms on the right hand side for which $\Lambda' = \Lambda$.

Lemma 3.2.10. *For any ${}^\circ\mathcal{Z}' \in \pi_{\mathrm{irr}}({}^\circ\mathcal{Z}(\Lambda))$ we have*

$$m(\Lambda, {}^\circ\mathcal{Z}') \neq 0 \iff \mathcal{Z}' = \mathcal{Z}.$$

Proof. By construction we have

$$I_p(\Lambda, {}^\circ\mathcal{Z}) \subset I(\Lambda, {}^\circ\mathcal{Z})_{\mathbb{F}_p},$$

and so $m(\Lambda, {}^\circ\mathcal{Z}) \neq 0$. Conversely, if $m(\Lambda, {}^\circ\mathcal{Z}') \neq 0$ then

$$I_p(\Lambda, {}^\circ\mathcal{Z}') \subset I(\Lambda, {}^\circ\mathcal{Z})_{\mathbb{F}_p},$$

and hence ${}^\circ\mathcal{Z}'_{\mathbb{F}_p} \subset I(\Lambda, {}^\circ\mathcal{Z})_{\mathbb{F}_p}$. In particular, ${}^\circ\mathcal{Z}'$ intersects the closure of ${}^\circ\mathcal{Z}$ in $\mathcal{Z}(\Lambda)$, and so ${}^\circ\mathcal{Z}' = {}^\circ\mathcal{Z}$ by the final claim of Lemma 3.2.8. □

If we take the image of (3.7) in $\mathrm{CH}^1(\mathcal{Z}(\Lambda))$, the left hand side vanishes because it the Weil divisor of the rational function on $\mathcal{Z}(\Lambda)$ that is p on $I(\Lambda, {}^\circ\mathcal{Z})$ and 1 on all other irreducible components. On the right hand side we have proven the vanishing of every term with $\Lambda \subsetneq \Lambda'$, and of every term with $\Lambda' = \Lambda$ and ${}^\circ\mathcal{Z}' \neq {}^\circ\mathcal{Z}$. Thus

$$0 = m(\Lambda, {}^\circ\mathcal{Z}) \cdot I_p(\Lambda, {}^\circ\mathcal{Z})$$

in the Chow group. As our Chow groups have rational coefficients, it follows that $I_p(\Lambda, {}^\circ\mathcal{Z}) = 0$, completing the proof of Proposition 3.2.7. □

3.3. **Geometric properties in low codimension: the general case.** We now return to the consideration of the general case where L is not necessarily self-dual.

Let $r = r(L)$ and $L \hookrightarrow L^\sharp$ be as in Definition 3.1.1. Thus $L^\sharp$ is a self-dual quadratic $\mathbb{Z}$-module of signature $(n+r, 2)$, containing L as a $\mathbb{Z}$-module direct summand. As in Remark 2.2.7, there is an induced finite morphism

$$\mathcal{M} \to \mathcal{M}^\sharp$$

of normal integral models over $\mathbb{Z}[\Sigma^{-1}]$. The target comes with its own Kuga-Satake abelian scheme $\mathcal{A}^\sharp \to \mathcal{M}^\sharp$, and its own family of special cycles

$$\mathcal{Z}^\sharp(T^\sharp, \mu^\sharp) \to \mathcal{M}^\sharp.$$

Of course we must have $\mu^\sharp = 0$, by the self-duality of $L^\sharp$, so we abbreviate

$$\mathcal{Z}^\sharp(T^\sharp) = \mathcal{Z}^\sharp(T^\sharp, 0).$$

At last, we arrive at the main result of §3.

Proposition 3.3.1. *Fix $T \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu \in (L^\vee/L)^d$, and suppose*

$$\mathrm{rank}(T) \leqslant \frac{n - 2r - 4}{3}.$$

The special cycle $\mathcal{Z}(T, \mu)$ is a flat, reduced, local complete intersection over $\mathbb{Z}[\Sigma^{-1}]$, and is equidimensional of codimension $\mathrm{rank}(T)$ *in $\mathcal{M}$. Moreover, restriction to the generic fiber*

$$\mathrm{CH}^1(\mathcal{Z}(T, \mu)) \to \mathrm{CH}^1(Z(T, \mu))$$

is injective.

Proof. If we set $n^\sharp = n + r$, then every $T^\sharp \in \mathsf{S}$ in Lemma 3.3.2 satisfies

$$\mathrm{rank}(T^\sharp) = \mathrm{rank}(T) + r \leqslant \frac{n^\sharp - 4}{3}.$$

Let $W^\sharp$ be the positive definite quadratic space of rank $\mathrm{rank}(T^\sharp)$ associated to $T^\sharp$ as in (2.10), and let $\Lambda^\sharp \subset W^\sharp$ be the $\mathbb{Z}$-lattice spanned by the distinguished generators $e_1, \ldots, e_{n+r} \in W^\sharp$. As in §3.2, there is an associated Deligne-Mumford stack

$$\mathcal{Z}^\sharp(\Lambda^\sharp) \to \mathcal{M}^\sharp$$

parametrizing isometric embeddings of $\Lambda^\sharp$ into $V(\mathcal{A}^\sharp)$.

As the tuple $e^\sharp = (e_1, \ldots, e_{n+r})$ has moment matrix $T^\sharp = Q(e^\sharp)$ by construction, Remark 3.2.1 provides us with a canonical isomorphism of $\mathcal{M}^\sharp$-stacks $\mathcal{Z}^\sharp(\Lambda^\sharp) \cong \mathcal{Z}^\sharp(T^\sharp)$.

We now need the following lemma.

Lemma 3.3.2. *There exists a finite subset*

$$\mathsf{S} \subset \mathrm{Sym}_{r+d}(\mathbb{Q})$$

of positive semi-definite matrices of rank $r + \mathrm{rank}(T)$ *such that there is an open and closed immersion of* $\mathcal{M}^\sharp$*-stacks*

$$\mathcal{Z}(T, \mu) \hookrightarrow \bigsqcup_{T^\sharp \in \mathsf{S}} \mathcal{Z}^\sharp(T^\sharp).$$

Proof. Let $\Lambda \subset L^\sharp$ be the orthogonal to $L \subset L^\sharp$. By the self-duality of $L^\sharp$ there are canonical bijections

$$L^\vee/L \leftarrow L^\sharp/(L \oplus \Lambda) \to \Lambda^\vee/\Lambda.$$

It follows that there is a (unique) $\nu \in (\Lambda^\vee/\Lambda)^d$ such that $\mu + \nu = 0$ as elements of $L^\sharp/(L \oplus \Lambda)$. If we fix a lift $\tilde{\nu} \in (\Lambda^\vee)^d$ and set

$$T_1 = T + Q(\tilde{\nu}) \in \mathrm{Sym}_d(\mathbb{Q}),$$

then Proposition 2.4.7 implies that there is an open and closed immersion

$$\mathcal{Z}(T, \mu) \hookrightarrow \mathcal{Z}^\sharp(T_1) \times_{\mathcal{M}^\sharp} \mathcal{M}. \tag{3.8}$$

As in the proof of Proposition 2.4.7, for any scheme $S \to \mathcal{M}$ there is a canonical isometric embedding

$$V(\mathcal{A}_S) \oplus \Lambda \hookrightarrow V(\mathcal{A}^\sharp_S)$$

for any scheme $S \to \mathcal{M}$, extending $\mathbb{Q}$-linearly to an isomorphism

$$V(\mathcal{A}_S)_\mathbb{Q} \oplus \Lambda_\mathbb{Q} \cong V(\mathcal{A}^\sharp_S)_\mathbb{Q}.$$

In particular, we have a canonical embedding $\Lambda \to V(\mathcal{A}^\sharp_\mathcal{M})$. A choice of basis $y_1, \ldots, y_r \in \Lambda$ therefore determines a morphism

$$\mathcal{M} \to \mathcal{Z}^\sharp(T_2) \tag{3.9}$$

of $\mathcal{M}^\sharp$-stacks, where $T_2 = Q(y)$ is the moment matrix of the tuple

$$y = (y_1, \ldots, y_r) \in \Lambda^r \subset V(\mathcal{A}^\sharp_\mathcal{M})^r.$$

This map is in fact an open and closed immersion. Since it is known to be finite, it is enough to know that it is an open immersion. For this, note that both source and target are normal Deligne-Mumford stacks, flat over

$\mathbb{Z}_{(p)}$: for $\mathcal{Z}^\sharp(T_2)$, this follows from our numerical hypotheses and Proposition 3.2.6. By [Mad25, Lemma 5.1.12], it now suffices to check that the map is generically an open immersion. This can be checked using the argument in [Mad16, Lemma 7.1].

Combining (3.8) and (3.9) yields an open and closed immersion

$$\mathcal{Z}(T,\mu) \to \mathcal{Z}^\sharp(T_1) \times_{\mathcal{M}^\sharp} \mathcal{Z}^\sharp(T_2) \cong \bigsqcup_{T^\sharp = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix}} \mathcal{Z}^\sharp(T^\sharp),$$

where we have used the product formula from Proposition 2.4.6. Explicitly, for any scheme $S \to \mathcal{M}$, a point of $\mathcal{Z}(T,\mu)(S)$ is given by a tuple $x \in \prod_{i=1}^d V_{\mu_i}(\mathcal{A}_S)$ satisfying $Q(x) = T$. The immersion sends x to the tuple

$$x^\sharp = (x + \tilde{\nu}, y) \in V(\mathcal{A}_S^\sharp)^d \times V(\mathcal{A}_S^\sharp)^r = V(\mathcal{A}_S^\sharp)^{d+r}$$

in the factor indexed by $T^\sharp = Q(x^\sharp)$.

It remains to show that $\mathcal{Z}(T,\mu)$ only meets those $\mathcal{Z}^\sharp(T^\sharp)$ with $T^\sharp$ positive semi-definite and

$$\mathrm{rank}(T^\sharp) = \mathrm{rank}(T) + r = \mathrm{rank}(T) + \mathrm{rank}_{\mathbb{Z}}(\Lambda).$$

This follows from Remark 2.2.12 and the observation (noting that every component of $\tilde{\nu} \in \Lambda_{\mathbb{Q}}^d$ is a $\mathbb{Q}$-linear combination of $y_1, \ldots, y_r \in \Lambda$) that the components of $x^\sharp$ and the components of (x,y) generate the same subspace of $V(\mathcal{A}_S^\sharp)_{\mathbb{Q}} = V(\mathcal{A}_S)_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}}$. □

Using Lemma 3.3.2, the desired properties of $\mathcal{Z}(T,\mu)$ follow immediately from the corresponding properties of $\mathcal{Z}^\sharp(\Lambda^\sharp)$ proved in Proposition 3.2.6 and Proposition 3.2.7. □

4. Modularity in low codimension

Keep the quadratic lattice $L \subset V$ and the integral model $\mathcal{M}$ over $\mathbb{Z}[\Sigma^{-1}]$ as in §2.1 and §2.2. We consider the family of special cycles $\mathcal{Z}(T',\mu')$ on $\mathcal{M}$ indexed by those $T' \in \mathrm{Sym}_{d+1}(\mathbb{Q})$ whose upper left $d \times d$ block is a fixed $T \in \mathrm{Sym}_d(\mathbb{Q})$. Roughly speaking, our goal is show that if d is small then these special cycles are the Fourier coefficients of a Jacobi form of index T, valued in the codimension $d+1$ Chow group of $\mathcal{M}$.

Such a result was proved in the generic fiber (without restriction on d) in the thesis of W. Zhang, by reducing it to a modularity result of Borcherds for generating series of divisors. It is the crucial Proposition 3.3.1 that will allow us to deduce the analogous result on the integral model from the results of Borcherds in the generic fiber.

4.1. **Jacobi forms.** We recall just enough of the theory of Jacobi forms to fix our conventions, as these differ slightly from [Zha09] and [BWR15].

Fix an integer $g \geqslant 1$. The Siegel modular group $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$ acts on the Siegel half-space $\mathcal{H}_g \subset \mathrm{Sym}_g(\mathbb{C})$ via the usual formula

$$\gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1},$$

where we have written

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g.$$

Let $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ act on the space of matrices $M_{2,g-1}(\mathbb{Z})$ by left multiplication. Following [BWR15], we regard the *Jacobi group*

$$J_g \stackrel{\text{def}}{=} \Gamma_1 \ltimes M_{2,g-1}(\mathbb{Z})$$

as a sub*set* of Γ_g using the injective function (*not* a group homomorphism)

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} {}^t x \\ {}^t y \end{pmatrix} \right) \mapsto \left(\begin{array}{cc|cc} 1_{g-1} & -y & 0_{g-1} & x \\ 0 & a & a\,{}^t x + b\,{}^t y & b \\ \hline 0_{g-1} & 0 & 1_{g-1} & 0 \\ 0 & c & c\,{}^t x + d\,{}^t y & d \end{array} \right) \tag{4.1}$$

for column vectors $x, y \in \mathbb{Z}^{g-1}$. Note that the restriction of this injective map to the subgroup $\Gamma_1 \subset J_g$ is actually a group homomorphism.

Remark 4.1.1. As in [WR15, §4], there is an *extended Jacobi group* J_g^{ext}, which can be realized both as a subgroup of Γ_g, and as a central extension

$$1 \to \mathbb{Z} \to J_g^{\text{ext}} \to J_g \to 1$$

with the property that the surjection to J_g admits a set-theoretic section whose image generates J_g^{ext}. The use of the function (4.1) is a convenient way of hiding the presence of the larger extended Jacobi group, as the subset $J_g \subset \Gamma_g$ generates a subgroup isomorphic to J_g^{ext}.

The metaplectic double cover of the Siegel modular group is denoted $\widetilde{\Gamma}_g$. Its elements are pairs

$$\tilde{\gamma} = (\gamma, j_\gamma) \in \widetilde{\Gamma}_g, \tag{4.2}$$

consisting of a $\gamma \in \Gamma_g$ and a holomorphic function $j_\gamma(\tau)$ on $\mathcal{H}_g$ whose square is $\det(C\tau + D)$. As in [BWR15, (5)], the *metaplectic Jacobi group*

$$\widetilde{J}_g \stackrel{\text{def}}{=} \widetilde{\Gamma}_1 \ltimes M_{2,g-1}(\mathbb{Z})$$

can be identified with a subset of $\widetilde{\Gamma}_g$, using an injection lifting (4.1). Moreover, the restriction of this embedding to $\widetilde{\Gamma}_1$ is also a group homomorphism. In this way, we can view $\widetilde{\Gamma}_1$ as a *subgroup* of $\widetilde{\Gamma}_g$.

In the following definition, taken from [BWR15, §2.2], we write elements of the Siegel half-space as

$$\tau = \begin{pmatrix} \tau'' & z \\ {}^t z & \tau' \end{pmatrix} \in \mathcal{H}_g \quad \text{with} \quad \tau' \in \mathcal{H}_1, \quad \tau'' \in \mathcal{H}_{g-1},$$

and $z \in \mathbb{C}^{g-1}$ a column vector.

Definition 4.1.2. Suppose $\rho : \widetilde{\Gamma}_g \to \mathrm{GL}(V)$ is a finite dimensional representation with finite kernel. A holomorphic function

$$\phi_T : \mathcal{H}_1 \times \mathbb{C}^{g-1} \to V$$

is a *Jacobi form of half-integral weight k, index $T \in \mathrm{Sym}_{g-1}(\mathbb{Q})$, and representation ρ* if the function

$$\Phi_T(\tau) \stackrel{\text{def}}{=} \phi_T(\tau', z) \cdot e^{2\pi i \mathrm{Tr}(T \cdot \tau'')} \tag{4.3}$$

on $\mathcal{H}_g$ satisfies the transformation law

$$\Phi_T(\gamma \cdot \tau) = j_\gamma(\tau)^{2k} \rho(\tilde{\gamma}) \cdot \Phi_T(\tau)$$

for all elements (4.2) in the subset $\widetilde{J}_g \subset \widetilde{\Gamma}_g$, and if for all $\alpha, \beta \in \mathbb{Q}^{g-1}$ the function $\phi_T(\tau', \tau'\alpha + \beta)$ of $\tau' \in \mathcal{H}_1$ is holomorphic at ∞.

Any Jacobi form of representation ρ has a Fourier expansion

$$\phi_T(\tau', z) = \sum_{\substack{m \in \mathbb{Q} \\ \alpha \in \mathbb{Q}^{g-1}}} c(m, \alpha) \cdot q^m \xi_1^{\alpha_1} \cdots \xi_{g-1}^{\alpha_{g-1}},$$

where we have set $q^m = e^{2\pi i m \tau'}$ and $\xi_i^{\alpha_i} = e^{2\pi i \alpha_i z_i}$.

Remark 4.1.3. Suppose $\rho : \widetilde{\Gamma}_g \to \mathrm{GL}(V)$ is a finite dimensional representation with finite kernel. Given a holomorphic Siegel modular form $\phi : \mathcal{H}_g \to V$ of half-integer weight k and representation ρ, there is a Fourier-Jacobi expansion

$$\phi(\tau) = \sum_{T \in \mathrm{Sym}_{g-1}(\mathbb{Q})} \phi_T(\tau', z) \cdot e^{2\pi i \mathrm{Tr}(T \cdot \tau'')}$$

in which each coefficient ϕ_T is a Jacobi form of the weight k, index T, and representation ρ.

In practice, the representation ρ will always be a form of the Weil representation. Let N be a free $\mathbb{Z}$-module of finite rank endowed with a (nondegenerate) quadratic form Q. Denote by

$$S_{N,g} = \mathbb{C}[(N^\vee/N)^g] \cong \mathbb{C}[N^\vee/N]^{\otimes g}$$

the finite dimensional vector space of $\mathbb{C}$-valued functions on $(N^\vee/N)^g$, and by $S_{N,g}^*$ its $\mathbb{C}$-linear dual. For any $\mu \in (N^\vee/N)^g$ we denote by $\phi_\mu \in S_{N,g}$ the characteristic function of μ. As μ varies these form a basis of $S_{N,g}$, and we denote by $\phi_\mu^* \in S_{N,g}^*$ the dual basis vectors.

Denote by

$$\omega_{N,g} : \widetilde{\Gamma}_g \to \mathrm{GL}(S_{N,g}) \quad \text{and} \quad \omega_{N,g}^* : \widetilde{\Gamma}_g \to \mathrm{GL}(S_{N,g}^*) \tag{4.4}$$

the Weil representation and its contragredient. To resolve some confusion in the literature, we now pin down the precise normalization of (4.4). Suppose N has signature (p, q), and $\tilde{\gamma} = (\gamma, j_\gamma)$ is as in (4.2).

(1) If $\gamma = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ then, noting that j_γ is a constant function (in fact a square root of $\det(A) = \det(D) = \pm 1$),

$$\omega_{N,g}(\tilde{\gamma}) \cdot \phi_\mu = j_\gamma^{p-q} \cdot \phi_{\mu \cdot A^{-1}}.$$

(2) If $\gamma = \left(\begin{smallmatrix} I & B \\ 0 & I \end{smallmatrix}\right)$ and $j_\gamma = 1$ then

$$\omega_{N,g}(\tilde{\gamma}) \cdot \phi_\mu = e^{-2\pi i \mathrm{Tr}(Q(\tilde{\mu})B)} \cdot \phi_\mu,$$

where $Q(\tilde{\mu}) \in \mathrm{Sym}_g(\mathbb{Q})$ is the moment matrix of any lift $\tilde{\mu} \in (N^\vee)^g$.

(3) If $\gamma = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ and the square root $j_\gamma(\tau) = \sqrt{\det(\tau)}$ is the standard branch determined by $j_\gamma(iI_g) = e^{g\pi i/4}$, then

$$\omega_{N,g}(\tilde{\gamma}) \cdot \phi_\mu = \frac{e^{\pi i g(p-q)/4}}{[N^\vee : N]^{g/2}} \sum_{\nu \in (N^\vee/N)^g} e^{2\pi i [\mu,\nu]} \cdot \phi_\nu.$$

These relations determine $\omega_{N,g}$ uniquely. The normalization is such that if N positive definite of rank n, the theta series

$$\vartheta_{N,g}(\tau) = \sum_{\mu \in (N^\vee/N)^g} \left(\sum_{x \in \mu + N^g} e^{2\pi i \mathrm{Tr}(\tau Q(x))} \right) \phi_\mu^*$$

is a Siegel modular form of weight $n/2$ and representation $\omega_{N,g}^*$.

Remark 4.1.4. Our Weil representation does not agree with the Weil representation $\rho_{N,g}$ of [Zha09, Definition 2.2]. Instead, the isomorphism

$$S_{N,g} \xrightarrow{\phi_\mu \mapsto \phi_\mu^*} S_{N,g}^*$$

identifies $\rho_{N,g}$ with $\omega_{N,g}^*$. Alternatively, our $\omega_{N,g}$ agrees with Zhang's $\rho_{-N,g}$, where $-N$ has the same underlying $\mathbb{Z}$-module as N, but is endowed with the signature (q,p) quadratic form $-Q$.

Remark 4.1.5. There is a recurring error in [Zha09], originating in the proof of [Zha09, Theorem 2.9]. That proof claims that a certain generating series of Borcherds is a modular form of representation $\rho_{L,1}^*$, where L is a quadratic lattice of signature $(n,2)$. In fact, this modular form has representation $\omega_{L,1}^*$; see the proof of Proposition 4.3.3. The point is that Borcherds works with a lattice L of signature $(2,n)$, and to reformulate the theorem for signature $(n,2)$ one must replace the quadratic form by its negative. This does not change the space $S_{L,1}$, but it does change the Weil representation (see the previous remark).

Remark 4.1.6. When $g = 1$ we omit it from the notation, so that

$$\widetilde{\Gamma} = \widetilde{\Gamma}_1 \quad \text{and} \quad \mathcal{H} = \mathcal{H}_1,$$

and the Weil representation and its contragredient are

$$\omega_N : \widetilde{\Gamma} \to \mathrm{GL}(S_N) \quad \text{and} \quad \omega_N^* : \widetilde{\Gamma} \to \mathrm{GL}(S_N^*).$$

4.2. **Statement of modularity in low codimension.** Throughout the remainder of §4 we impose the following two hypotheses.

Hypothesis 4.2.1. The integral model $\mathcal{M}$ over $\mathbb{Z}[\Sigma^{-1}]$ is regular. See Proposition 2.2.4 for conditions on Σ that guarantee this.

Hypothesis 4.2.2. Recalling the integer $r(L) \geqslant 0$ of Definition 3.1.1, we assume that d is a positive integer satisfying

$$d+1 \leqslant \frac{n-2r(L)-4}{3}.$$

The first hypothesis is needed[3] to make sense of (4.5) below, which requires a well-defined intersection product on the rational Chow groups of $\mathcal{M}$. This is available to us if $\mathcal{M}$ is regular, as explained in §A.2. The second hypothesis is imposed so that we may make use of Proposition 3.3.1.

Suppose $T' \in \mathrm{Sym}_{d+1}(\mathbb{Q})$ and $\mu' \in (L^\vee/L)^{d+1}$. Hypothesis 4.2.2 and Proposition 3.3.1 imply that the special cycle

$$\mathcal{Z}(T',\mu') \to \mathcal{M}$$

is flat over $\mathbb{Z}[\Sigma^{-1}]$, and equidimensional of codimension $\mathrm{rank}(T')$ in $\mathcal{M}$. By Definition A.1.4, there is an associated *naive cycle class*

$$[\mathcal{Z}(T',\mu')] \in \mathrm{CH}^{\mathrm{rank}(T')}(\mathcal{M}).$$

Define the *corrected cycle class*

$$\mathcal{C}(T',\mu') \stackrel{\mathrm{def}}{=} \underbrace{c_1(\omega^{-1})\cdots c_1(\omega^{-1})}_{d+1-\mathrm{rank}(T')} \cdot [\mathcal{Z}(T',\mu')] \in \mathrm{CH}^{d+1}(\mathcal{M}), \tag{4.5}$$

where $c_1(\omega^{-1})$ is the image of the inverse tautological line bundle (2.6) under the first Chern class map of Definition A.3.1. Abbreviate

$$\mathcal{C}(T') = \sum_{\mu' \in (L^\vee/L)^{d+1}} \mathcal{C}(T',\mu') \otimes \phi^*_{\mu'} \in \mathrm{CH}^{d+1}(\mathcal{M}) \otimes S^*_{L,d+1}.$$

The remainder of §4 is devoted to the proof of the following result.

Proposition 4.2.3. *For any fixed $T \in \mathrm{Sym}_d(\mathbb{Q})$, the formal generating series*

$$\sum_{\substack{m\in\mathbb{Q}\\ \alpha\in\mathbb{Q}^d}} \mathcal{C}\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} \cdot q^m \xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}$$

with coefficients in $\mathrm{CH}^{d+1}(\mathcal{M}) \otimes S^*_{L,d+1}$ is a Jacobi form of index T, weight $1+\frac{n}{2}$, and representation

$$\omega^*_{L,d+1} : \widetilde{\Gamma}_{d+1} \to \mathrm{GL}(S^*_{L,d+1}).$$

[3]If we knew that there was a well-defined intersection product on the rational Chow group of a normal (but not necessarily regular) stack, then Hypothesis 4.2.1 would be unnecessary throughout §4. In fact, it would be sufficient to know that there is a well-defined intersection product between *Cartier divisors* and arbitrary cycle classes on a normal stack.

Here Jacobi modularity is understood, as in Theorem A, after applying any $\mathbb{Q}$*-linear functional* $\mathrm{CH}^{d+1}(\mathcal{M}) \to \mathbb{C}$.

Proposition 4.2.3, when combined with the main result of [BWR15], is already enough to show that

$$\sum_{T' \in \mathrm{Sym}_{d+1}(\mathbb{Q})} \mathcal{C}(T') \cdot q^{T'}$$

is the q-expansion of a Siegel modular form of representation $\omega^*_{L,d+1}$. See Theorem 6.2.1 and its proof for details.

4.3. **Auxiliary cycle classes.** In this subsection we work with a fixed positive semi-definite $T \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu = (\mu_1, \dots, \mu_d) \in (L^\vee/L)^d$.

Given a $T' \in \mathrm{Sym}_{d+1}(\mathbb{Q})$ with upper left $d \times d$ block T, and a $\mu_{d+1} \in L^\vee/L$, there is a morphism

$$\mathcal{Z}(T', \mu') \to \mathcal{Z}(T, \mu) \tag{4.6}$$

of special cycles on $\mathcal{M}$, where $\mu' = (\mu_1, \dots, \mu_{d+1})$. This morphism sends an S-valued point

$$(x_1, \dots, x_{d+1}) \in V_{\mu_1}(\mathcal{A}_S) \times \cdots \times V_{\mu_{d+1}}(\mathcal{A}_S)$$

of the source to its truncation

$$(x_1, \dots, x_d) \in V_{\mu_1}(\mathcal{A}_S) \times \cdots \times V_{\mu_d}(\mathcal{A}_S).$$

The morphism (4.6) is finite and unramified, by the same argument as in the proof of [AGHM17, Proposition 2.7.2].

Recall from (2.10) that T determines a positive definite quadratic space W of dimension $\mathrm{rank}(T)$, together with distinguished generators $e_1, \dots, e_d \in W$. As in (2.11), a functorial point $S \to \mathcal{Z}(T, \mu)$ determines an isometric embedding

$$W \xrightarrow{e_i \mapsto x_i} V(\mathcal{A}_S)_{\mathbb{Q}}. \tag{4.7}$$

This allows us to define a family of finite unramified stacks

$$\mathcal{Y}(m, \mu_{d+1}, w) \to \mathcal{Z}(T, \mu) \tag{4.8}$$

indexed by $m \in \mathbb{Q}$, $\mu_{d+1} \in L^\vee/L$, and $w \in W$, with functor of points

$$\mathcal{Y}(m, \mu_{d+1}, w)(S) = \left\{ x_{d+1} \in V_{\mu_{d+1}}(\mathcal{A}_S) : \begin{array}{c} Q(x_{d+1} - w) = m \\ [x_{d+1}, e_i] = [w, e_i] \ \forall 1 \leqslant i \leqslant d \end{array} \right\}.$$

Note that the conditions $[x_{d+1}, e_i] = [w, e_i]$ are equivalent to w being the orthogonal projection of x_{d+1} to $W \subset V(\mathcal{A}_S)_{\mathbb{Q}}$.

The following Proposition shows that the new stacks (4.8) are not really new at all. They are special cycles we already know, but indexed in a different way.

Proposition 4.3.1. *There is an isomorphism of $\mathcal{Z}(T,\mu)$-stacks*

$$\mathcal{Y}(m,\mu_{d+1},w) \cong \mathcal{Z}(T',\mu'), \tag{4.9}$$

where $\mu' = (\mu_1,\ldots,\mu_{d+1})$, and

$$T' = \begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m+Q(w) \end{pmatrix},$$

for the column vector $\alpha \in \mathbb{Q}^d$ with components $\alpha_i = [w,e_i]$. Moreover,

$$m \neq 0 \iff \operatorname{rank}(T') = \operatorname{rank}(T)+1,$$

and when these conditions hold both (4.6) *and* (4.8) *are generalized Cartier divisors (Definition 2.4.1).*

Proof. Let W' be the quadratic space of dimension $\operatorname{rank}(T')$ determined by T', exactly as the space W of (2.10) was determined by T. As we do not assume that T' is positive semi-definite, the quadratic space W' need not be positive definite, but it is nondegenerate (by construction). There are distinguished vectors $e_1,\ldots,e_{d+1}$ that span W', the vectors $e_1,\ldots,e_d$ span a positive definite subspace $W \subset W'$ isometric to (2.10), and the tuples

$$e = (e_1,\ldots,e_d) \in W^d \quad \text{and} \quad e' = (e_1,\ldots,e_{d+1}) \in (W')^{d+1}$$

satisfy $Q(e) = T$ and $Q(e') = T'$. Using the relation between T' and w, one checks first that w is the orthogonal projection of e_{d+1} to W, and then that

$$0 = [e_{d+1}-w,w] = Q(e_{d+1}) - Q(e_{d+1}-w) - Q(w).$$

In particular

$$Q(e_{d+1}-w) = m. \tag{4.10}$$

Now we construct the isomorphism (4.9). Fix an $\mathcal{M}$-scheme S and a tuple

$$x = (x_1,\ldots,x_d) \in V_{\mu_1}(\mathcal{A}_S) \times \cdots \times V_{\mu_d}(\mathcal{A}_S).$$

with moment matrix $Q(x) = T$. This determines a point $x \in \mathcal{Z}(T,\mu)(S)$, and an isometric embedding $W \to V(\mathcal{A}_S)_{\mathbb{Q}}$ by (4.7).

A lift of x to $\mathcal{Z}(T',\mu')(S)$ determines a special quasi-endomorphism

$$x_{d+1} \in V_{\mu_{d+1}}(\mathcal{A}_S),$$

which then determines an extension of $W \to V(\mathcal{A}_S)_{\mathbb{Q}}$ to

$$W' \xrightarrow{e_i \mapsto x_i} V(\mathcal{A}_S)_{\mathbb{Q}}.$$

The calculation (4.10) shows that $Q(x_{d+1}-w) = m$, and combining this with $[x_{d+1},e_i] = [x_{d+1},x_i] = \alpha_i$ shows that x_{d+1} defines a lift of x to $\mathcal{Y}(m,\mu_{d+1},w)(S)$.

Conversely, any lift of x to $\mathcal{Y}(m,\mu_{d+1},w)(S)$ corresponds to an $x_{d+1} \in V_{\mu_{d+1}}(\mathcal{A}_S)$ with the property that $Q(x_{d+1}-w) = m$, and the orthogonal projection of x_{d+1} to $W \subset V(\mathcal{A}_S)_{\mathbb{Q}}$ is w. An elementary linear algebra argument shows that $x' = (x_1,\ldots,x_{d+1})$ has moment matrix T', so determines a lift of x to $\mathcal{Z}(T',\mu')(S)$. This establishes the isomorphism (4.9).

If $m = 0$ then (4.10) implies that $e_{d+1} - w$ is an isotropic vector orthogonal to W, which is therefore contained in the radical of the quadratic form on W'. As this radical is trivial, $e_{d+1} = w \in W$. It follows that $W' = W$, and $\mathrm{rank}(T') = \mathrm{rank}(T)$.

Now suppose $m \neq 0$. It follows from (4.10) that $e_{d+1} \neq w$, hence $e_{d+1} \neq W$ and

$$\mathrm{rank}(T') = \dim(W') = \dim(W) + 1 = \mathrm{rank}(T) + 1.$$

It remains to show that (4.8) is a generalized Cartier divisor. In light of the isomorphism (4.9), it suffices to show the same for (4.6).

By Remark 2.2.12 we may assume that T' is positive semi-definite, for otherwise $\mathcal{Z}(T', \mu') = \varnothing$ and the claim is vacuous. This assumption implies that W' is a positive definite quadratic space, and so (4.10) implies $m > 0$. In particular $m + Q(w) > 0$. By Proposition 2.4.3, the right vertical arrow in

$$\begin{array}{ccc} \mathcal{Z}(T,\mu) \times_{\mathcal{M}} \mathcal{Z}(m + Q(w), \mu_{d+1}) & \longrightarrow & \mathcal{Z}(m + Q(w), \mu_{d+1}) \\ \downarrow & & \downarrow \\ \mathcal{Z}(T,\mu) & \longrightarrow & \mathcal{M} \end{array}$$

is a generalized Cartier divisor. Proposition 2.4.6 allows us to realize $\mathcal{Z}(T', \mu')$ as an open and closed substack of the upper left corner, and so there exists an étale cover $U \to \mathcal{Z}(T, \mu)$ such that $\mathcal{Z}(T', \mu')_U \to U$ is a disjoint union of closed immersions $Z_i \to U$, each of which is locally defined by the vanishing of a section of $\mathcal{O}_U$. To see that Z_i is an effective Cartier divisor on U, we must show that this section is not a zero divisor. This relies on the following lemma of commutative algebra.

Lemma 4.3.2. *Suppose that S is a Cohen-Macaulay local Noetherian ring with maximal ideal $\mathfrak{m}$; then an element $a \in \mathfrak{m}$ is a non-zero divisor if and only if $\dim S/(a) = \dim S - 1$.*

Proof. By Krull's Hauptidealsatz [Sta22, Tag 00KV], we have

$$\dim(S/(a)) \geqslant \dim(S) - 1,$$

with equality holding exactly when a is not contained in any minimal prime of S. On the other hand, saying that a is a non-zero divisor is equivalent to saying that a is not contained in any associated prime of S. Since S is Cohen-Macaulay, its associated primes are precisely its minimal ones, and so the lemma follows. □

Recall that Hypothesis (4.2.2) guarantees that $\mathcal{Z}(T', \mu')$ has dimension

$$\dim(\mathcal{M}) - \mathrm{rank}(T') = \dim(\mathcal{M}) - \mathrm{rank}(T) - 1 = \dim(\mathcal{Z}(T, \mu)) - 1.$$

As $\mathcal{Z}(T, \mu)$ is Cohen-Macaulay by Proposition 3.3.1, the desired conclusion now follows from Lemma 4.3.2. □

If $m \neq 0$, Proposition 4.3.1 allows us to define, using Remark 2.4.2,

$$[\mathcal{Y}(m, \mu_{d+1}, w)] \in \mathrm{CH}^1(\mathcal{Z}(T, \mu)) \tag{4.11}$$

as the cycle class associated to the generalized Cartier divisor (4.8). More precisely, it is the first Chern class (Definition A.3.1) of the associated line bundle.

We next extend the definition to $m = 0$. In this case the condition $Q(x_{d+1} - w) = 0$ imposed in the definition of the domain of (4.8) can be satisfied by at most the vector $x_{d+1} = w$, and so

$$\mathcal{Y}(0, \mu_{d+1}, w)(S) = \begin{cases} \mathcal{Z}(T, \mu)(S) & \text{if } w \in V_{\mu_{d+1}}(\mathcal{A}_S) \\ \varnothing & \text{otherwise.} \end{cases}$$

This implies that the morphism (4.8) is a closed immersion. On the other hand, Proposition 4.3.1 tells us that there is a distinguished choice of (T', μ') for which $\mathrm{rank}(T) = \mathrm{rank}(T')$ and

$$\mathcal{Y}(0, \mu_{d+1}, w) \cong \mathcal{Z}(T', \mu').$$

We deduce that for this choice of (T', μ') the morphism (4.6) is a closed immersion between stacks of the same dimension. Now form the first Chern class

$$c_1\left(\omega^{-1}|_{\mathcal{Z}(T', \mu')}\right) \in \mathrm{CH}^1(\mathcal{Z}(T', \mu')),$$

where $\omega \in \mathrm{Pic}(\mathcal{M})$ is the tautological bundle, and define

$$[\mathcal{Y}(0, \mu_{d+1}, w)] \in \mathrm{CH}^1(\mathcal{Z}(T, \mu)) \tag{4.12}$$

to be its push-forward via (4.6).

Let $Y(m, \mu_{d+1}, w)$ be the generic fiber of (4.8), and let

$$[Y(m, \mu_{d+1}, w)] \in \mathrm{CH}^1(Z(T, \mu)) \tag{4.13}$$

be the restriction of (4.11) and (4.12) to the generic fiber of $\mathcal{Z}(T, \mu)$. The following proposition is our version of [Zha09, Proposition 2.6]. One should regard it as a corollary of a theorem of Borcherds [Bor99].

Proposition 4.3.3. *The formal generating series*

$$\sum_{\substack{m \in \mathbb{Q} \\ w \in W \\ \mu_{d+1} \in L^\vee / L}} [Y(m, \mu_{d+1}, w)] \otimes \phi^*_{\mu_{d+1}} \cdot q^{m + Q(w)} \xi_1^{[w, e_1]} \cdots \xi_d^{[w, e_d]}$$

converges[4] *to a holomorphic function*

$$\phi_T(\tau', z) : \mathcal{H} \times \mathbb{C}^d \to \mathrm{CH}^1(Z(T, \mu)) \otimes S_L^*.$$

The corresponding function (4.3) *on* $\mathcal{H}_{d+1}$ *satisfies*

$$\Phi_T(\gamma \cdot \tau) = j_\gamma(\tau)^{2+n} \omega_L^*(\tilde{\gamma}) \cdot \Phi_T(\tau)$$

for all elements $(\gamma, j_\gamma) \in \widetilde{\Gamma} \subset \widetilde{\Gamma}_{d+1}$ *as in* (4.2).

[4]Convergence is understood in the sense of Theorem A. That is to say, after applying any $\mathbb{C}$-linear functional $\mathrm{CH}^1(Z(T, \mu)) \to \mathbb{C}$.

Proof. The claim is vacuously true if $Z(T,\mu)$ is empty. Hence, as in §2.3, we may fix an orthogonal decomposition $V = V^\flat \oplus W$ and use this to express

$$Z(T,\mu) = \bigsqcup_{g \in G^\flat(\mathbb{Q})\backslash \Xi(T,\mu)/K} M_g^\flat. \tag{4.14}$$

as a disjoint union of smaller Shimura varieties. As in (2.12), each $g \in \Xi(T,\mu)$ determines lattices $L_g^\flat \subset V^\flat$ and $\Lambda_g \subset W$.

The Shimura variety $M_g^\flat$ has its own tautological line bundle $\omega_g^\flat$, its own Kuga-Satake abelian scheme $A_g^\flat \to M_g^\flat$, and its own family of special cycles

$$Z_g^\flat(m,\nu) \to M_g^\flat$$

indexed by $m \in \mathbb{Q}$ and $\nu \in L_g^{\flat,\vee}/L_g^\flat$. When $m \neq 0$ there is an associated class

$$[Z_g^\flat(m,\nu)] \in \mathrm{CH}^1(M_g^\flat)$$

by Remark 2.4.2 and Proposition 2.4.3. When $m = 0$ we define

$$[Z_g^\flat(0,\nu)] = \begin{cases} c_1(\omega_g^{\flat,-1}) & \text{if } \nu = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.3.4. At a prime $p \notin \Sigma$ the lattice $L_g^\flat$ may be far from maximal. Fortunately, we have no need for any integral model of $M_g^\flat$ over $\mathbb{Z}[\Sigma^{-1}]$. All constructions and proofs from §2.2 and §2.4 can be carried out (usually with less effort) directly on the canonical model over $\mathbb{Q}$.

Lemma 4.3.5. *For every $g \in \Xi(T,\mu)$ there is a canonical isomorphism of $M_g^\flat$-stacks*

$$Y(m,\mu_{d+1},w) \times_{Z(T,\mu)} M_g^\flat \cong \bigsqcup_{\substack{\nu \in L_g^{\flat,\vee}/L_g^\flat \\ \nu + w \in g\cdot(\mu_{d+1} + L_{\widehat{\mathbb{Z}}})}} Z_g^\flat(m,\nu). \tag{4.15}$$

Here we regard $\mu_{d+1} + L_{\widehat{\mathbb{Z}}} \subset V_{\mathbb{A}_f}$. Moreover,

$$[Y(m,\mu_{d+1},w)]|_{M_g^\flat} = \sum_{\substack{\nu \in L_g^{\flat,\vee}/L_g^\flat \\ \nu + w \in g\cdot(\mu_{d+1} + L_{\widehat{\mathbb{Z}}})}} [Z_g^\flat(m,\nu)] \in \mathrm{CH}^1(M_g^\flat) \tag{4.16}$$

where the left hand side is the projection of (4.13) *to the g-summand in*

$$\mathrm{CH}^1(Z(T,\mu)) \cong \bigoplus_{g \in G^\flat(\mathbb{Q})\backslash \Xi(T,\mu)/K} \mathrm{CH}^1(M_g^\flat).$$

Proof. Applying the proof of Proposition 2.4.7 to the morphism $M_g^\flat \to M$ of Remark 2.5, we see that that for any $M_g^\flat$-scheme S there is a canonical isometry

$$V(A_S)_\mathbb{Q} = V(A_{g,S}^\flat)_\mathbb{Q} \oplus W,$$

which restricts to a bijection

$$V_{\mu_{d+1}}(A_S) = \bigsqcup_{\substack{\nu\in L_g^{\flat,\vee}/L_g^\flat \\ \lambda\in\Lambda_g^\vee/\Lambda_g \\ \nu+\lambda\in g\cdot(\mu_{d+1}+L_{\widehat{\mathbb{Z}}})}} V_\nu(A^\flat_{g,S}) \times (\lambda+\Lambda_g). \tag{4.17}$$

This isometric embedding $W \to V(A_S)_\mathbb{Q}$ agrees with that of (4.7).

A S-point of the left hand side of (4.15) is an S-point of $M_g^\flat \subset Z(T,\mu)$, together with a special quasi-endomorphism

$$x_{d+1} \in V_{\mu_{d+1}}(A_S)$$

whose orthogonal projection to W is w, and such that $Q(x_{d+1}-w)=m$. Using (4.17), we find that there is a unique pair of cosets

$$\nu \in L_g^{\flat,\vee}/L_g^\flat \quad \text{and} \quad \lambda\in\Lambda_g^\vee/\Lambda_g$$

such that $\nu+\lambda \in g\cdot(\mu_{d_1}+L_{\widehat{\mathbb{Z}}})$, and such that

$$x_{d+1}-w \in V_\nu(A^\flat_{g,S}) \quad \text{and} \quad w\in\lambda+\Lambda_g.$$

In particular, $x_{d+1}-w$ determines an S-point of $Z_g^\flat(m,\nu)$.

This construction establishes the isomorphism (4.15), from which (4.16) follows directly. We note that when $m=0$ both sides of (4.16) are equal to

$$\begin{cases} c_1(\omega^{-1}_{M_g^\flat}) & \text{if } w\in g\cdot(\mu_{d+1}+L_{\widehat{\mathbb{Z}}}) \\ 0 & \text{otherwise.}\end{cases} \qquad \square$$

Dualizing the tautological map $S_{\Lambda_g}\otimes S^*_{\Lambda_g}\to\mathbb{C}$ yields a homomorphism

$$\mathbb{C}\to S^*_{\Lambda_g}\otimes S_{\Lambda_g} \tag{4.18}$$

sending $1\mapsto \sum_{w\in\Lambda_g^\vee/\Lambda_g}\phi_w^*\otimes\phi_w$. On the other hand, if we abbreviate

$$L_g = V\cap gL_{\widehat{\mathbb{Z}}},$$

we may use the inclusions

$$L_g^\flat\oplus\Lambda_g\subset L_g\subset L_g^\vee\subset L_g^{\flat,\vee}\oplus\Lambda_g^\vee$$

to define a homomorphism $S^*_{L_g^\flat}\otimes S^*_{\Lambda_g}\to S^*_{L_g}$ by

$$\phi_\nu^*\otimes\phi_w^*\mapsto\begin{cases}\phi^*_{\nu+w} & \text{if } \nu+w\in L_g^\vee \\ 0 & \text{otherwise}\end{cases}$$

for all $\nu\in L_g^{\flat,\vee}/L_g^\flat$ and $w\in\Lambda_g^\vee/\Lambda_g$. This defines the second arrow in the $\widetilde{\Gamma}_1$-equivariant composition

$$S^*_{L_g^\flat}\xrightarrow{(4.18)} S^*_{L_g^\flat}\otimes S^*_{\Lambda_g}\otimes S_{\Lambda_g}\to S^*_{L_g}\otimes S_{\Lambda_g}\to S^*_L\otimes S_{\Lambda_g} \tag{4.19}$$

The third arrow is defined using the isomorphism $S^*_{L_g}\cong S^*_L$ induced by $g^{-1}: L_g^\vee/L_g\to L^\vee/L$.

It is a theorem of Borcherds [Bor99] that the formal generating series

$$\sum_{\substack{m \geqslant 0 \\ \nu \in L_g^{\flat,\vee}/L_g^\flat}} [Z_g^\flat(m,\nu)] \otimes \phi_\nu^* \cdot q^m$$

with coefficients in $\mathrm{CH}^1(M_g^\flat) \otimes S_{L_g^\flat}^*$ is a modular form on $\mathcal{H}$ of weight $1 + \frac{n^\flat}{2}$ and representation

$$\omega_{L_g^\flat}^* : \widetilde{\Gamma} \to \mathrm{GL}(S_{L_g^\flat}^*).$$

More precisely, as Borcherds works only in the complex fiber, one should use [HM20, Theorem B] for the corresponding modularity statement on the canonical model.

Applying (4.19) to this last generating series coefficient-by-coefficient yields the formal generating series

$$\sum_{\substack{m \geqslant 0 \\ \nu \in L_g^{\flat,\vee}/L_g^\flat}} \sum_{\substack{\mu_{d+1} \in L^\vee/L \\ w \in \Lambda_g^\vee/\Lambda_g \\ \nu + w \in g \cdot (\mu_{d+1} + L_{\widehat{\mathbb{Z}}})}} [Z_g^\flat(m,\nu)] \otimes \phi_{\mu_{d+1}}^* \otimes \phi_w \cdot q^m \tag{4.20}$$

with coefficients in $\mathrm{CH}^1(M_g^\flat) \otimes S_L^* \otimes S_{\Lambda_g}$, which is therefore a modular form on $\mathcal{H}$ of weight $1 + \frac{n^\flat}{2}$ and representation

$$\omega_L^* \otimes \omega_{\Lambda_g} : \widetilde{\Gamma} \to \mathrm{GL}(S_L^* \otimes S_{\Lambda_g}).$$

Consider the theta function $\mathcal{H} \times \mathbb{C}^d \to S_{\Lambda_g}^*$ defined by

$$\vartheta_w(\tau', z) = \sum_{w \in \Lambda_g^\vee} \phi_w^* \cdot q^{Q(w)} \xi_1^{[w,e_1]} \cdots \xi_d^{[w,e_d]}. \tag{4.21}$$

If, as in Definition 4.1.2, we define a function on $\mathcal{H}_{d+1}$ by

$$\Theta_w(\tau) = \vartheta_w(\tau', z) \cdot e^{2\pi i \mathrm{Tr}(T \cdot \tau'')}$$

then [Zha09, Lemma 2.8] implies the equality

$$\Theta_w(\gamma \cdot \tau) = j_\gamma(\tau)^{n - n^\flat} \omega_{\Lambda_g}^*(\tilde{\gamma}) \cdot \Theta_w(\tau)$$

for all (4.2) in the subgroup $\widetilde{\Gamma} \subset \widetilde{\Gamma}_{d+1}$.

Now use the tautological pairing $S_{\Lambda_g} \otimes S_{\Lambda_g}^* \to \mathbb{C}$ to multiply (4.20) with (4.21). Lemma 4.3.5 implies that the resulting generating series is

$$\sum_{\substack{m \in \mathbb{Q} \\ w \in W \\ \mu_{d+1} \in L^\vee/L}} [Y(m, \mu_{d+1}, w)]|_{M_g^\flat} \otimes \phi_{\mu_{d+1}}^* \cdot q^{m+Q(w)} \xi_1^{[w,e_1]} \cdots \xi_d^{[w,e_d]},$$

which therefore satisfies the transformation law stated in Proposition 4.3.3. Varying g and using (4.14) completes the proof of that proposition. □

Corollary 4.3.6. *The generating series*

$$\sum_{\substack{m\in\mathbb{Q}\\ w\in W\\ \mu_{d+1}\in L^\vee/L}} [\mathcal{Y}(m,\mu_{d+1},w)]\otimes\phi^*_{\mu_{d+1}}\cdot q^{m+Q(w)}\xi_1^{[w,e_1]}\cdots\xi_d^{[w,e_d]}$$

defines a holomorphic function

$$\mathcal{H}\times\mathbb{C}^d\to \mathrm{CH}^1(\mathcal{Z}(T,\mu))\otimes S_L^*$$

satisfying the same transformation law under $\widetilde{\Gamma}\subset\widetilde{\Gamma}_{d+1}$ *as the generating series of Proposition 4.3.3.*

Proof. Proposition 3.3.1 (which applies, thanks to Hypothesis 4.2.2) implies the injectivity of the restriction map

$$\mathrm{CH}^1(\mathcal{Z}(T,\mu))\to\mathrm{CH}^1(Z(T,\mu)),$$

and so the claim is immediate from Proposition 4.3.3. □

4.4. **Proof of Proposition 4.2.3.** Proposition 4.2.3 is vacuously true if $T\in\mathrm{Sym}_d(\mathbb{Q})$ is not positive semi-definite. Indeed, in this case $\mathcal{Z}(T,\mu)=\varnothing$ by Remark 2.2.12, and it follows from (4.5) and (4.6) that the generating series of Proposition 4.2.3 vanishes coefficient-by-coefficient. Thus we may assume, as in §4.3 that $T\in\mathrm{Sym}_d(\mathbb{Q})$ is positive semi-definite, and let $e_1,\ldots,e_d\in W$ be as in (2.10).

By Hypothesis 4.2.2 and Proposition 3.3.1, for any $\mu=(\mu_1,\ldots,\mu_d)\in(L^\vee/L)^d$ the canonical finite unramified map

$$f^{(T,\mu)}:\mathcal{Z}(T,\mu)\to\mathcal{M}$$

has equidimensional image of codimension $\mathrm{rank}(T)$, and so induces (Proposition A.1.3) a pushforward

$$f_*^{(T,\mu)}:\mathrm{CH}^1(\mathcal{Z}(T,\mu))\to\mathrm{CH}^{\mathrm{rank}(T)+1}(\mathcal{M}).$$

The following lemma relates the images of the cycle classes (4.11) and 4.12 under this map to the coefficients appearing in Proposition 4.2.3.

Lemma 4.4.1. *For any* $m\in\mathbb{Q}$, $w\in W$, *and* $\mu_{d+1}\in L^\vee/L$ *we have the equality*

$$\underbrace{c_1(\omega^{-1})\cdots c_1(\omega^{-1})}_{d-\mathrm{rank}(T)}\cdot f_*^{(T,\mu)}[\mathcal{Y}(m,\mu_{d+1},w)]=\mathcal{C}(T',\mu')$$

in $\mathrm{CH}^{d+1}(\mathcal{M})$. *On the right,* $T'\in\mathrm{Sym}_{d+1}(\mathbb{Q})$ *and* $\mu'\in(L^\vee/L)^{d+1}$ *have the same meaning as in Proposition 4.3.1.*

Proof. First suppose $m\neq 0$, so that $\mathrm{rank}(T')=\mathrm{rank}(T)+1$ by Proposition 4.3.1. It follows directly from the definitions and the commutative diagram

$$\begin{array}{ccc} \mathcal{Y}(m,\mu_{d+1},w) & \overset{(4.9)}{=\!=\!=\!=\!=} & \mathcal{Z}(T',\mu') \\ & {\scriptstyle (4.8)}\searrow \quad \swarrow{\scriptstyle (4.6)} & \\ & \mathcal{Z}(T,\mu) & \\ & \Big\downarrow {\scriptstyle f^{(T,\mu)}} & \\ & \mathcal{M} & \end{array}$$

that

$$f_*^{(T,\mu)}[\mathcal{Y}(m,\mu_{d+1},w)] = [\mathcal{Z}(T',\mu')]$$

in the codimension $\mathrm{rank}(T)+1$ Chow group of $\mathcal{M}$, and intersecting both sides with $d-\mathrm{rank}(T)$ copies of $c_1(\omega^{-1})$ proves the claim.

Suppose now that $m=0$, so that $\mathrm{rank}(T') = \mathrm{rank}(T)$ by Proposition 4.3.1. In this case

$$f_*^{(T,\mu)}[\mathcal{Y}(0,\mu_{d+1},w)] = f_*^{(T',\mu')}\left(\omega^{-1}|_{\mathcal{Z}(T',\mu')}\right) = c_1(\omega^{-1})\cdot[\mathcal{Z}(T',\mu')]$$

holds in the codimension $\mathrm{rank}(T)+1$ Chow group, where the first equality is by the definition of (4.12), and the second is by Proposition A.3.3. Once again, the claim follows. □

Lemma 4.4.2. *Suppose $m\in\mathbb{Q}$ and $\alpha\in\mathbb{Q}^d$. If*

$$\mathcal{C}\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} \neq 0$$

then there exists a unique $w\in W$ such that $\alpha_i = [w,e_i]$ for all $i=1,\ldots,d$.

Proof. Our assumption implies that there is some $\mu'\in(L^\vee/L)^{d+1}$ for which

$$\mathcal{Z}\left(\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix},\mu'\right)\neq\varnothing.$$

Any non-empty scheme S mapping to it determines special quasi-endomorphisms

$$x_1,\ldots,x_{d+1}\in V(\mathcal{A}_S)_{\mathbb{Q}}.$$

The first d-coordinates $x=(x_1,\ldots,x_d)$ satisfy $Q(x)=T$, and so determine an isometric embedding

$$W \xrightarrow{e_i\mapsto x_i} V(\mathcal{A}_S)_{\mathbb{Q}}.$$

Using the relation $[x_{d+1},x_i]=\alpha_i$ for $i=1,\ldots,d$, we see that the orthogonal projection of x_{d+1} to $W\subset V(\mathcal{A}_S)_{\mathbb{Q}}$ is a vector $w\in W$ with the desired properties. The uniqueness is clear, as $e_1,\ldots,e_d$ span the positive definite quadratic space W. □

Lemma 4.4.3. *For any $m\in\mathbb{Q}$ and column vectors $\alpha\in\mathbb{Q}^d$ and $x,y\in\mathbb{Z}^d$ we have*

$$\omega_{L,d+1}^*(\tilde{\gamma})\cdot\mathcal{C}\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} = e^{2\pi i\,{}^t x\alpha}\cdot\mathcal{C}\begin{pmatrix} T & Ty+\frac{\alpha}{2} \\ {}^ty\,{}^tT+\frac{{}^t\alpha}{2} & {}^ty\,T\,y+{}^ty\,\alpha+m \end{pmatrix},$$

where

$$\tilde{\gamma} = \left(\mathrm{id}_{\widetilde{\Gamma}}, \begin{pmatrix} {}^t x \\ {}^t y \end{pmatrix}\right) \in \widetilde{\Gamma} \ltimes M_{2,d}(\mathbb{Z}) = \widetilde{J}_{d+1}. \tag{4.22}$$

Proof. By the explicit formulas for (4.4), if we set

$$A = \begin{pmatrix} I_d & y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{d+1}(\mathbb{Z})$$

then $\tilde{\gamma}$ acts on $S^*_{L,d+1}$ as

$$\omega^*_{L,d+1}(\tilde{\gamma}) \cdot \phi^*_{\mu'} = e^{2\pi i x_1[\mu_1,\mu_{d+1}]} \cdots e^{2\pi i x_d[\mu_d,\mu_{d+1}]} \cdot \phi^*_{\mu' A}$$

for any $\mu' \in (L^\vee/L)^{d+1}$. It follows that

$$\mathcal{C}\left(\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix}, \mu'\right) \otimes \left(\omega^*_{L,d+1}(\tilde{\gamma}) \cdot \phi^*_{\mu'}\right) = e^{2\pi i\, {}^t x \alpha} \mathcal{C}\left(\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix}, \mu'\right) \otimes \phi^*_{\mu' A}.$$

Indeed, the essential point here is that

$$\mathcal{C}\left(\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix}, \mu'\right) \neq 0 \implies \mathcal{Z}\left(\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix}, \mu'\right) \neq \varnothing,$$

which implies, using Remark 2.2.8, that $[\mu_i, \mu_{d+1}] \equiv \alpha_i \pmod{\mathbb{Z}}$.

The preceding paragraph allows us to compute

$$\begin{aligned}
\omega^*_{L,d+1}(\tilde{\gamma}) \cdot \mathcal{C}\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} &= e^{2\pi i\, {}^t x \alpha} \sum_{\mu' \in (L^\vee/L)^{d+1}} \mathcal{C}\left(\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix}, \mu'\right) \otimes \phi^*_{\mu' A} \\
&= e^{2\pi i\, {}^t x \alpha} \sum_{\mu' \in (L^\vee/L)^{d+1}} \mathcal{C}\left(\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix}, \mu' A^{-1}\right) \otimes \phi^*_{\mu'} \\
&= e^{2\pi i\, {}^t x \alpha} \sum_{\mu' \in (L^\vee/L)^{d+1}} \mathcal{C}\left({}^t A \begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} A, \mu'\right) \otimes \phi^*_{\mu'} \\
&= e^{2\pi i\, {}^t x \alpha} \cdot \mathcal{C}\left({}^t A \begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} A\right).
\end{aligned}$$

In the third equality we have used the linear invariance of special cycles proved in Proposition 2.4.5, which implies the same invariance for the corrected cycle classes (4.5). □

Proof of Proposition 4.2.3. For a fixed $\mu \in (L^\vee/L)^d$, consider the generating series

$$\sum_{\substack{m \in \mathbb{Q} \\ w \in W \\ \mu_{d+1} \in L^\vee/L}} f_*^{(T,\mu)}[\mathcal{Y}(m, \mu_{d+1}, w)] \otimes \phi^*_{\mu_{d+1}} \cdot q^{m+Q(w)} \xi_1^{[w,e_1]} \cdots \xi_d^{[w,e_d]}.$$

This agrees with the pushforward via $\mathcal{Z}(T,\mu) \to \mathcal{M}$ of the generating series of Corollary 4.3.6, and so converges to a holomorphic function

$$\mathcal{H} \times \mathbb{C}^d \to \mathrm{CH}^{\mathrm{rank}(T)+1}(\mathcal{M}) \otimes S^*_L$$

transforming under $\widetilde{\Gamma} \subset \widetilde{J}_{d+1}$ as in Proposition 4.3.3.

The linear map $S_L^* \to S_{L,d}^* \otimes S_L^* \cong S_{L,d+1}^*$ sending

$$\phi_{\mu_{d+1}}^* \mapsto \phi_\mu^* \otimes \phi_{\mu_{d+1}}^*$$

is $\widetilde{\Gamma}$-equivariant, where the action on the source is via ω_L^*, and the action on the target is via the restriction of $\omega_{L,d+1}^*$ to $\widetilde{\Gamma} \subset \widetilde{\Gamma}_{d+1}$. Applying this map to the above generating series, summing over μ, and using Lemma 4.4.1, we find that

$$\sum_{\substack{m \in \mathbb{Q} \\ w \in W}} \mathcal{C}\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m + Q(w) \end{pmatrix} \cdot q^{m+Q(w)} \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \tag{4.23}$$

(inside the sum, $\alpha \in \mathbb{Q}^d$ has components $\alpha_i = [w, e_i]$) defines a holomorphic function

$$\mathcal{H} \times \mathbb{C}^d \to \mathrm{CH}^{d+1}(\mathcal{M}) \otimes S_{L,d+1}^*$$

transforming under the subgroup $\widetilde{\Gamma} \subset \widetilde{\Gamma}_{d+1}$ in the same way as a Jacobi form of index T, weight $1 + \frac{n}{2}$, and representation

$$\omega_{L,d+1}^* : \widetilde{\Gamma}_{d+1} \to \mathrm{GL}(S_{L,d+1}^*).$$

Using the change of variables $m \mapsto m - Q(w)$ and Lemma 4.4.2, we may rewrite (4.23) as

$$\phi_T(\tau', z) \stackrel{\mathrm{def}}{=} \sum_{\substack{m \in \mathbb{Q} \\ \alpha \in \mathbb{Q}^d}} \mathcal{C}\begin{pmatrix} T & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} \cdot q^m \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d},$$

which therefore transforms under $\widetilde{\Gamma} \subset \widetilde{\Gamma}_{d+1}$ in the same way.

To complete the proof, it only remains to check that this function also transforms correctly under any $\tilde{\gamma} \in M_{2,d}(\mathbb{Z}) \subset \widetilde{J}_{d+1}$, which we write in the form (4.22). Using Lemma 4.4.3, an elementary manipulation of the sum defining ϕ_T shows that

$$\omega_{L,d+1}^*(\tilde{\gamma}) \cdot \phi_T(\tau', z) = e^{2\pi i \tau' ({}^t y T y)} e^{-4\pi i ({}^t z + {}^t x) T y} \cdot \phi_T(\tau', z - y\tau' + x).$$

Unpacking Definition 4.1.2 shows that this is precisely the transformation law satisfied by a Jacobi form of the desired index, weight, and representation. □

5. Corrected cycle classes

Keep the quadratic lattice $L \subset V$ and the integral model $\mathcal{M}$ over $\mathbb{Z}[\Sigma^{-1}]$ as in §2.1 and §2.2. In this subsection we construct from the naive special cycles $\mathcal{Z}(T, \mu)$, which need not be equidimensional, canonical cycle classes $\mathcal{C}(T, \mu)$ in the Chow group of $\mathcal{M}$.

5.1. **Construction of the classes.** We will make essential use of Theorems A.2.6 and A.2.7, as well as Proposition A.4.4. For this reason we assume, throughout the entirety of §5, that $\mathcal{M}$ is regular (see Proposition 2.2.4).

Suppose $\mathcal{Z}$ is a Deligne-Mumford stack equipped with a finite morphism $\pi : \mathcal{Z} \to \mathcal{M}$. In §A.2 we associate to this data a $\mathbb{Q}$-vector space $G_0(\mathcal{Z})_{\mathbb{Q}}$, with the descending filtration

$$F^d G_0(\mathcal{Z})_{\mathbb{Q}} = \bigcup_{\substack{\text{closed substacks } \mathcal{Y} \subset \mathcal{Z} \\ \mathrm{codim}_{\mathcal{M}}(\pi(\mathcal{Y})) \geqslant d}} \mathrm{Image}\big(G_0(\mathcal{Y})_{\mathbb{Q}} \to G_0(\mathcal{Z})_{\mathbb{Q}}\big)$$

from (A.13). By Remark A.2.3, any coherent $\mathcal{O}_{\mathcal{Z}}$-module $\mathcal{F}$ determines an

$$[\mathcal{F}] \in G_0(\mathcal{Z})_{\mathbb{Q}}.$$

Fix a $T \in \mathrm{Sym}_d(\mathbb{Q})$ with $d \geqslant 1$, and a tuple of cosets $\mu = (\mu_1, \dots, \mu_d) \in (L^\vee/L)^d$. We have defined in (2.8) a finite and unramified morphism

$$\mathcal{Z}(T, \mu) \to \mathcal{M}.$$

If we denote by $t_1, \dots, t_d \in \mathbb{Q}$ the diagonal entries of T, there are forgetful morphisms $\mathcal{Z}(T, \mu) \to \mathcal{Z}(t_i, \mu_i)$, sending an S-point

$$x = (x_1, \dots, x_d) \in V_{\mu_1}(\mathcal{A}_S) \times \cdots \times V_{\mu_d}(\mathcal{A}_S)$$

to its i^{th} coordinate $x_i \in V_{\mu_i}(\mathcal{A}_S)$. The product of these maps defines a morphism of $\mathcal{M}$-stacks

$$\mathcal{Z}(T, \mu) \to \mathcal{Z}(t_1, \mu_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(t_d, \mu_d), \tag{5.1}$$

which presents $\mathcal{Z}(T, \mu)(S)$ as the locus of S-points of the codomain for which the moment matrix $Q(x)$ of $x = (x_1, \dots, x_d) \in \prod_i V_{\mu_i}(\mathcal{A}_S)$ is equal to T. The moment matrix $Q(x)$ is locally constant on S, and hence (5.1) is an open and closed immersion, by the rigidity lemma for endomorphisms of abelian schemes: if an endomorphism of $\mathcal{A}_S$ is 0 at some point of S, then it is 0 on the entire connected component of S containing that point [MFK94, Corollary 6.2].

By iterating the pairing of Proposition A.4.4, we obtain a multilinear map

$$\begin{array}{c} F^1 G_0(\mathcal{Z}(t_1, \mu_1))_{\mathbb{Q}} \otimes \cdots \otimes F^1 G_0(\mathcal{Z}(t_d, \mu_d))_{\mathbb{Q}} \\ \Big\downarrow {\scriptstyle z_1 \otimes \cdots \otimes z_d \mapsto z_1 \cap \cdots \cap z_d} \\ F^d G_0\big(\mathcal{Z}(t_1, \mu_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(t_d, \mu_d)\big)_{\mathbb{Q}} \\ \Big\downarrow \\ F^d G_0(\mathcal{Z}(T, \mu))_{\mathbb{Q}}, \end{array} \tag{5.2}$$

where the second arrow is restriction via (5.1).

The multilinear map just defined has a distinguished input $z_1 \otimes \cdots \otimes z_d$. If $(t_i, \mu_i) \neq (0, 0)$ we define

$$z_i = [\mathcal{O}_{\mathcal{Z}(t_i, \mu_i)}] \in G_0(\mathcal{Z}(t_i, \mu_i))_{\mathbb{Q}}.$$

If $(t_i, \mu_i) = (0,0)$ then $\mathcal{Z}(0,0) = \mathcal{M}$ by Proposition 2.4.3, and we define

$$z_i = [\mathcal{O}_{\mathcal{M}}] - [\omega] \in G_0(\mathcal{M})_{\mathbb{Q}},$$

where ω is the tautological bundle (2.6).

Lemma 5.1.1. *For all* $1 \leqslant i \leqslant d$ *we have* $z_i \in F^1 G_0(\mathcal{Z}(t_i, \mu_i))_{\mathbb{Q}}$.

Proof. If $(t_i, \mu_i) \neq (0,0)$ then Proposition 2.4.3 implies that the image of $\mathcal{Z}(t_i, \mu_i) \to \mathcal{M}$ is either empty or of codimension 1. Hence

$$F^1 G_0(\mathcal{Z}(t_i, \mu_i))_{\mathbb{Q}} = G_0(\mathcal{Z}(t_i, \mu_i))_{\mathbb{Q}},$$

and the claim is vacuously true. If $(t_i, \mu_i) = (0,0)$ the claim follows from the proof of Lemma A.3.2, especially the relation (A.12). □

Remark 5.1.2. The motivation for the definition of z_i in the case $(t_i, \mu_i) = (0,0)$ comes from Lemma A.3.2, which implies that the image of $[\mathcal{O}_{\mathcal{M}}] - [\omega]$ under

$$F^1 G_0(\mathcal{M})_{\mathbb{Q}} \xrightarrow{\text{(A.14)}} F^1 K_0(\mathcal{M})_{\mathbb{Q}} \xrightarrow{\text{(A.10)}} \mathrm{CH}^1(\mathcal{M})$$

is the first Chern class $c_1(\omega^{-1})$.

Definition 5.1.3. The *derived fundamental class*

$$[\mathcal{O}^{\text{derived}}_{\mathcal{Z}(T,\mu)}] \in F^d G_0(\mathcal{Z}(T,\mu))_{\mathbb{Q}}$$

is the image of $z_1 \otimes \cdots \otimes z_d$ under (5.2). The *corrected cycle class*

$$\mathcal{C}(T,\mu) \in \mathrm{CH}^d_{\mathcal{Z}(T,\mu)}(\mathcal{M})$$

is the image of the derived fundamental class under the composition

$$F^d G_0(\mathcal{Z}(T,\mu))_{\mathbb{Q}} \xrightarrow{\text{(A.14)}} F^d K_0^{\mathcal{Z}(T,\mu)}(\mathcal{M})_{\mathbb{Q}} \xrightarrow{\text{(A.10)}} \mathrm{CH}^d_{\mathcal{Z}(T,\mu)}(\mathcal{M}).$$

Remark 5.1.4. A somewhat fancier way to understand this construction is to calculate the fiber product

$$\mathcal{Z}(t_1, \mu_1) \times_{\mathcal{M}} \times \cdots \times_{\mathcal{M}} \mathcal{Z}(t_d, \mu_d)$$

in a *derived* sense, and hence as a *derived stack* over $\mathcal{M}$.

When forming this derived fiber product, one should interpret any factor of the form $\mathcal{Z}(t_i, \mu_i) = \mathcal{Z}(0,0)$ as itself being a derived stack. More precisely, inside the total space of the cotautological line bundle on $\mathcal{M}$, one can construct the derived self-intersection of the zero section. The result is a derived stack whose underlying classical stack is canonically identified with $\mathcal{M}$, but with virtual dimension $\dim(\mathcal{M}) - 1$. Every instance of $\mathcal{Z}(0,0)$ in the above fiber product should be replaced by this derived stack.

The underlying classical stack of this derived fiber product is of course just the classical fiber product, and the open and closed substack $\mathcal{Z}(T,\mu)$ of this classical stack lifts canonically to an open and closed substack of the derived fiber product, which we denote by $\mathcal{Z}^{\text{der}}(T,\mu)$.

By construction, $\mathcal{Z}^{\text{der}}(T,\mu)$ is *quasi-smooth* over $\mathcal{M}$ of virtual codimension d: this is just the derived analogue of a local complete intersection of codimension d. A general result of Khan [Kha22, §6]—a derived generalization of results found in [Gro71] for local complete intersections—now shows that the structure sheaf of $\mathcal{Z}^{\mathrm{der}}(T,\mu)$ defines a class in $F^dK_0(\mathcal{M})_{\mathbb{Q}}$, whose image recovers the corrected class $\mathcal{C}(T,\mu)$ defined above.

Note that the derived interpretation as explained here does not (yet) shed any light on why Theorems C, D, and E from the introduction are true.

We will not need this perspective in this paper, but these ideas are explored further in [Mad22], where it is shown that the derived stack $\mathcal{Z}^{\mathrm{der}}(T,\mu)$ admits a moduli interpretation, which leads to alternate proofs of Theorems C, D and E.

The remainder of §5 is devoted to studying the properties of $\mathcal{C}(T,\mu)$.

5.2. **Intersections of cycle classes.** We first explain how our corrected cycle classes behave under the intersection pairing in the Chow ring, as this is one of the few properties that follow directly from the definition.

Fix positive integers d' and d'', symmetric matrices

$$T' \in \mathrm{Sym}_{d'}(\mathbb{Q}) \quad \text{and} \quad T'' \in \mathrm{Sym}_{d''}(\mathbb{Q}),$$

and tuples $\mu' \in (L^\vee/L)^{d'}$ and $\mu'' \in (L^\vee/L)^{d''}$.

Proposition 5.2.1. *The corrected cycle classes*

$$\mathcal{C}(T',\mu') \in \mathrm{CH}^{d'}_{\mathcal{Z}(T',\mu')}(\mathcal{M}) \quad \textit{and} \quad \mathcal{C}(T'',\mu'') \in \mathrm{CH}^{d''}_{\mathcal{Z}(T'',\mu'')}(\mathcal{M}).$$

of Definition 5.1.3 satisfy the intersection formula

$$\mathcal{C}(T',\mu') \cdot \mathcal{C}(T'',\mu'') = \sum_{T = \left(\begin{smallmatrix} T' & * \\ * & T'' \end{smallmatrix}\right)} \mathcal{C}(T,\mu),$$

where $\mu = (\mu',\mu'')$ is the concatenation of μ' and μ'', and the product

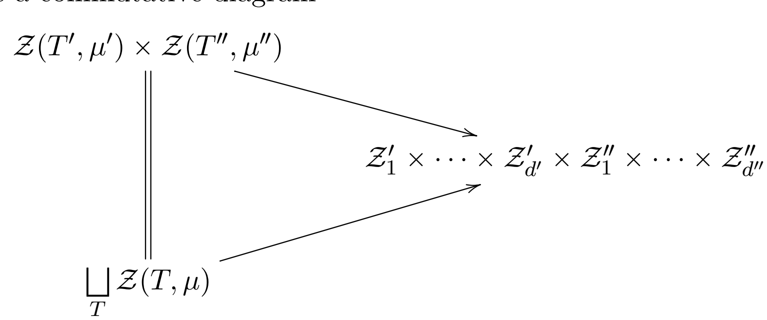
$$\mathrm{CH}^{d'}_{\mathcal{Z}(T',\mu')}(\mathcal{M}) \otimes \mathrm{CH}^{d''}_{\mathcal{Z}(T'',\mu'')}(\mathcal{M}) \to \mathrm{CH}^{d'+d''}_{\mathcal{Z}(T',\mu')\times_{\mathcal{M}}\mathcal{Z}(T'',\mu'')}(\mathcal{M})$$

on the left is the intersection pairing of §A.2.

Proof. Let $t'_1,\ldots,t'_{d'}$ and $t''_1,\ldots,t''_{d''}$ be the diagonal entries of T' and T''. If we abbreviate

$$\mathcal{Z}'_i = \mathcal{Z}(t'_i,\mu'_i) \quad \text{and} \quad \mathcal{Z}''_i = \mathcal{Z}(t''_i,\mu''_i),$$

there is a commutative diagram



in which the vertical = is the canonical isomorphism of Proposition 2.4.6, and both diagonal arrows are open and closed immersions. Here and below, all fiber products are over $\mathcal{M}$.

Using the pairing (A.18), the constructions of §5.1 provide us with a class

$$z_1' \cap \cdots \cap z_{d'}' \cap z_1'' \cap \cdots \cap z_{d''}'' \in F^{d'+d''} G_0(\mathcal{Z}_1' \times \cdots \times \mathcal{Z}_{d'}' \times \mathcal{Z}_1'' \times \cdots \times \mathcal{Z}_{d''}'')_{\mathbb{Q}}$$

whose restriction along the top diagonal arrow is

$$[\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T',\mu')}] \cap [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T'',\mu'')}] \in G_0(Z(T',\mu') \times Z(T'',\mu''))_{\mathbb{Q}},$$

and whose restriction along the bottom diagonal arrow is

$$\sum_T [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T,\mu)}] \in \bigoplus_T G_0(\mathcal{Z}(T,\mu))_{\mathbb{Q}} = G_0\Big(\bigsqcup_T \mathcal{Z}(T,\mu)\Big)_{\mathbb{Q}}.$$

In particular, the vertical = in the above diagram identifies

$$[\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T',\mu')}] \cap [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T'',\mu'')}] = \sum_T [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T,\mu)}].$$

The proposition follows by applying the composition

$$F^{d'+d''} G_0\left(Z(T',\mu') \times Z(T'',\mu'')\right)_{\mathbb{Q}} \xrightarrow{\text{(A.14)}} F^{d'+d''} K_0^{Z(T',\mu') \times Z(T'',\mu'')}(\mathcal{M})_{\mathbb{Q}}$$
$$\xrightarrow{\text{(A.10)}} \mathrm{CH}^{d'+d''}_{Z(T',\mu') \times Z(T'',\mu'')}(\mathcal{M})$$

to both sides of this last equality. □

5.3. **An alternate construction.** In this subsection we will give a different characterization of the derived fundamental classes of Definition 5.1.3. This will be used in the proof of Proposition 5.4.1 below.

Consider again the situation of §5.1, where we have fixed $\mu \in (L^\vee/L)^d$ and $t_1, \ldots, t_d \in \mathbb{Q}$ are the diagonal entries of $T \in \mathrm{Sym}_d(\mathbb{Q})$.

Fix an étale surjection $U \to \mathcal{M}$ with U scheme. The special divisors

$$\mathcal{Z}(t_i, \mu_i) \to \mathcal{M}$$

are finite and unramified, and so, as in Definition 2.4.1, we may assume that U is chosen so that the natural map $Z_i \to U$ is a closed immersion of schemes for every $1 \leqslant i \leqslant d$ and every connected component

$$Z_i \subset \mathcal{Z}(t_i, \mu_i)_U.$$

Fix a tuple $(Z_1, \ldots, Z_d)$ with each $Z_i \subset \mathcal{Z}(t_i, \mu_i)_U$ a connected component. If $(t_i, \mu_i) \neq (0,0)$ then $Z_i \subset U$ is an effective Cartier divisor (Proposition 2.4.3), and its ideal sheaf $I_{Z_i} \subset \mathcal{O}_U$ determines a chain complex of locally free $\mathcal{O}_U$-modules

$$C_{Z_i} = (\cdots \to 0 \to I_{Z_i} \to \mathcal{O}_U \to 0 \to \cdots)$$

supported in degrees 1 and 0. If $(t_i, \mu_i) = (0,0)$, so that $Z_i = U$, we instead define

$$C_{Z_i} = (\cdots \to 0 \to \omega|_U \xrightarrow{0} \mathcal{O}_U \to 0 \to \cdots).$$

The tensor product $C_{Z_1} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{Z_d}$ is a complex of locally free $\mathcal{O}_U$-modules, whose ℓ^{th} homology

$$H_\ell(C_{Z_1} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{Z_d}) \tag{5.3}$$

is a coherent sheaf on U annihilated by the ideal sheaf of the closed subscheme $Z_1 \times_U \cdots \times_U Z_d \subset U$. We may therefore view this sheaf as a coherent sheaf on this closed subscheme.

By varying the tuple $(Z_1, \ldots, Z_d)$, we obtain from (5.3) a coherent sheaf

$$H_\ell\big(C_{\mathcal{Z}(t_1,\mu_1)_U} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{\mathcal{Z}(t_d,\mu_d)_U}\big) \tag{5.4}$$

(this is just notation; no complex $C_{\mathcal{Z}(t_i,\mu_i)_U}$ of $\mathcal{O}_U$-modules will be defined) on the disjoint union

$$\mathcal{Z}(t_1,\mu_1)_U \times_U \cdots \times_U \mathcal{Z}(t_d,\mu_d)_U = \bigsqcup_{(Z_1,\ldots,Z_d)} Z_1 \times_U \cdots \times_U Z_d,$$

which admits a canonical descent to a coherent sheaf

$$H_\ell\big(C_{\mathcal{Z}(t_1,\mu_1)} \otimes_{\mathcal{O}_\mathcal{M}} \cdots \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(t_d,\mu_d)}\big) \tag{5.5}$$

(again, no complex $C_{\mathcal{Z}(t_i,\mu_i)}$ of $\mathcal{O}_\mathcal{M}$-modules will be defined) on

$$\mathcal{Z}(t_1,\mu_1) \times_\mathcal{M} \cdots \times_\mathcal{M} \mathcal{Z}(t_d,\mu_d).$$

This last sheaf may be restricted to the open and closed substack (5.1).

Proposition 5.3.1. *The derived fundamental class of Definition 5.1.3 is equal to*

$$[\mathcal{O}^{\text{derived}}_{\mathcal{Z}(T,\mu)}] = \sum_{\ell \geqslant 0} (-1)^\ell \cdot \big[H_\ell\big(C_{\mathcal{Z}(t_1,\mu_1)} \otimes_{\mathcal{O}_\mathcal{M}} \cdots \otimes_{C_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(t_d,\mu_d)}\big)|_{\mathcal{Z}(T,\mu)}\big].$$

Proof. An elementary but tedious exercise in homological algebra shows that

$$z_1 \cap \cdots \cap z_d = \sum_{\ell \geqslant 0} (-1)^\ell \cdot \big[H_\ell(\mathcal{O}_{\mathcal{Z}(t_1,\mu_1)} \otimes_{\mathcal{O}_\mathcal{M}} \cdots \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(t_d,\mu_d)})\big]$$

as elements of

$$G_0(\mathcal{Z}(t_1,\mu_1) \times_\mathcal{M} \cdots \times_\mathcal{M} \mathcal{Z}(t_d,\mu_d))_\mathbb{Q},$$

where each $z_i \in G_0(\mathcal{Z}(t_i,\mu_i))$ is as in §5.1, and the intersection on the left is obtained by iterating the pairing of Lemma A.4.1. The claim follows immediately from this.

We point out that verifying the equality above does not require unpacking the use of derived algebraic geometry or the sheaf of spectra $\mathbf{G}_{Z_1 \times_M Z_2}$ in the proof of Lemma A.4.1. One need only verify the same equality in the naive Grothendieck group G_0^{naive} of Remark A.2.1, with the $\cap$ on the left defined by (A.17), and use the commutativity of the diagram in Lemma A.4.1. □

Remark 5.3.2. The slightly complicated constructions above are done solely to account for the failure of the special divisors $\mathcal{Z}(t_i,\mu_i) \to \mathcal{M}$ to be closed immersions. If they were closed immersions, we would simply have defined complexes of locally free $\mathcal{O}_{\mathcal{M}}$-modules

$$C_{\mathcal{Z}(t_i,\mu_i)} = \begin{cases} (\cdots \to 0 \to I_{\mathcal{Z}(t_i,\mu_i)} \to \mathcal{O}_{\mathcal{M}} \to 0 \to \cdots) & \text{if } (t_i,\mu_i) \neq (0,0) \\ \\ (\cdots \to 0 \to \omega \xrightarrow{0} \mathcal{O}_{\mathcal{M}} \to 0 \to \cdots) & \text{if } (t_i,\mu_i) = (0,0). \end{cases}$$

The sheaf (5.5) would then be understood in the literal sense, as the ℓ^{th} homology of the tensor product of complexes.

5.4. **Linear invariance.** Suppose X is any abelian group. Given a d-tuple $x \in X^d$ and an $A \in \mathrm{GL}_d(\mathbb{Z})$, we define $xA \in X^d$ using the habitual rule for multiplication of a row vector by a matrix.

Fix a matrix $T \in \mathrm{Sym}_d(\mathbb{Q})$ and a tuple $\mu \in (L^\vee/L)^d$. Fix also a matrix $A \in \mathrm{GL}_d(\mathbb{Z})$, and set

$$(T', \mu') = ({}^tATA, \mu A).$$

By Proposition 2.4.5 there is an isomorphism of $\mathcal{M}$-stacks

$$\mathcal{Z}(T,\mu) \cong \mathcal{Z}(T',\mu'), \tag{5.6}$$

sending an S-point $x \in V(\mathcal{A}_S)^d_{\mathbb{Q}}$ of the left hand side to the S-point $xA \in V(\mathcal{A}_S)^d_{\mathbb{Q}}$ of the right hand side. Hence the special cycles in (5.6) have the same images in $\mathcal{M}$, and there is an equality

$$\mathrm{CH}^d_{\mathcal{Z}(T,\mu)}(\mathcal{M}) = \mathrm{CH}^d_{\mathcal{Z}(T',\mu')}(\mathcal{M}) \tag{5.7}$$

of Chow groups with support.

Proposition 5.4.1. *The isomorphism* (5.6) *identifies the derived fundamental classes*

$$[\mathcal{O}^{\text{derived}}_{\mathcal{Z}(T,\mu)}] \in G_0(\mathcal{Z}(T,\mu)) \quad \textit{and} \quad [\mathcal{O}^{\text{derived}}_{\mathcal{Z}(T',\mu')}] \in G_0(\mathcal{Z}(T',\mu')).$$

In particular, the equality $\mathcal{C}(T,\mu) = \mathcal{C}(T',\mu')$ holds in (5.7).

Proof. Using the alternate construction of Proposition 5.3.1, we are reduced to proving the existence, for every $\ell \geqslant 0$, of an isomorphism

$$\begin{aligned} &H_\ell\big(C_{\mathcal{Z}(t_1,\mu_1)} \otimes_{\mathcal{O}_{\mathcal{M}}} \cdots \otimes_{C_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(t_d,\mu_d)}\big)|_{\mathcal{Z}(T,\mu)} \\ &\cong H_\ell\big(C_{\mathcal{Z}(t'_1,\mu'_1)} \otimes_{\mathcal{O}_{\mathcal{M}}} \cdots \otimes_{C_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(t'_d,\mu'_d)}\big)|_{\mathcal{Z}(T',\mu')} \end{aligned} \tag{5.8}$$

of coherent sheaves on (5.6). Here $t_1,\ldots,t_d$ and $t'_1,\ldots,t'_d$ are the diagonal entries of T and T'.

Moreover, it suffices to treat the case in which $d \geqslant 2$ and

$$A = \begin{pmatrix} 1 & 0 & \\ 1 & 1 & \\ & & I_{d-2} \end{pmatrix}. \tag{5.9}$$

Indeed, the group $\mathrm{GL}_d(\mathbb{Z})$ is generated by A, the permutation matrices, and the diagonal matrices, and the claim is easily proved in the latter two cases.

As in the constructions of §5.3, choose an étale surjection $U \to \mathcal{M}$ fine enough that the morphisms

$$\mathcal{Z}(t_i, \mu_i)_U \to U \quad \text{and} \quad \mathcal{Z}(t'_i, \mu'_i)_U \to U,$$

for all $1 \leqslant i \leqslant d$, restrict to closed immersions on all connected components

$$Z_i \subset \mathcal{Z}(t_i, \mu_i)_U \quad \text{and} \quad Z'_i \subset \mathcal{Z}(t'_i, \mu'_i)_U.$$

To simplify notation, we abbreviate $\mathcal{Z}$ for the $\mathcal{M}$-stack (5.6). Fix a geometric point z of $\mathcal{Z}$. The finite étale z-scheme z_U defined by the cartesian diagram

$$\begin{array}{ccc} z_U & \longrightarrow & \mathcal{Z}_U \\ \downarrow & & \downarrow \\ z & \longrightarrow & \mathcal{Z} \end{array}$$

decomposes as a finite disjoint union of points $z_U = \bigsqcup y$.

Fix one connected component $y \subset z_U$. Its image in $\mathcal{Z}_U$ lands on some connected component $Z \subset \mathcal{Z}_U$, whose images under the two maps

$$\begin{array}{ccccc} \mathcal{Z}(T, \mu)_U & = & \mathcal{Z}_U & = & \mathcal{Z}(T', \mu')_U \\ \downarrow & & & & \downarrow \\ \mathcal{Z}(t_i, \mu_i)_U & & & & \mathcal{Z}(t'_i, \mu'_i)_U \end{array}$$

are then contained in unique connected components

$$Z_i \subset \mathcal{Z}(t_i, \mu_i)_U \quad \text{and} \quad Z'_i \subset \mathcal{Z}(t'_i, \mu'_i)_U.$$

The natural map $Z \to U$ is a closed immersion, realizing Z as a connected component of both intersections

$$Z_1 \times_U \cdots \times_U Z_d \quad \text{and} \quad Z'_1 \times_U \cdots \times_U Z'_d.$$

The construction (5.3) gives us coherent sheaves

$$H_\ell(C_{Z_1} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{Z_d}) \quad \text{and} \quad H_\ell(C_{Z'_1} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{Z'_d})$$

on these two intersections, and we will construct a Zariski open neighborhood $y \in V_y \subset Z$ together with a canonical isomorphism

$$H_\ell(C_{Z_1} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{Z_d})|_{V_y} \cong H_\ell(C_{Z'_1} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{Z'_d})|_{V_y} \tag{5.10}$$

of coherent sheaves on V_y.

Before we construct (5.10), we explain how it implies the existence of the desired isomorphism (5.8), and hence completes the proof of Proposition 5.4.1. Recalling from (5.1) that $\mathcal{Z}_U$ is an open and closed subscheme of both

$$\mathcal{Z}(t_1, \mu_1)_U \times_U \cdots \times_U \mathcal{Z}(t_d, \mu_d)_U$$

and

$$\mathcal{Z}(t'_1, \mu'_1)_U \times_U \cdots \times_U \mathcal{Z}(t'_d, \mu'_d)_U,$$

varying the connected component $y \in \pi_0(z_U)$ in (5.10) defines an isomorphism

$$H_\ell\big(C_{\mathcal{Z}(t_1,\mu_1)_U} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{\mathcal{Z}(t_d,\mu_d)_U}\big)|_{V_z} \cong H_\ell\big(C_{\mathcal{Z}(t'_1,\mu'_1)_U} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{\mathcal{Z}(t'_d,\mu'_d)_U}\big)|_{V_z}$$

between the sheaves of (5.4), after restriction to the Zariski open neighborhood

$$V_z \stackrel{\text{def}}{=} \bigsqcup_{y \in \pi_0(z_U)} V_y \subset \mathcal{Z}_U$$

of the image of $z_U \to \mathcal{Z}_U$. Varying the geometric point z and gluing over the resulting Zariski open cover $\{V_z\}_z$ of $\mathcal{Z}_U$ defines an isomorphism

$$H_\ell\big(C_{\mathcal{Z}(t_1,\mu_1)_U} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{\mathcal{Z}(t_d,\mu_d)_U}\big)|_{\mathcal{Z}_U} \cong H_\ell\big(C_{\mathcal{Z}(t'_1,\mu'_1)_U} \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} C_{\mathcal{Z}(t'_d,\mu'_d)_U}\big)|_{\mathcal{Z}_U},$$

and finally étale descent via $\mathcal{Z}_U \to \mathcal{Z}$ defines the desired isomorphism (5.8).

We now turn to the construction of (5.10). Consider the first order infinitesimal neighborhood

$$Z \subset \widetilde{Z} \subset U$$

of the closed subscheme $Z \subset U$. In other words, if $I_Z \subset \mathcal{O}_U$ is the ideal sheaf defining Z, then $\widetilde{Z}$ is defined by the ideal sheaf I_Z^2. Similarly, denote by

$$Z_i \subset \widetilde{Z}_i \subset U, \qquad Z'_i \subset \widetilde{Z}'_i \subset U$$

the first order infinitesimal neighborhoods of Z_i and Z'_i. Clearly $\widetilde{Z}$ is contained in both $\widetilde{Z}_i$ and $\widetilde{Z}'_i$.

The following is the analogue of [How19, Theorem 5.1].

Lemma 5.4.2. *For every $1 \leqslant i \leqslant d$ there are canonical sections*

$$s_i \in H^0\big(\widetilde{Z}_i, \omega|^{-1}_{\widetilde{Z}_i}\big) \quad \text{and} \quad s'_i \in H^0\big(\widetilde{Z}'_i, \omega|^{-1}_{\widetilde{Z}'_i}\big)$$

with scheme-theoretic zero loci $Z_i \subset \widetilde{Z}_i$ and $Z'_i \subset \widetilde{Z}'_i$, respectively. After restriction to $\widetilde{Z}$, these sections are related by $s'_1 = s_1 + s_2$, and $s'_i = s_i$ when $i > 1$.

Proof. By virtue of the moduli problem defining $\mathcal{Z}(t_i, \mu_i)$, there is a canonical special endomorphism $x_i \in V_{\mu_i}(\mathcal{A}_{Z_i})$. The desired section

$$s_i = \mathrm{obst}_{x_i} \in H^0\big(\widetilde{Z}_i, \omega|^{-1}_{\widetilde{Z}_i}\big)$$

is the obstruction to deforming x_i, as in the proof of Proposition 2.4.3 (if $x_i = 0$ we understand $\mathrm{obst}_{x_i} = 0$, because there is no obstruction to deforming the 0 endomorphism). The section s'_i is defined similarly.

Because of the particular choice of matrix (5.9), after restriction to Z the special quasi-endomorphisms x_i and x'_i are related by $x'_1 = x_1 + x_2$, and $x'_i = x_i$ if $i > 1$. This leads to similar relations between s_i and s'_i. ☐

Lemma 5.4.3. *Around every point of Z one can find a Zariski open affine neighborhood $V \subset U$ over which $\omega|_V \cong \mathcal{O}_V$, and sections*

$$\sigma_1, \sigma_2 \in H^0(V, \omega|_V^{-1}) \quad \textit{and} \quad \alpha \in H^0(V, \mathcal{O}_V)$$

such that

(i) σ_1 has zero locus $Z_1 \cap V$ and agrees with s_1 on $\widetilde{Z}_1 \cap V$,
(ii) σ_2 has zero locus $Z_2 \cap V$ and agrees with s_2 on $\widetilde{Z}_2 \cap V$,
(iii) α restricts to the constant function 1 on $Z_2 \cap V$,
(iv) the section

$$\sigma_1' \stackrel{\mathrm{def}}{=} \sigma_1 + \alpha\sigma_2$$

has zero locus $Z_1' \cap V$ and agrees with s_1' on the closed formal subscheme, lying between $Z_1' \cap V$ and $\widetilde{Z}_1' \cap V$, defined by the ideal sheaf

$$I_{Z_1' \cap V} \cdot \big(I_{Z_1' \cap V} + I_{Z_2 \cap V} \big) \subset \mathcal{O}_V.$$

Proof. The proof is identical to that of [How19, Lemma 5.2], and makes crucial use of the fact that $\mathcal{M}$ is regular and Noetherian. □

Choose a Zariski open $V \subset U$ as in Lemma 5.4.3 containing the image of the geoemtric point $y \to Z \subset U$. The sections of Lemma 5.4.3 determine chain complexes of locally free $\mathcal{O}_V$-modules

$$\begin{aligned} D_{Z_1} &= (\cdots \to 0 \to \omega|_V \xrightarrow{\sigma_1} \mathcal{O}_V \to 0) \\ D_{Z_2} &= (\cdots \to 0 \to \omega|_V \xrightarrow{\sigma_2} \mathcal{O}_V \to 0) \\ D_{Z_1'} &= (\cdots \to 0 \to \omega|_V \xrightarrow{\sigma_1'} \mathcal{O}_V \to 0), \end{aligned}$$

and there are canonical isomorphisms

$$D_{Z_1} \cong C_{Z_1}|_V, \quad D_{Z_2} \cong C_{Z_2}|_V, \quad D_{Z_1'} \cong C_{Z_1'}|_V.$$

Indeed, if $t_1 \neq 0$, so that $Z_1 \subset U$ is a Cartier divisor and $\sigma_1 \neq 0$, the first isomorphism is

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \omega|_V & \xrightarrow{\sigma_1} & \mathcal{O}_V & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma_1} & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I_{Z_1 \cap V} & \longrightarrow & \mathcal{O}_V & \longrightarrow & 0 \end{array}$$

If $t_1 = 0$, so that $Z_1 = U$ and $\sigma_1 = 0$, then $D_{Z_1} = C_{Z_1}|_V$ by definition. The other isomorphisms are entirely similar.

Using the relation $\sigma_1' = \sigma_1 + \alpha\sigma_2$, we see that the diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \omega|_V \otimes \omega|_V & \xrightarrow{\partial_2} & \omega|_V \oplus \omega|_V & \xrightarrow{\partial_1} & \mathcal{O}_V & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle g_1} & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \omega|_V \otimes \omega|_V & \xrightarrow{\partial_2^*} & \omega|_V \oplus \omega|_V & \xrightarrow{\partial_1^*} & \mathcal{O}_V & \longrightarrow & 0 \end{array}$$

determined by $g_1(\eta_1,\eta_2)=(\eta_1,\eta_2-\alpha\eta_1)$ and

$$\begin{aligned}\partial_1(\eta_1,\eta_2)&=\sigma_1(\eta_1)+\sigma_2(\eta_2)\\ \partial_1^*(\eta_1,\eta_2)&=\sigma_1'(\eta_1)+\sigma_2(\eta_2)\\ \partial_2(\eta_1\otimes\eta_2)&=\big(\sigma_2(-\eta_2)\eta_1,\sigma_1(\eta_1)\eta_2\big)\\ \partial_2^*(\eta_1\otimes\eta_2)&=\big(\sigma_2(-\eta_2)\eta_1,\sigma_1'(\eta_1)\eta_2\big)\end{aligned}$$

commutes, and defines the middle isomorphism in

$$C_{Z_1}|_V\otimes C_{Z_2}|_V\cong D_{Z_1}\otimes D_{Z_2}\cong D_{Z_1'}\otimes D_{Z_2}\cong C_{Z_1'}|_V\otimes C_{Z_2}|_V.$$

As our choice of (5.9) implies $Z_i=Z_i'$ and $C_{Z_i}=C_{Z_i'}$ for all $i>1$, we obtain an isomorphism

$$(C_{Z_1}\otimes\cdots\otimes C_{Z_d})|_V\cong(C_{Z_1'}\otimes\cdots\otimes C_{Z_d'})|_V,$$

of complexes of locally free $\mathcal{O}_V$-modules. This isomorphism depends on the choices of sections in Lemma 5.4.3, which are not unique. However, exactly as in the proof of [How19, Lemma 5.2], the conditions of that lemma imply that different choices yield homotopic isomorphisms, and so the induced isomorphism

$$H_\ell(C_{Z_1}\otimes\cdots\otimes C_{Z_d})|_V\cong H_\ell(C_{Z_1'}\otimes\cdots\otimes C_{Z_d'})|_V$$

is independent of the choices.

In this last isomorphism both sides are coherent sheaves on V annihilated by the ideal sheaf of the closed subscheme

$$V_y\overset{\text{def}}{=}V\cap Z,$$

yielding the desired isomorphism (5.10). □

5.5. **Comparison with the naive cycle.** The following result shows that the corrected cycle class $\mathcal{C}(T,\mu)$ agrees with the class obtained by imitating the construction of (1.1), whenever that construction makes sense. We remark that the proof uses the linear invariance property of Proposition 5.4.1 in an essential way.

Proposition 5.5.1. *Fix $T\in\mathrm{Sym}_d(\mathbb{Q})$ and $\mu\in(L^\vee/L)^d$. If the naive special cycle $\mathcal{Z}(T,\mu)$ is equidimensional with*

$$\dim(\mathcal{Z}(T,\mu))=\dim(\mathcal{M})-\mathrm{rank}(T),$$

so that the naive cycle class

$$[\mathcal{Z}(T,\mu)]\in\mathrm{CH}^{\mathrm{rank}(T)}_{\mathcal{Z}(T,\mu)}(\mathcal{M})$$

is defined (Definition A.1.4), then

$$\mathcal{C}(T,\mu)=\underbrace{c_1(\omega^{-1})\cdots c_1(\omega^{-1})}_{d-\mathrm{rank}(T)}\cdot[\mathcal{Z}(T,\mu)]\in\mathrm{CH}^d_{\mathcal{Z}(T,\mu)}(\mathcal{M}).$$

Proof. We may assume that T is positive semi-definite, for otherwise the Chow group with support $\mathrm{CH}^d_{\mathcal{Z}(T,\mu)}(\mathcal{M})$ is trivial by Remark 2.2.12.

First suppose $\mathrm{rank}(T) = d$. In particular, T is positive definite, so has all diagonal entries nonzero. Recalling the open and closed immersion (5.1), consider a closed geometric point

$$s \to \mathcal{Z}(T,\mu) \subset \mathcal{Z}(t_1,\mu_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(t_d,\mu_d).$$

For every $1 \leqslant i \leqslant d$, Proposition 2.4.3 implies that the natural map

$$\mathcal{O}^{\mathrm{et}}_{\mathcal{M},s} \to \mathcal{O}^{\mathrm{et}}_{\mathcal{Z}(t_i,\mu_i),s}$$

on étale local rings is surjective with kernel generated by a single element f_i. As

$$\mathcal{O}^{\mathrm{et}}_{\mathcal{Z}(T,\mu),s} \cong \mathcal{O}^{\mathrm{et}}_{\mathcal{M},s}/(f_1,\ldots,f_d),$$

our assumptions imply that $f_1,\ldots,f_d \in \mathcal{O}^{\mathrm{et}}_{\mathcal{M},s}$ is a regular sequence.

For every $1 \leqslant e \leqslant d$ the étale local ring at s of

$$\mathcal{Y}_e = \mathcal{Z}(t_1,\mu_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(t_e,\mu_e)$$

is therefore Cohen-Macaulay of dimension $\dim(\mathcal{M}) - e$, and a result of Serre [Ser00, Section V.B.6] implies that

$$\mathrm{Tor}_\ell^{\mathcal{O}^{\mathrm{et}}_{\mathcal{M},s}}\big(\mathcal{O}^{\mathrm{et}}_{\mathcal{Y}_e,s}, \mathcal{O}^{\mathrm{et}}_{\mathcal{Z}(t_{e+1},\mu_{e+1}),s}\big) = 0$$

for all $\ell > 0$. Using (A.17) and the commutative diagram of Lemma A.4.1, one sees by induction on e that the intersection

$$[\mathcal{O}_{\mathcal{Z}(t_1,\mu_1)}] \cap \cdots \cap [\mathcal{O}_{\mathcal{Z}(t_e,\mu_e)}] \in F^e G_0(\mathcal{Y}_e)_{\mathbb{Q}},$$

has the form

$$[\mathcal{O}_{\mathcal{Z}(t_1,\mu_1)}] \cap \cdots \cap [\mathcal{O}_{\mathcal{Z}(t_e,\mu_e)}] = [\mathcal{O}_{\mathcal{Y}_e}] + [\mathcal{F}_e] - [\mathcal{G}_e]$$

for coherent sheaves $\mathcal{F}_e$ and $\mathcal{G}_e$ on $\mathcal{Y}_e$ with trivial stalks at any closed geometric point $s \to \mathcal{Z}(T,\mu) \to \mathcal{Y}_e$.

Taking $d = e$ shows that

$$[\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T,\mu)}] = [\mathcal{O}_{\mathcal{Z}(T,\mu)}] \in F^d G_0(\mathcal{Z}(T,\mu)),$$

as both are equal to the image of the class

$$[\mathcal{O}_{\mathcal{Z}(t_1,\mu_1)}] \cap \cdots \cap [\mathcal{O}_{\mathcal{Z}(t_d,\mu_d)}] \in F^d G_0\big(\mathcal{Z}(t_1,\mu_1) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(t_d,\mu_d)\big)_{\mathbb{Q}}$$

under the second arrow in (5.2). The equality of cycle classes

$$\mathcal{C}(T,\mu) = [\mathcal{Z}(T,\mu)]$$

now follows from Theorem A.2.7.

Now consider the other extreme, in which $T = 0_d$ has rank 0. In this case

$$\mathcal{Z}(0_d,\mu) = \begin{cases} \mathcal{M} & \text{if } \mu = 0 \\ \varnothing & \text{if } \mu \neq 0. \end{cases}$$

If $\mu \neq 0$ the the proposition is vacuously true, as the Chow group with support vanishes. On the other hand, by construction $\mathcal{C}(0_d, 0)$ is the image of

$$\underbrace{([\mathcal{O}_{\mathcal{M}}] - [\omega]) \cap \cdots \cap ([\mathcal{O}_{\mathcal{M}}] - [\omega])}_{d} \in F^d G_0(\mathcal{M})_{\mathbb{Q}}$$

under

$$F^d G_0(\mathcal{M})_{\mathbb{Q}} \xrightarrow{(A.8)} F^d K_0(\mathcal{M})_{\mathbb{Q}} \xrightarrow{(A.10)} \mathrm{CH}^d(\mathcal{M}).$$

It follows from Lemma A.3.2 that this image is

$$\mathcal{C}(0_d, 0) = \underbrace{c_1(\omega^{-1}) \cdots c_1(\omega^{-1})}_{d} \in \mathrm{CH}^d(\mathcal{M}),$$

as desired.

For the general case, let $r = \mathrm{rank}(T)$. As the cycle classes $[\mathcal{Z}(T, \mu)]$ and $\mathcal{C}(T, \mu)$ satisfy the same linear invariance property (Proposition 2.4.5 and Proposition 5.4.1), we may reduce to the case in which

$$T = \begin{pmatrix} T' & 0 \\ 0 & 0_{d-r} \end{pmatrix}$$

for a positive definite $r \times r$-matrix T'. We may further assume that

$$\mu_{r+1} = \cdots = \mu_d = 0.$$

Indeed, if some $\mu_i \neq 0$ with $r < i \leqslant d$ then $\mathcal{Z}(0, \mu_i) = \varnothing$ by Proposition 2.4.3, and so $\mathcal{Z}(T, \mu) = \varnothing$ by (5.1).

Set $\mu' = (\mu_1, \ldots, \mu_r)$. Directly from the moduli interpretation we see

$$\mathcal{Z}(T, \mu) \cong \mathcal{Z}(T', \mu')$$

as $\mathcal{M}$-stacks. Combining this with the positive definite and rank 0 cases already proved yields the first equality in

$$\begin{aligned} [\mathcal{Z}(T, \mu)] \cdot \underbrace{c_1(\omega^{-1}) \cdots c_1(\omega^{-1})}_{d-r} &= \mathcal{C}(T', \mu') \cdot \mathcal{C}\big(0_{d-r}, 0\big) \\ &= \sum_S \mathcal{C}(S, \mu). \end{aligned}$$

The second equality is by the intersection formula of Proposition 5.2.1, and the sum runs over all matrices of the form

$$S = \begin{pmatrix} T' & * \\ * & 0_{d-r} \end{pmatrix} \in \mathrm{Sym}_d(\mathbb{Q}).$$

The only nonzero terms come from positive semi-definite S, and the only such S is $S = T$. This completes the proof of Proposition 5.5.1. □

Corollary 5.5.2. *For any $T \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu \in (L^\vee/L)^d$, restriction to the generic fiber*

$$\mathrm{CH}^d_{\mathcal{Z}(T,\mu)}(\mathcal{M}) \to \mathrm{CH}^d_{Z(T,\mu)}(M)$$

sends the corrected cycle class $\mathcal{C}(T, \mu)$ to the class $C(T, \mu)$ of (1.1).

Proof. For a fixed pair (T,μ), it suffices to prove the claim after enlarging the finite set of primes Σ that we have inverted on the base. By adding to Σ all primes p for which $\mathcal{Z}(T,\mu)$ has an irreducible component supported in characteristic p, we may assume that no such primes exist.

As the generic fiber $Z(T,\mu)$ is equidimensional of codimension $\operatorname{rank}(T)$ in the generic fiber M, for example by Proposition 2.3.2, also $\mathcal{Z}(T,\mu)$ is equidimensional of codimension $\operatorname{rank}(T)$ in $\mathcal{M}$. The claim now follows from Proposition 5.5.1. □

5.6. **Pullbacks of cycle classes.** We will now consider the setup of (2.4), so that we have an isometric embedding $L \to L^\sharp$ of quadratic lattices inducing a morphism $M \to M^\sharp$ of canonical models of Shimura varieties. Assume our finite set of primes Σ is chosen so that both L_p and $L_p^\sharp$ are maximal at all $p \notin \Sigma$, so that the above morphism of canonical models extends to a finite morphism

$$f : \mathcal{M} \to \mathcal{M}^\sharp$$

of integral models over $\mathbb{Z}[\Sigma^{-1}]$ as in Remark (2.2.7). Assume further that both integral models $\mathcal{M}$ and $\mathcal{M}^\sharp$ are regular, so that the corrected cycle classes of Definition 5.1.3 are defined for both integral models.

The results of §A.2 provide us with a pullback

$$f^* : \mathrm{CH}^d_{Z^\sharp}(\mathcal{M}^\sharp) \to \mathrm{CH}^d_{Z^\sharp \times_{\mathcal{M}^\sharp} \mathcal{M}}(\mathcal{M})$$

for any finite morphism $\mathcal{Z}^\sharp \to \mathcal{M}^\sharp$. Given a pair $(T^\sharp,\mu^\sharp)$ with $T^\sharp \in \mathrm{Sym}_d(\mathbb{Q})$ and $\mu^\sharp \in (L^{\sharp,\vee}/L^\sharp)^d$, we can form the corrected cycle class

$$\mathcal{C}^\sharp(T^\sharp,\mu^\sharp) \in \mathrm{CH}^d_{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)}(\mathcal{M}^\sharp),$$

and ask how its pullback is related to the corrected cycle classes on $\mathcal{M}$.

The answer to this equation is exactly what one would expect given the decomposition

$$\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp) \times_{\mathcal{M}^\sharp} \mathcal{M} \cong \bigsqcup_{\substack{T \in \mathrm{Sym}_d(\mathbb{Q}) \\ \mu \in (L^\vee/L)^d}} \bigsqcup_{\substack{\nu \in (\Lambda^\vee/\Lambda)^d \\ \mu+\nu=\mu^\sharp}} \bigsqcup_{\substack{y \in \nu+\Lambda^d \\ T+Q(y)=T^\sharp}} \mathcal{Z}(T,\mu), \tag{5.11}$$

of Proposition 2.4.7. Recall that here $\Lambda \subset L^\sharp$ is the positive definite quadratic lattice of vectors orthogonal to $L \subset L^\sharp$, and the relation $\mu+\nu=\mu^\sharp$ means that the natural map

$$(L^\vee \oplus \Lambda^\vee)/(L \oplus \Lambda) \to (L^\vee \oplus \Lambda^\vee)/L^\sharp$$

sends

$$\mu+\nu \mapsto \mu^\sharp \in L^{\sharp,\vee}/L^\sharp \subset (L^\vee \oplus \Lambda^\vee)/L^\sharp.$$

Proposition 5.6.1. *The equality of cycle classes*

$$f^*\mathcal{C}^\sharp(T^\sharp,\mu^\sharp) = \sum_{\substack{T \in \mathrm{Sym}_d(\mathbb{Q}) \\ \mu \in (L^\vee/L)^d}} \sum_{\substack{\nu \in (\Lambda^\vee/\Lambda)^d \\ \mu+\nu=\mu^\sharp}} \sum_{\substack{y \in \nu+\Lambda^d \\ T+Q(y)=T^\sharp}} \mathcal{C}(T,\mu)$$

holds in $\mathrm{CH}^d_{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)\times_{\mathcal{M}^\sharp}\mathcal{M}}(\mathcal{M})$.

Proof. Fix one $\mathcal{Z}(T,\mu)$ appearing in the right hand side of (5.11), in the part of the decomposition indexed by some $\nu \in (\Lambda^\vee/\Lambda)^d$ and $y \in \nu + \Lambda^d$. Let $t_1,\ldots,t_d$ be the diagonal entries of T, let $t_1^\sharp,\ldots,t_d^\sharp$ be the diagonal entries of $T^\sharp$, and abbreviate

$$\mathcal{Z}_i = \mathcal{Z}(t_i,\mu_i) \quad \text{and} \quad \mathcal{Z}_i^\sharp = \mathcal{Z}^\sharp(t_i^\sharp,\mu_i^\sharp)$$

for the associated special divisors. We must have T positive semi-definite (for otherwise $\mathcal{C}(T,\mu)=0$), and hence all $t_i \geqslant 0$.

For every $1 \leqslant i \leqslant d$, the codimension one case of Proposition 2.4.7 provides us with a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_i & \longrightarrow & \mathcal{Z}_i^\sharp \\ \big\downarrow & & \big\downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{M}^\sharp, \end{array} \tag{5.12}$$

which defines an open and closed immersion

$$j : \mathcal{Z}_i \hookrightarrow \mathcal{Z}_i^\sharp \times_{\mathcal{M}^\sharp} \mathcal{M}$$

On moduli, this sends a special quasi-endomorphism $x_i \in V_{\mu_i}(\mathcal{A}_S)$ to

$$x_i^\sharp = x_i + y_i \in V_{\mu_i^\sharp}(\mathcal{A}_S^\sharp). \tag{5.13}$$

In particular, there is a homomorphism

$$G_0(\mathcal{Z}_i^\sharp)_{\mathbb{Q}} \xrightarrow{\cap[\mathcal{O}_{\mathcal{M}}]} G_0(\mathcal{Z}_i^\sharp \times_{\mathcal{M}^\sharp} \mathcal{M})_{\mathbb{Q}} \xrightarrow{j^*} G_0(\mathcal{Z}_i)_{\mathbb{Q}} \tag{5.14}$$

obtained by composing the intersection pairing

$$G_0(\mathcal{Z}_i^\sharp)_{\mathbb{Q}} \otimes G_0(\mathcal{M})_{\mathbb{Q}} \xrightarrow{\cap} G_0(\mathcal{Z}_i^\sharp \times_{\mathcal{M}^\sharp} \mathcal{M})_{\mathbb{Q}}$$

of Lemma A.4.1 with restriction along j.

Lemma 5.6.2. *Recall from §5.1 the distinguished classes*

$$z_i^\sharp \in G_0(\mathcal{Z}_i^\sharp) \quad \textit{and} \quad z_i \in G_0(\mathcal{Z}_i).$$

The homomorphism (5.14) *sends* $z_i^\sharp \mapsto z_i$.

Proof. First suppose $(t_i^\sharp,\mu_i^\sharp)=(0,0)$. As

$$0 = t_i^\sharp = t_i + Q(y_i),$$

both $t_i = 0$ and $y_i = 0$, and the latter implies $\nu_i = 0$. Thus $\mathcal{Z}_i^\sharp = \mathcal{M}^\sharp$ and $\mathcal{Z}_i = \mathcal{M}$, and we have

$$z_i^\sharp = [\mathcal{O}_{\mathcal{M}^\sharp}] - [\omega^\sharp] \quad \text{and} \quad z_i = [\mathcal{O}_{\mathcal{M}}] - [\omega].$$

Using (A.17) and the fact that the tautological bundle $\omega^\sharp \in \mathrm{Pic}(\mathcal{M}^\sharp)$ pulls back to the tautological bundle $\omega \in \mathrm{Pic}(\mathcal{M})$, we see that (5.14) sends

$$[\mathcal{O}_{\mathcal{M}^\sharp}] \mapsto [\mathcal{O}_{\mathcal{M}}] \quad \text{and} \quad [\omega^\sharp] \mapsto [\omega].$$

The lemma follows immediately from this.

Next assume that $(t_i^\sharp, \mu_i^\sharp) \neq (0,0)$ and $(t_i, \mu_i) \neq (0,0)$. Fix a geometric point $y \to \mathcal{Z}_i$, which we can also view as a point on $\mathcal{M}, \mathcal{M}^\sharp$ and $\mathcal{Z}_i^\sharp$. As both

$$\mathcal{Z}_i \to \mathcal{M} \quad \text{and} \quad \mathcal{Z}_i^\sharp \to \mathcal{M}^\sharp$$

are generalized Cartier divisors (Proposition 2.4.3), we can write

$$\mathcal{O}^{\mathrm{et}}_{\mathcal{Z}_i^\sharp, y} \cong \mathcal{O}^{\mathrm{et}}_{\mathcal{M}^\sharp, y}/(g)$$

for a nonzero $g \in \mathcal{O}^{\mathrm{et}}_{\mathcal{M}^\sharp, y}$ whose image in $\mathcal{O}^{\mathrm{et}}_{\mathcal{M}, y}$ satisfies

$$\mathcal{O}^{\mathrm{et}}_{\mathcal{Z}_i, y} \cong \mathcal{O}^{\mathrm{et}}_{\mathcal{M}, y}/(g).$$

It follows that

$$\mathrm{Tor}_\ell^{\mathcal{O}^{\mathrm{et}}_{\mathcal{M}^\sharp, y}}(\mathcal{O}^{\mathrm{et}}_{\mathcal{Z}_i^\sharp, y}, \mathcal{O}^{\mathrm{et}}_{\mathcal{M}, y}) \cong \begin{cases} \mathcal{O}^{\mathrm{et}}_{\mathcal{Z}_i, y} & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0. \end{cases}$$

Allowing y to vary shows that

$$\underline{\mathrm{Tor}}_\ell^{\mathcal{O}_{\mathcal{M}^\sharp}}(\mathcal{O}_{\mathcal{Z}_i^\sharp}, \mathcal{O}_{\mathcal{M}})|_{\mathcal{Z}_i} \cong \begin{cases} \mathcal{O}_{\mathcal{Z}_i} & \text{if } \ell = 0 \\ 0 & \text{if } \ell > 0, \end{cases}$$

and hence (5.14) sends $z_i^\sharp = [\mathcal{O}_{\mathcal{Z}_i^\sharp}]$ to $z_i = [\mathcal{O}_{\mathcal{Z}_i}]$, as desired.

Finally, we treat the subtle case in which $(t_i^\sharp, \mu_i^\sharp) \neq (0,0)$ and $(t_i, \mu_i) = (0,0)$. This is the case that accounts for improper intersection between the images of $\mathcal{M} \to \mathcal{M}^\sharp$ and $\mathcal{Z}^\sharp(T^\sharp, \mu^\sharp) \to \mathcal{M}^\sharp$. The left vertical arrow in (5.12) is an isomorphism

$$\mathcal{Z}_i \cong \mathcal{M},$$

and the top horizontal arrow is identified with the closed immersion

$$i : \mathcal{M} \to \mathcal{Z}_i^\sharp$$

sending a functorial point $S \to \mathcal{M}$ to the point $S \to \mathcal{Z}_i^\sharp$ determined by the special quasi-endomorphism $y_i \in V_{\mu_i^\sharp}(\mathcal{A}_S^\sharp)$ of (5.13). This induces the open and closed immersion

$$j : \mathcal{M} \cong \mathcal{Z}_i \hookrightarrow \mathcal{Z}_i^\sharp \times_{\mathcal{M}^\sharp} \mathcal{M},$$

and the composition (5.14) factors as

(5.15)
$$\begin{array}{ccc} G_0(\mathcal{Z}_i^\sharp)_{\mathbb{Q}} & \xrightarrow{\cap[\mathcal{O}_{\mathcal{M}}]} & G_0(\mathcal{Z}_i^\sharp \times_{\mathcal{M}^\sharp} \mathcal{M})_{\mathbb{Q}} \\ \downarrow{\scriptstyle \cap[\mathcal{O}_{\mathcal{Z}_i^\sharp}]} & & \\ G_0(\mathcal{Z}_i^\sharp \times_{\mathcal{M}^\sharp} \mathcal{Z}_i^\sharp)_{\mathbb{Q}} & & \downarrow{\scriptstyle j^*} \\ \downarrow{\scriptstyle \Delta^*} & & \\ G_0(\mathcal{Z}_i^\sharp)_{\mathbb{Q}} & \longrightarrow & G_0(\mathcal{M})_{\mathbb{Q}}. \end{array}$$

Here $\Delta : \mathcal{Z}_i^\sharp \to \mathcal{Z}_i^\sharp \times_{\mathcal{M}^\sharp} \mathcal{Z}_i^\sharp$ is the diagonal morphism, which is both an open and closed immersion, and the bottom horizontal arrow is the derived pullback

$$[\mathcal{F}] \mapsto \sum_{\ell \geqslant 0} (-1)^\ell \cdot [\underline{\mathrm{Tor}}_\ell^{\mathcal{O}_{\mathcal{Z}_i^\sharp}}(\mathcal{F}, i_*\mathcal{O}_\mathcal{M})] \tag{5.16}$$

along the closed immersion i.

Abbreviating

$$(S_0, \eta_0) = \left(\begin{pmatrix} t_i^\sharp & 0 \\ 0 & 0 \end{pmatrix}, (\mu_i^\sharp, 0) \right) \quad \text{and} \quad (S_0', \eta_0') = \left(\begin{pmatrix} t_i^\sharp & t_i^\sharp \\ t_i^\sharp & t_i^\sharp \end{pmatrix}, (\mu_i^\sharp, \mu_i^\sharp) \right),$$

there are canonical isomorphisms

$$\mathcal{Z}^\sharp(S_0, \eta_0) \cong \mathcal{Z}_i^\sharp \cong \mathcal{Z}^\sharp(S_0', \eta_0').$$

Under the moduli interpretations, a special quasi-endomorphism x of $\mathcal{A}^\sharp$ representing a point of the stack in the middle is sent to $(x, 0)$ on the left, and (x, x) on the right. Proposition 2.4.6 realizes

$$\mathcal{Z}_i^\sharp \cong \mathcal{Z}^\sharp(S_0', \eta_0') \subset \mathcal{Z}_i^\sharp \times_\mathcal{M} \mathcal{Z}_i^\sharp$$

as an open and closed substack, and this agrees with the diagonal embedding denoted Δ above.

The linear invariance proved in Proposition 5.4.1 implies the equality of derived fundamental classes

$$[\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}^\sharp(S_0, \eta_0)}] = [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}^\sharp(S, \eta)}] \in G_0(\mathcal{Z}_i^\sharp)_\mathbb{Q},$$

Unpacking their definitions shows that the composition of the left vertical arrows in (5.15) sends

$$[\mathcal{O}_{\mathcal{Z}_i^\sharp}] \mapsto [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}^\sharp(S_0', \eta_0')}] = [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}^\sharp(S_0, \eta_0)}] = [\mathcal{O}_{\mathcal{Z}_i^\sharp}] - [\omega^\sharp|_{\mathcal{O}_{\mathcal{Z}_i^\sharp}}],$$

The bottom horizontal arrow (5.16), which simplifies to $[\mathcal{F}] \mapsto [i^*\mathcal{F}]$ when $\mathcal{F}$ is a vector bundle on $\mathcal{Z}_i^\sharp$, then sends

$$[\mathcal{O}_{\mathcal{Z}_i^\sharp}] - [\omega^\sharp|_{\mathcal{O}_{\mathcal{Z}_i^\sharp}}] \mapsto [\mathcal{O}_\mathcal{M}] - [\omega].$$

Combining these calculations with the commutativity of the diagram shows that (5.14) sends $z_i^\sharp = [\mathcal{O}_{\mathcal{Z}_i^\sharp}]$ to $z_i = [\mathcal{O}_\mathcal{M}] - [\omega]$, as desired. □

Now consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}(T, \mu) & \longrightarrow & \mathcal{Z}(T^\sharp, \mu^\sharp) \\
\downarrow & & \downarrow \\
\mathcal{Z}_1 \times_\mathcal{M} \cdots \times_\mathcal{M} \mathcal{Z}_d & \longrightarrow & \mathcal{Z}_1^\sharp \times_{\mathcal{M}^\sharp} \cdots \times_{\mathcal{M}^\sharp} \mathcal{Z}_d^\sharp \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{f} & \mathcal{M}^\sharp.
\end{array}$$

The upper vertical arrows are open and closed immersions. The middle horizontal arrow identifies

$$\mathcal{Z}_1 \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}_d \subset (\mathcal{Z}_1^\sharp \times_{\mathcal{M}^\sharp} \cdots \times_{\mathcal{M}^\sharp} \mathcal{Z}_d^\sharp) \times_{\mathcal{M}^\sharp} \mathcal{M}$$

as an open and closed substack, and induces a morphism

$$G_0(\mathcal{Z}_1^\sharp \times_{\mathcal{M}^\sharp} \cdots \times_{\mathcal{M}^\sharp} \mathcal{Z}_d^\sharp) \to G_0(\mathcal{Z}_1 \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}_d)$$

exactly as in (5.14). It follows from Lemma 5.6.2 that this morphism sends

$$z_1^\sharp \cap \cdots \cap z_d^\sharp \mapsto z_1 \cap \cdots \cap z_d,$$

and so the commutativity of the diagram

$$\begin{array}{ccc} G_0(\mathcal{Z}_1^\sharp \times_{\mathcal{M}^\sharp} \cdots \times_{\mathcal{M}^\sharp} \mathcal{Z}_d^\sharp) & \longrightarrow & G_0(\mathcal{Z}_1 \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}_d) \\ \downarrow & & \downarrow \\ G_0(\mathcal{Z}(T^\sharp, \mu^\sharp)) & \longrightarrow & G_0(\mathcal{Z}(T, \mu)) \end{array}$$

implies that the bottom horizontal arrow sends

$$[\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)}] \mapsto [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T,\mu)}].$$

Allowing $\mathcal{Z}(T, \mu)$ to vary over the right hand side of (5.11), we see that in the commutative diagram

$$\begin{array}{ccc} G_0\big(\mathcal{Z}^\sharp(T^\sharp, \mu^\sharp)\big)_{\mathbb{Q}} & \xrightarrow{\cap[\mathcal{O}_{\mathcal{M}}]} & G_0\big(\mathcal{Z}^\sharp(T^\sharp, \mu^\sharp) \times_{\mathcal{M}^\sharp} \mathcal{M}\big)_{\mathbb{Q}} \\ {\scriptstyle (A.8)}\downarrow & & \downarrow{\scriptstyle (A.8)} \\ K_0^{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)}(\mathcal{M}^\sharp)_{\mathbb{Q}} & \xrightarrow{f^*} & K_0^{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)\times_{\mathcal{M}^\sharp}\mathcal{M}}(\mathcal{M})_{\mathbb{Q}}, \end{array}$$

the top horizontal arrow sends

$$[\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)}] \mapsto \sum_{\substack{T \in \mathrm{Sym}_d(\mathbb{Q}) \\ \mu \in (L^\vee/L)^d}} \sum_{\substack{\nu \in (\Lambda^\vee/\Lambda)^d \\ \mu+\nu=\mu^\sharp}} \sum_{\substack{y \in \nu + \Lambda^d \\ T+Q(y)=T^\sharp}} [\mathcal{O}^{\mathrm{derived}}_{\mathcal{Z}(T,\mu)}].$$

Proposition 5.6.1 follows immediately from this and the diagram

$$\begin{array}{ccc} F^d K_0^{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)}(\mathcal{M}^\sharp)_{\mathbb{Q}} & \xrightarrow{f^*} & F^d K_0^{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)\times_{\mathcal{M}^\sharp}\mathcal{M}}(\mathcal{M})_{\mathbb{Q}} \\ {\scriptstyle (A.10)}\downarrow & & \downarrow{\scriptstyle (A.10)} \\ \mathrm{CH}^d_{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)}(\mathcal{M}^\sharp) & \xrightarrow[f^*]{} & \mathrm{CH}^d_{\mathcal{Z}^\sharp(T^\sharp,\mu^\sharp)\times_{\mathcal{M}^\sharp}\mathcal{M}}(\mathcal{M}), \end{array}$$

which commutes by the very definition of the bottom horizontal arrow. □

6. Modularity in all codimensions

In this section we prove our main result. We remind the reader that V is a quadratic space of signature $(n,2)$ with $n \geqslant 1$, $L \subset V$ is a lattice on which the quadratic form is $\mathbb{Z}$-valued, Σ is a finite set of primes containing all primes for which L_p is not maximal (an assumption that will be strengthened below), and

$$\mathcal{M} \to \mathrm{Spec}(\mathbb{Z}[\Sigma^{-1}])$$

is the integral model of dimension $n+1$ from §2.2.

6.1. Siegel theta series.

Let (Λ, Q) be a positive definite quadratic space over $\mathbb{Z}$, satisfying $\Lambda^\vee = \Lambda$. The self-duality condition implies that the rank of Λ is even, say

$$\mathrm{rank}(\Lambda) = 2k.$$

For any positive integer d, the *genus d Siegel theta series*

$$\vartheta_{\Lambda,d}(\tau) = \sum_{y \in \Lambda^d} q^{Q(y)} \tag{6.1}$$

is Siegel modular form of genus d and weight k. Here $\tau \in \mathcal{H}_d \subset \mathrm{Sym}_d(\mathbb{C})$ is the variable on the Siegel half-space of genus d, $Q(y) \in \mathrm{Sym}_d(\mathbb{Q})$ is the moment matrix as in (2.9), and $q^{Q(y)} = e^{2\pi i \mathrm{Tr}(\tau Q(y))}$.

Theorem 6.1.1 (Böcherer [Böc89])**.** *If $4 \mid k$ and $k > 2d$, the space of $\mathbb{C}$-valued Siegel modular forms of genus d and weight k is spanned by the genus d Siegel theta series* (6.1) *as Λ varies over all self-dual positive definite $\mathbb{Z}$-quadratic spaces of rank $2k$.*

6.2. The main result.

We now extend the modularity of generating series proved in [BWR15] from complex Shimura varieties to their integral models.

Assume throughout that Σ contains

- all odd primes p such that p^2 divides $[L^\vee : L]$, and
- $p = 2$, if L_2 is not hyperspecial.

In particular $\mathcal{M}$ is regular by Proposition 2.2.4, allowing us to define the derived cycle classes

$$\mathcal{C}(T,\mu) \in \mathrm{CH}^d(\mathcal{M})$$

of Definition 5.1.3. By Proposition 5.5.1, these are given by the elementary formula

$$\mathcal{C}(T,\mu) = \underbrace{c_1(\omega^{-1}) \cdots c_1(\omega^{-1})}_{d-\mathrm{rank}(T)} \cdot [\mathcal{Z}(T,\mu)] \in \mathrm{CH}^d(\mathcal{M}) \tag{6.2}$$

whenever the naive special cycle $\mathcal{Z}(T,\mu) \to \mathcal{M}$ is equidimensional of codimension $\mathrm{rank}(T)$. Using the notation of §4.1, abbreviate

$$\mathcal{C}(T) = \sum_{\mu \in (L^\vee/L)^d} \mathcal{C}(T,\mu) \otimes \phi_\mu^* \in \mathrm{CH}^d(\mathcal{M}) \otimes S_{L,d}^*.$$

Theorem 6.2.1. *For every integer $1 \leqslant d \leqslant n+1$, the formal generating series*

$$\phi(\tau) = \sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} \mathcal{C}(T) \cdot q^T \tag{6.3}$$

valued in $\mathrm{CH}^d(\mathcal{M}) \otimes S^*_{L,d}$ *converges to a Siegel modular form of weight* $1 + \frac{n}{2}$ *and representation*

$$\omega^*_{L,d} : \widetilde{\Gamma}_d \to \mathrm{GL}(S^*_{L,d}). \tag{6.4}$$

The convergence and modularity are understood in the sense of Theorem A: they hold after applying any $\mathbb{Q}$*-linear functional* $\mathrm{CH}^d(\mathcal{M}) \to \mathbb{C}$*.*

Proof. When $d = 1$ the desired modularity is [HM20, Theorem B]. We remark that the isomorphism $S_L \cong S^*_L$ sending $\phi_\mu \mapsto \phi^*_\mu$ identifies the representation ρ_L in the statement of *loc. cit.* with the representation ω^*_L defined in §4.1. Henceforth we assume $d \geqslant 2$.

It is a theorem of Bruinier and Westerholt-Raum [BWR15] that (6.3) is modular of the stated weight and representation if and only if two conditions are satisfied:

(1) For every $T \in \mathrm{Sym}_d(\mathbb{Q})$ and $A \in \mathrm{GL}_d(\mathbb{Z})$, the coefficients satisfy the linear invariance relation
$$\mathcal{C}(T) = \mathcal{C}({}^t ATA).$$

(2) For every $T_0 \in \mathrm{Sym}_{d-1}(\mathbb{Q})$, the generating series
$$\sum_{\substack{m \in \mathbb{Q} \\ \alpha \in \mathbb{Q}^{d-1}}} \mathcal{C}\begin{pmatrix} T_0 & \frac{\alpha}{2} \\ \frac{{}^t\alpha}{2} & m \end{pmatrix} \cdot q^m \xi_1^{\alpha_1} \cdots \xi_{d-1}^{\alpha_{d-1}}$$
with coefficients in $\mathrm{CH}^d(\mathcal{M}) \otimes S^*_{L,d}$ is a Jacobi form of weight $1 + \frac{n}{2}$, index T_0, and representation (6.4).

Let $r(L)$ be the integer of Definition 3.1.1. If we assume that
$$n \geqslant 3d + 2r(L) + 4$$
then the special cycles $\mathcal{Z}(T, \mu) \to \mathcal{M}$ indexed by $T \in \mathrm{Sym}_d(\mathbb{Q})$ are equidimensional of codimension $\mathrm{rank}(T)$ by Proposition 3.3.1, and so the equality (6.2) holds. The linear invariance of condition (1) is Proposition 5.4.1 The Jacobi modularity of condition (2) is (up to a change of notation) Proposition 4.2.3. Note that we are using (6.2) to compare the cycle classes of Definition 5.1.3 with the cycle classes (4.5) used throughout §4. This proves the theorem when $n \geqslant 3d + 2r(L) + 4$.

To treat the general case, let Λ be any positive definite self-dual quadratic lattice over $\mathbb{Z}$, chosen so that
$$\mathrm{rank}(\Lambda) \geqslant 3d + 2r(L) + 4.$$
The $\mathbb{Z}$-quadratic space $L^\sharp = L \oplus \Lambda$ has signature $(n^\sharp, 2)$ with
$$n^\sharp \geqslant \mathrm{rank}(\Lambda) \geqslant 3d + 2r(L^\sharp) + 4,$$

where we have used $r(L^\sharp) \leqslant r(L)$. The quadratic lattice $L^\sharp$ determines its own integral model $\mathcal{M}^\sharp$ over $\mathbb{Z}[\Sigma^{-1}]$, with its own corrected special cycles

$$\mathcal{C}^\sharp(T,\mu) \in \mathrm{CH}^d(\mathcal{M}^\sharp).$$

As in Remark 2.2.7, there is a finite morphism $f : \mathcal{M} \to \mathcal{M}^\sharp$ of regular stacks over $\mathbb{Z}[\Sigma^{-1}]$. The self-duality of Λ implies that $S_{L^\sharp,d} \cong S_{L,d}$, so there is a pullback (§A.2)

$$f^* : \mathrm{CH}^d(\mathcal{M}^\sharp) \otimes S^*_{L^\sharp,d} \to \mathrm{CH}^d(\mathcal{M}) \otimes S^*_{L,d}.$$

By the special case proved above, the generating series

$$\phi^\sharp(\tau) = \sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} \mathcal{C}^\sharp(T) \cdot q^T$$

valued in $\mathrm{CH}^d(\mathcal{M}^\sharp) \otimes S^*_{L^\sharp,d}$ is a Siegel modular form of genus d, weight $1 + \frac{n^\sharp}{2}$, and representation $\omega^*_{L^\sharp,d}$. On the other hand, Proposition 5.6.1 (more precisely, the special case stated in the introduction as Theorem E) implies the factorization of generating series

$$f^*\phi^\sharp(\tau) = \phi(\tau) \cdot \vartheta_{\Lambda,d}(\tau),$$

where the second factor on the right is the Siegel theta series (6.1). It follows that $\phi(\tau)$ is a *meromorphic* Siegel modular form of weight

$$1 + \frac{n^\sharp}{2} - \frac{\mathrm{rank}(\Lambda)}{2} = 1 + \frac{n}{2}$$

with poles supported on the zero locus of $\vartheta_{\Lambda,d}(\tau)$.

It remains to verify that $\phi(\tau)$ is holomorphic, which we do by allowing Λ to vary. Fix a point $\tau_0 \in \mathcal{H}_d$ in the genus d Siegel half-space. As in the arguments of [Bai58], it follows from the construction of Poincaré series found in [Car58] that there exists a Siegel modular form of genus d and some weight k that does not vanish at τ_0. Moreover, we may choose this form in such a way that its weight satisfies $4 \mid k$ and $2k \geqslant 3d + 2r(L) + 4$. By Theorem 6.1.1, there exists a self-dual Λ as above with

$$\mathrm{rank}(\Lambda) = 2k \geqslant 3d + 2r(L) + 4,$$

whose associated genus d Siegel theta series $\vartheta_{\Lambda,d}$ does not vanish at τ_0. The existence of such a Λ implies that $\phi(\tau)$ is holomorphic near τ_0.[5] □

[5] There is a subtlety in this proof pointed out to us by Steve Kudla. In the proof we are implicitly making use of the following fact: The ring of formal q-series satisfying the linear invariance property enjoyed by the series here is an integral domain, since it is the complete local ring of the *normal* Baily-Borel compactification of the Siegel modular variety. See the discussion in [Kud21, §8].

Appendix A. Chow groups

We need a working theory of Chow groups (always with $\mathbb{Q}$-coefficients) for Deligne-Mumford stacks M, and in greater generality than is usually found in the literature. For example, §3 and §4 make systematic use of Chow groups of stacks that are not locally integral.

Throughout this section we fix a ring S that is either a field or an excellent Dedekind domain (for example, $S = \mathbb{Z}$). The term *stack* will mean an equidimensional and separated Deligne-Mumford stack of finite type over S.

A.1. **Chow groups of Deligne-Mumford stacks.** Our goal in this subsection is to define Chow groups of stacks, and show that it is covariant with respect to finite morphisms. Most of what we need can be deduced directly from the results of [Gil84].

As in [Gil84, Definition 3.2] a stack M has an underlying topological space $|M|$, whose points are the integral closed substacks $Z \subset M$. Each open substack $U \subset M$ determines an open set $|U| \subset |M|$, whose points are those integral closed substacks $Z \subset M$ for which $Z \cap U \neq \emptyset$. Any morphism of stacks $M' \to M$ induces a continuous map

$$|M'| \to |M| \tag{A.1}$$

by [Gil84, Corollary 3.4].

We write $M^{(d)}$ for the subset of $|M|$ consisting of the those integral substacks of codimension d.

Remark A.1.1. A field-valued point $x \in M(k)$ determines a map of topological spaces $|\mathrm{Spec}(k)| \to |M|$ whose image is a single point. This point, the *Zariski closure* of x, is an integral closed substack denoted $\overline{\{x\}} \subset M$. Taking Zariski closures of field-valued points establishes a bijection between the topological space $|M|$ as defined above, and the more common definition in terms of equivalence classes of field-valued points [Sta22, Tag 04XE].

Recall from [Gil84, §3] that a stack ξ is *punctual* if it is reduced and $|\xi|$ is a single point. The ring of global functions

$$k(\xi) = H^0(\xi, \mathcal{O}_\xi)$$

of such a ξ is a field. Moreover, if $U \to \xi$ is an étale chart, then $U = \mathrm{Spec}(E)$ where E is a separable $k(\xi)$-algebra. If we write $U \times_\xi U = \mathrm{Spec}(E')$ then E' is a free E-module (for either of the two natural maps $E \to E'$), and we define the *ramification index* of ξ to be

$$e(\xi) = \frac{\mathrm{rank}_E(E')}{\dim_{k(\xi)}(E)}.$$

This is independent of the choice of chart.

According to [Sta22, Tag 0H22], one can associate to any $Z \in |M|$ a distinguished punctual stack ξ together with a map $\xi \to M$ such that the image of $|\xi| \to |M|$ is Z. This ξ is known as the *residual gerb* at Z, but we will refer to it below simply as the *generic point* of Z. We usually conflate $Z \in M^{(d)}$ with its generic point, for example by writing $\xi \in M^{(d)}$ and referring to ξ as a codimension d point of M.

If ξ is the generic point of an integral stack M, then we call $k(\xi)$ the *field of rational functions of* M, and also denote it by $k(M)$. If we write $\mathrm{Et}(M)$ for the category of étale maps $U \to M$ with U a scheme, then

$$k(M) = \varprojlim_{(U\to M)\in \mathrm{Et}(M)} k(U). \tag{A.2}$$

For an integer $d \geqslant 0$, define the *vector space of d-cycles* $\mathcal{Z}^d(M)$ as the free $\mathbb{Q}$-vector space on the set $M^{(d)}$ of codimension d integral closed substacks. In particular, each point $\xi \in M^{(d)}$ gives us a basis vector $[\xi] \in \mathcal{Z}^d(M)$. By [Gil84, Lemma 4.3], we have

$$\mathcal{Z}^d(M) = \varprojlim_{(U\to M)\in \mathrm{Et}(M)} \mathcal{Z}^d(U). \tag{A.3}$$

When M is an integral scheme there is a divisor map $\mathrm{div} : k(M) \to \mathcal{Z}^1(M)$ defined by

$$\mathrm{div}(f) = \sum_{\xi\in M^{(1)}} \mathrm{ord}_\xi(f)[\xi].$$

Here

$$\mathrm{ord}_\xi(f) = \mathrm{length}(R/(g)) - \mathrm{length}(R/(h))$$

where $R = \mathcal{O}_{M,\xi}$ is the local ring of M at ξ, and we have written $f = g/h$ in its field of fractions. Combining this construction with (A.2) and (A.3) allows us to extend the definition of the divisor map $\mathrm{div} : k(M) \to \mathcal{Z}^1(M)$ to any integral stack M.

For any finite morphism $\pi : M' \to M$ of stacks with $\dim(M') = \dim(M) - r$, there is a pushforward

$$\pi_* : \mathcal{Z}^{d-r}(M') \to \mathcal{Z}^d(M). \tag{A.4}$$

Indeed, given a codimension $d-r$ point $\xi' \in |M'|$, its image under (A.1) is a codimension d point $\xi \in |M|$, and there is a canonical finite morphism of punctual stacks $\xi' \to \xi$. This allows us to define

$$\pi_*[\xi'] = \frac{e(\xi)}{e(\xi')} \cdot \deg\big(k(\xi')/k(\xi)\big) \cdot [\xi],$$

where the degree is the usual degree of a field extension.

In particular, if $\xi \in M^{(d-1)}$ is the generic point of an integral closed substack $W \subset M$ there is a divisor map

$$\mathrm{div}_\xi : k(\xi)^\times = k(W)^\times \xrightarrow{\mathrm{div}} \mathcal{Z}^1(W) \to \mathcal{Z}^d(M),$$

where the final arrow is the pushforward along the inclusion $W \hookrightarrow M$.

Definition A.1.2. Setting

$$\mathcal{R}^d(M) = \bigoplus_{\xi\in M^{(d-1)}} k(\xi)^\times_{\mathbb{Q}},$$

define the ($\mathbb{Q}$-coefficient) *codimension d Chow group* of a stack M by

$$\mathrm{CH}^d(M) = \mathrm{coker}\left(\mathcal{R}^d(M) \xrightarrow{\sum_\xi \mathrm{div}_\xi} \mathcal{Z}^d(M)\right).$$

As usual, cycles in the kernel of the natural map $\mathcal{Z}^d(M) \to \mathrm{CH}^d(M)$ are said to be *rationally equivalent to* 0.

Proposition A.1.3. *Suppose $\pi : M' \to M$ is a finite morphism of stacks with* $\dim(M') = \dim(M) - r$*. The pushforward on cycles* (A.4) *descends to*

$$\pi_* : \mathrm{CH}^{d-r}(M') \to \mathrm{CH}^d(M).$$

Proof. Given a codimension $d - r + 1$ point $\xi' \in |M'|$ with image $\xi \in |M|$, and an $f \in k(\xi')^\times$, we need to check that $\pi_*\mathrm{div}_{\xi'}(f) \in \mathcal{Z}^d(M)$ is rationally equivalent to 0. This follows from the fact that $k(\xi')$ is a finite field extension of $k(\xi)$ satisfying

$$\pi_*\mathrm{div}_{\xi'}(f) = \mathrm{div}_\xi\big(\mathrm{Nm}_{k(\xi')/k(\xi)}(f)\big).$$ □

We need the notion of Chow groups with support from [Sou92, I.2]. For any finite map $\pi : Z \to M$ as in Definition A.1.4, set

$$\mathrm{CH}^d_Z(M) = \mathrm{CH}^{d-r}(\pi(Z)),$$

where $\pi(Z) \subset M$ is the stack theoretic image of π: the reduced closed substack characterized by the property that for every étale map $U \to M$ with U a scheme, the closed subscheme $U \times_M \pi(Z) \subset U$ is equal to the image of the finite morphism $U \times_M Z \to U$.

Definition A.1.4. Suppose $\pi : Z \to M$ is a finite morphism of stacks with Z equidimensional of dimension $\dim(Z) = \dim(M) - r$. Define

$$[Z] = \sum_{i=1}^n m_i \cdot \pi_*[\xi_i] \in \mathrm{CH}^r_Z(M),$$

where $\xi_1, \ldots, \xi_n \in Z^{(0)}$ are the generic points of the irreducible components of Z, and m_i is the length of the étale local ring $\mathcal{O}^{\mathrm{et}}_{Z,\xi_i}$,

A.2. **Chow groups and Grothendieck groups.** The Chow groups defined above are contravariant with respect to morphisms between regular stacks, and also admit a bilinear intersection pairing. The key to these properties are the results of [GS87, §8], [Sou92, Ch. I] and [Gil09], relating Chow groups to Grothendieck groups of locally free sheaves.

For a scheme M, let $K_0(M)$ be the quotient of the free abelian group generated by symbols $[\mathcal{Q}_\bullet]$, where $\mathcal{Q}_\bullet$ runs over finite complexes of vector bundles on M, by the relations

- $[\mathcal{Q}_\bullet] = [\mathcal{R}_\bullet]$ whenever $\mathcal{Q}_\bullet$ and $\mathcal{R}_\bullet$ are quasi-isomorphic,
- $[\mathcal{Q}_\bullet] = [\mathcal{P}_\bullet] + [\mathcal{R}_\bullet]$ whenever there is a short exact sequence

$$0 \to \mathcal{P}_\bullet \to \mathcal{Q}_\bullet \to \mathcal{R}_\bullet \to 0.$$

If $\pi : Z \to M$ is a finite morphism, the group $K_0^Z(M)$ is defined in exactly the same way, except that we only consider complexes that become exact after restriction to the open subscheme $M \smallsetminus \pi(Z)$.

Still assuming that M is a scheme, we similarly write $G_0(M)$ for the Grothendieck group of the category of coherent $\mathcal{O}_M$-modules. Thus $G_0(M)$ is the abelian group generated by symbols $[\mathcal{Q}]$ with $\mathcal{Q}$ is a coherent sheaf on M, subject to the relations $[\mathcal{Q}] = [\mathcal{P}] + [\mathcal{R}]$ whenever there is a short exact sequence

$$0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{R} \to 0.$$

Our $G_0(M)$ is the group denoted $K_0'(M)$ in [Sou92].

Remark A.2.1. One can naively imitate these definitions when M is a stack. For example, we define $G_0^{\mathrm{naive}}(M)$ to be the free group generated by symbols $[\mathcal{Q}]$ where $\mathcal{Q}$ is a coherent sheaf on M, modulo the relations $[\mathcal{Q}] = [\mathcal{P}] + [\mathcal{R}]$ whenever there is a short exact sequence as above. While $G_0^{\mathrm{naive}}(M)$ will be of use to us, the analogous naive extensions of $K_0^Z(M)$ and $K_0(M)$ will not. For example, Theorem A.2.7 below is false if one uses these naive definitions.

In light of the previous remark, we associate $\mathbb{Q}$-vector spaces $K_0(M)_{\mathbb{Q}}$ and $G_0(M)_{\mathbb{Q}}$ to a stack M following the more sophisticated constructions of [Gil09, §2]. This requires the machinery of K-theory and G-theory spectra as laid out in [TT90, §3]. Recall that $\mathrm{Et}(M)$ is the étale site of M, whose objects are schemes U equipped with an étale morphism $U \to M$.

Quillen K-theory defines a presheaf

$$\mathbf{K}_M(U \to M) \stackrel{\mathrm{def}}{=} K(U)_{\mathbb{Q}}$$

on $\mathrm{Et}(M)$ valued in spectra over the Eilenberg-MacLane spectrum $H\mathbb{Q}$ (we will call this a *rational spectrum* for concision), or, more prosaically, in the derived category of bounded below chain complexes of $\mathbb{Q}$-vector spaces; see [Tak04, §2.1] for an elementary and explicit representation as a chain complex. By a result of Thomason, this presheaf is in fact a sheaf, and one now defines the rational K-theory $K(M)_{\mathbb{Q}}$ to be its global sections. The vector space $K_0(M)_{\mathbb{Q}}$ is defined as the 0^{th} homology of $K(M)_{\mathbb{Q}}$.

A completely analogous construction, using the rational spectrum associated with the exact category of coherent sheaves, gives us a sheaf of spectra

$$\mathbf{G}_M(U \to M) \stackrel{\mathrm{def}}{=} G(U)_{\mathbb{Q}}$$

on $\mathrm{Et}(M)$. Taking global sections defines the rational G-theory space $G(M)_{\mathbb{Q}}$, and the vector space $G_0(M)_{\mathbb{Q}}$ is defined as its 0^{th} homology.

To get K_0-groups with support along a finite map $Z \to M$, one now repeats the construction using the presheaf

$$\mathbf{K}_M^Z(U \to M) \mapsto K^{Z \times_M U}(U)_{\mathbb{Q}}$$

associating to U the rational spectrum associated with the exact category of bounded complexes of vector bundles on U with cohomology sheaves supported on the image of $Z \times_M U \to U$. Taking the 0-th homology group of the global sections of this presheaf (which is, once again, actually a sheaf) defines the vector space $K_0^Z(M)_{\mathbb{Q}}$.

Still assuming that $Z \to M$ is a finite morphism of stacks, fix a morphism

$$f : M' \to M$$

and set $Z' = Z \times_M M'$. Given an étale morphism $(U \to M) \in \mathrm{Et}(M)$, if we set $U' = U \times_M M'$ then pullback via $U' \to U$ takes bounded complexes of vector bundles on U, acyclic outside the image of $Z \times_M U \to U$, to bounded complexes of vector bundles on U', acyclic outside the image of $Z' \times_{M'} U' \to U'$. This induces a pullback map on the corresponding sheaves of spectra, and hence a map

$$f^* : K_0^Z(M)_{\mathbb{Q}} \to K_0^{Z'}(M')_{\mathbb{Q}}. \tag{A.5}$$

Now suppose we have finite morphisms of stacks $Z_1 \to M$ and $Z_2 \to M$. For any $(U \to M) \in \mathrm{Et}(M)$ the tensor product of bounded complexes of vector bundles determines a map of rational spectra

$$K^{Z_1 \times_M U}(U)_{\mathbb{Q}} \otimes K^{Z_2 \times_M U}(U)_{\mathbb{Q}} \to K^{Z_1 \times_M Z_2 \times_M U}(U)_{\mathbb{Q}},$$

which in turn gives rise to a canonical pairing

$$K_0^{Z_1}(M)_{\mathbb{Q}} \otimes K_0^{Z_2}(M)_{\mathbb{Q}} \to K_0^{Z_1 \times_M Z_2}(M)_{\mathbb{Q}} \tag{A.6}$$

By construction, this pairing is compatible (in the obvious sense) with the pullback (A.5).

The above vector spaces $K_0(M)_{\mathbb{Q}}$, $K_0^Z(M)_{\mathbb{Q}}$, and $G_0(M)_{\mathbb{Q}}$ agree with those defined at the beginning of this subsection when M is a scheme, and the operations (A.5) and (A.6) are the obvious ones defined by pullbacks and tensor products of complexes of sheaves. For general stacks M these vector spaces do not admit obvious descriptions in terms of coherent sheaves on M.

Remark A.2.2. When ξ is a punctual stack, the rank map $K_0(\xi)_{\mathbb{Q}} \to \mathbb{Q}$ is an isomorphism. To see this one reduces to the case where ξ admits a finite étale cover $\mathrm{Spec}(L) \to \xi$ by a field L, and then uses [Gil09, Corollary 2.7], which shows that $K_0(\xi)_{\mathbb{Q}} \to K(\mathrm{Spec}(L))_{\mathbb{Q}} \cong \mathbb{Q}$ is an isomorphism.

Remark A.2.3. Every coherent sheaf $\mathcal{F}$ on M gives rise to[6] a map of sheaves of spectra $\underline{H\mathbb{Q}} \to \mathbf{G}_M$, which evaluates on global sections to a canonical class

$$[\mathcal{F}] \in G_0(M)_{\mathbb{Q}}.$$

Here $\underline{H\mathbb{Q}}$ is the locally constant sheaf of spectra assigning to every connected $(U \to M) \in \mathrm{Et}(M)$ the constant spectrum $H\mathbb{Q}$ (equivalently, the object $\mathbb{Q}[0]$ in the derived category of bounded below complexes of $\mathbb{Q}$-vector

[6]This can be seen for instance from the explicit chain complex from [Tak04, §2.1], where $G(X)_{\mathbb{Q}}$ is represented by a complex of $\mathbb{Q}$-vector spaces whose degree 0 component is a quotient of the free abelian group on the set of coherent sheaves on X.

spaces). Recalling the notation of Remark A.2.1, this construction defines a homomorphism of vector spaces

$$G_0^{\mathrm{naive}}(M)_{\mathbb{Q}} \to G_0(M)_{\mathbb{Q}}, \tag{A.7}$$

which is surjective by [Gil09, Lemma 2.5].

Remark A.2.4. The homomorphism (A.7) need not be an isomorphism. Consider the punctual stack

$$M = [\mathrm{Spec}(\mathbb{C})/H]$$

determined by a finite group H acting trivially on $\mathbb{C}$. In this case $G_0^{\mathrm{naive}}(M)_{\mathbb{Q}}$ is the free $\mathbb{Q}$-module on the finite set of isomorphism classes of irreducible representations of H, while $G_0(M)_{\mathbb{Q}} \cong \mathbb{Q}$ by Remark A.2.2. The map (A.7) sends an irreducible representation to its dimension.

Proposition A.2.5. *Assume M is regular. Any finite morphism of stacks $\pi : Z \to M$ induces a pushforward homomorphism*

$$\pi_* : G_0(Z)_{\mathbb{Q}} \to K_0^Z(M)_{\mathbb{Q}}. \tag{A.8}$$

It is an isomorphism if M is a scheme and $\pi : Z \to M$ is a closed immersion.

Proof. In the case where M is a scheme, this is [Sou92, I.3.1 Lemma 4]. The pushforward homomorphism sends the class of a coherent sheaf $[\mathcal{F}] \in G_0(Z)$ to any finite resolution of $\pi_*\mathcal{F}$ by vector bundles on M.

In general, for every $(M' \to M) \in \mathrm{Et}(M)$ we have a map

$$\pi'_* : G(Z')_{\mathbb{Q}} \to K^{Z'}(M')_{\mathbb{Q}}$$

of spectra arising from a functor of exact categories. The right hand side can be identified with the rational spectrum associated with the category of perfect complexes on M' that are acyclic outside of Z' (see the argument in [TT90, Theorem 3.21], and the left hand side is associated with the exact category of bounded complexes of coherent sheaves on $Z' = Z \times_M M'$. The functor is now induced by pushforward along $\pi' : Z' \to M'$. □

Any finite morphisms of stacks $Y \to Z \to M$ induce maps

$$K_0^Y(M)_{\mathbb{Q}} \to K_0^Z(M)_{\mathbb{Q}}.$$

Define the *coniveau filtration* on $K_0^Z(M)_{\mathbb{Q}}$ by

$$F^d K_0^Z(M)_{\mathbb{Q}} = \bigcup_{\substack{\text{closed substacks } Y \subset Z \\ \mathrm{codim}_M(Y) \geqslant d}} \mathrm{Image}\big(K_0^Y(M)_{\mathbb{Q}} \to K_0^Z(M)_{\mathbb{Q}}\big), \tag{A.9}$$

and denote by

$$\mathrm{Gr}^d_\gamma K_0^Z(M)_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} F^d K_0^Z(M)_{\mathbb{Q}} / F^{d+1} K_0^Z(M)_{\mathbb{Q}}$$

the graded pieces of the filtration. For schemes, the following theorems of Gillet-Soulé are proved in [GS87] and [Sou92]. The extensions to stacks are addressed in [Gil09, §2.4].

Theorem A.2.6 (Gillet-Soulé)**.** *Suppose M is a regular stack.*

(1) *Given finite morphisms $Z_1 \to M$ and $Z_2 \to M$, the pairing* (A.6) *restricts to a bilinear pairing*

$$F^{d_1} K_0^{Z_1}(M)_{\mathbb{Q}} \otimes F^{d_2} K_0^{Z_2}(M)_{\mathbb{Q}} \to F^{d_1+d_2} K_0^{Z_1 \times_M Z_2}(M)_{\mathbb{Q}}.$$

(2) *Given a morphism $f : M' \to M$ with M' another regular stack, and a finite morphism $Z \to M$, the pullback* (A.5) *restricts to*

$$f^* : F^d K_0^Z(M)_{\mathbb{Q}} \to F^d K_0^{Z \times_M M'}(M')_{\mathbb{Q}}.$$

Theorem A.2.7 (Gillet-Soulé)**.** *Let M be a regular stack. For any finite morphism*

$$\pi : Z \to M$$

with $\dim(Z) = \dim(M) - r$*, there is a canonical isomorphism*

$$\mathrm{CH}^d_Z(M) \cong \mathrm{Gr}^d_\gamma K_0^Z(M)_{\mathbb{Q}} \tag{A.10}$$

carrying the class $[Z] \in \mathrm{CH}^r_Z(M)$ of Definition A.1.4 to the image of $[\mathcal{O}_Z]$ under

$$G_0(Z)_{\mathbb{Q}} \xrightarrow{(A.8)} K_0^Z(M)_{\mathbb{Q}}.$$

Proof. In the case of schemes, the existence of this isomorphism is [GS87, Theorem 8.2], and this argument is generalized to stacks in [Gil09, Theorem 2.8]. For the convenience of the reader, we recall some key inputs into these proofs. This will also help us justify the last assertion about the relationship between the cycle class $[Z]$ and the G-theory class $[\mathcal{O}_Z]$, since this is not made completely explicit in the references cited.

The starting point is the Brown-Gersten-Quillen spectral sequence with first page

$$E_1^{p,q} = \bigoplus_{\xi \in M^{(p)} \cap \pi(Z)} K_{-p-q}(\xi)_{\mathbb{Q}},$$

converging to the (higher) K-groups with support $K^Z_{-p-q}(M)_{\mathbb{Q}}$. This converges to the coniveau filtration on $K_0^Z(M)_{\mathbb{Q}}$. See [Sou92, Theorem 6] or [Gil09, Theorem 2.8] .

Next, we have Bloch's formula (due to Quillen), which shows that, on the second page, we have $E_2^{p,p} \cong \mathrm{CH}^p_Z(M)$. More precisely, we obtain the composition

$$\bigoplus_{\xi \in M^{(p-1)} \cap \pi(Z)} k(\xi)^\times_{\mathbb{Q}} \cong \bigoplus_{\xi \in M^{(p-1)} \cap \pi(Z)} K_1(\xi)_{\mathbb{Q}} = E_1^{p-1,p} \to E_1^{p,p}$$
$$= \bigoplus_{\eta \in M^{(p)} \cap \pi(Z)} K_0(\eta)_{\mathbb{Q}} \cong \mathcal{Z}^p_Z(M).$$

The arrow in the middle is the differential in the spectral sequence, and Quillen shows that this is exactly the divisor map whose cokernel is $\mathrm{CH}^p_Z(M)$ (for the case of stacks, we also need the observation from Remark A.2.2).

Finally, the interaction between this spectral sequence and Adams operations is used to show that the spectral sequence stabilizes on the second page (see [Sou92, §6.4], [GS87, Theorem 8.2] and [Gil09, §2.4]), which establishes the isomorphisms

$$\mathrm{CH}^p_Z(M) \cong E_2^{p,p} \cong \mathrm{Gr}^p_\gamma K_0^Z(M)_{\mathbb{Q}}.$$

Here, for $\xi \in M^{(p)} \cap \pi(Z)$, the associated map

$$K_0(\xi)_{\mathbb{Q}} \to \mathrm{Gr}^p_\gamma K_0^Z(M)_{\mathbb{Q}} \tag{A.11}$$

can be described as follows. Let $Y \subset M$ be the integral substack with generic point ξ. Then we have $F^p K_0^Y(M)_{\mathbb{Q}} = K_0^Y(M)_{\mathbb{Q}}$, as well as an exact sequence

$$F^{p+1}K_0^Y(M)_{\mathbb{Q}} \to K_0^Y(M)_{\mathbb{Q}} \to K_0(\xi)_{\mathbb{Q}} \to 0,$$

where the second map is just the map $K_0^Y(M) \to K_0(\xi)$ obtained by restriction to the generic point. This arises from the *localization sequence* for K-theory spectra [TT90, Theorems 6.8, 7.4, 7.6], which shows that, for every closed substack $Z \subset Y$, we have a fiber sequence of sheaves of spectra

$$\mathbf{K}^Z_M \to \mathbf{K}^Y_M \to \mathbf{K}^{Y \smallsetminus Z}_{M \smallsetminus Z}.$$

Taking global sections and then looking at H_0 gives us an exact sequence

$$K_0^Z(M)_{\mathbb{Q}} \to K_0^Y(M)_{\mathbb{Q}} \to K_0^{Y\smallsetminus Z}(M \smallsetminus Z)_{\mathbb{Q}} \to 0$$

To finish, we need to observe that

$$\mathrm{colim}_{\substack{Z \subset Y \\ Z \neq Y}} K_0^{Y \smallsetminus Z}(M \smallsetminus Z)_{\mathbb{Q}} \cong K_0(\xi)_{\mathbb{Q}},$$

which can be checked on the level of the corresponding sheaves of spectra.

This gives us an isomorphism

$$K_0(\xi)_{\mathbb{Q}} \cong \mathrm{Gr}^p_\gamma K_0^Y(M)_{\mathbb{Q}},$$

and composing it with the natural map

$$\mathrm{Gr}^p_\gamma K_0^Y(M)_{\mathbb{Q}} \to \mathrm{Gr}^p_\gamma K_0^Z(M)_{\mathbb{Q}}$$

now yields (A.11).

It still remains to verify the assertion about the class $[Z]$. That $[\pi_*\mathcal{O}_Z] \in F^r K_0^Z(M)_{\mathbb{Q}}$ is immediate from the definitions. That its image in

$$\mathrm{Gr}^r_\gamma K_0^Z(M)_{\mathbb{Q}} \cong \mathrm{CH}^r_Z(M)$$

is $[Z]$ comes down to the fact that for any generic point $\zeta \in Z^{(0)}$ with image $\xi \in M^{(r)} \cap \pi(Z)$, the image of $\mathcal{O}_\zeta$ in $K_0(\xi)_{\mathbb{Q}} \cong \mathbb{Q}$ is

$$\frac{e(\xi)}{e(\zeta)} \cdot \deg\big(k(\zeta)/k(\xi)\big). \qquad \square$$

Combining Theorems A.2.6 and A.2.7 yields intersection pairings and pullbacks on Chow groups of regular stacks. If M is a regular stack and $Z_1, Z_2 \to M$ are finite morphisms, there is a canonical bilinear intersection pairing

$$\mathrm{CH}^{d_1}_{Z_1}(M) \otimes \mathrm{CH}^{d_2}_{Z_2}(M) \to \mathrm{CH}^{d_1+d_2}_{Z_1 \times_M Z_2}(M).$$

For any morphism $M' \to M$ between regular stacks and any finite morphism $Z \to M$, there is a pullback

$$\mathrm{CH}^d_Z(M) \to \mathrm{CH}^d_{Z'}(M'),$$

where $Z' = Z \times_M M'$.

A.3. **Line bundles and divisor classes.** Suppose $\mathcal{L}$ is a line bundle on an integral scheme M. A *rational trivialization* s of $\mathcal{L}$ is an equivalence class of pairs (U, ξ), where $U \subset M$ is a dense open subscheme, and $\xi : \mathcal{O}_U \xrightarrow{\cong} \mathcal{L}|_U$ is a trivialization; two such pairs (U_1, ξ_1) and (U_2, ξ_2) are equivalent if the trivializations ξ_1, ξ_2 agree on the intersection $U_1 \cap U_2$. Write $k(\mathcal{L})^\times$ for the set of such rational trivializations.

The *divisor* of $s \in k(\mathcal{L})^\times$ is the cycle

$$\mathrm{div}(s) \stackrel{\mathrm{def}}{=} \sum_{\xi \in M^{(1)}} \mathrm{ord}_\xi(s)[\xi] \in \mathcal{Z}^1(M),$$

where the integer $\mathrm{ord}_\xi(s)$ is defined as follows. Let $R = \mathcal{O}_{M,\xi}$, and choose an isomorphism $\mathcal{L}|_{\mathrm{Spec}(R)} \cong \mathcal{O}_{\mathrm{Spec}(R)}$. Via this isomorphism s corresponds to a rational function f/g in the fraction field of R, and

$$\mathrm{ord}_\xi(s) = \mathrm{length}(R/(f)) - \mathrm{length}(R/(g)).$$

More generally, for a line bundle $\mathcal{L}$ on an integral stack M define

$$k(\mathcal{L})^\times \stackrel{\mathrm{def}}{=} \varprojlim_{(U \to M) \in \mathrm{Et}(M)} k(\mathcal{L}|_U)^\times.$$

From the case of schemes discussed above, we obtain a map

$$\mathrm{div} : k(\mathcal{L})^\times \cong \varprojlim_{(U \to M) \in \mathrm{Et}(M)} k(\mathcal{L}|_U)^\times \to \varprojlim_{(U \to M) \in \mathrm{Et}(M)} \mathcal{Z}^1(U) \cong \mathcal{Z}^1(M).$$

Note that $k(\mathcal{O}_M)^\times = k(M)^\times$ is the set of nonzero elements in (A.2).

If M is any (not necessarily integral) stack, let $Z_1, \ldots, Z_r$ be its irreducible components. Viewing these as integral stacks, we define

$$k(\mathcal{L})^\times = \prod_{i=1}^r k(\mathcal{L}|_{Z_i})^\times$$

and

$$\mathrm{div}(s) = \sum_i \mathrm{div}(s_i) \in \mathcal{Z}^1(M)$$

for any $s = (s_1, \ldots, s_r) \in k(\mathcal{L})^\times$. It is easy to see that the class of $\mathrm{div}(s)$ in $\mathrm{CH}^1(M)$ depends only on the isomorphism class of $\mathcal{L}$, and not on the particular choice of s. This allows us to make the following definition.

Definition A.3.1. The *first Chern class map*

$$c_1 : \mathrm{Pic}(M) \to \mathrm{CH}^1(M)$$

sends a line bundle $\mathcal{L}$ to the cycle class $[\mathrm{div}(s)]$ for any $s \in k(\mathcal{L})^\times$.

Suppose that $D \subset M$ is an effective Cartier divisor. In other words, D a closed substack whose ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_M$ is a line bundle. We have two ways of associating to D a class in $\mathrm{CH}^1(M)$. First, we can take the class $[D]$ as in Definition A.1.4. Second, we can take the first Chern class of the line bundle

$$\mathcal{L}(D) \stackrel{\text{def}}{=} \mathcal{I}_D^{-1}.$$

These two constructions agree, as the canonical section $\mathcal{O}_M \to \mathcal{L}(D)$ determines an $s \in k(\mathcal{L})^\times$ with

$$c_1(\mathcal{L}(D)) = [\mathrm{div}(s)] = [D].$$

Lemma A.3.2. *If M is regular, the composition*

$$\mathrm{Pic}(M) \xrightarrow{c_1} \mathrm{CH}^1(M) \xrightarrow{(A.10)} \mathrm{Gr}^1_\gamma K_0(M)_{\mathbb{Q}}$$

sends $\mathcal{L} \mapsto [\mathcal{O}_M]-[\mathcal{L}^{-1}]$. By slight abuse of notation, we are here identifying $[\mathcal{O}_M]$ and $[\mathcal{L}^{-1}]$ with their images under

$$G_0(M)_{\mathbb{Q}} \xrightarrow{(A.8)} K_0(M)_{\mathbb{Q}}.$$

Proof. If we write $\mathcal{L}^{-1} \cong \mathcal{I}_D \otimes \mathcal{I}_E^{-1}$ for effective Cartier divisors $D, E \subset M$ with $D \cap E \subset M$ of codimension $\geqslant 2$, then $c_1(\mathcal{L}) \in \mathrm{CH}^1(M)$ is represented by the class of the associated Weil divisor $D - E$.

Tensoring the short exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{I}_E^{-1} \to \mathcal{I}_E^{-1}/\mathcal{O}_M \to 0$$

with I_D shows that

$$[\mathcal{L}^{-1}] = [\mathcal{I}_D] + [\mathcal{I}_D \otimes (\mathcal{I}_E^{-1}/\mathcal{O}_M)]$$

holds in $G_0^{\mathrm{naive}}(M)$, which we rewrite as

$$[\mathcal{O}_M] - [\mathcal{L}^{-1}] = [\mathcal{O}_M/\mathcal{I}_D] - [\mathcal{I}_D \otimes (\mathcal{I}_E^{-1}/\mathcal{O}_M)].$$

Using the assumption that $D \cap E$ has codimension $\geqslant 2$, and the fact that $\mathcal{I}_E$ is locally principal, one can check that the equalities

$$[\mathcal{I}_D \otimes (\mathcal{I}_E^{-1}/\mathcal{O}_M)] = [\mathcal{I}_E^{-1}/\mathcal{O}_M] = [\mathcal{O}_M/\mathcal{I}_E]$$

hold in $G_0^{\mathrm{naive}}(M)$, up to a linear combination of classes $[\mathcal{F}]$ with $\mathcal{F}$ the pushforward to M of a coherent sheaf on a codimension two closed substack (contained in E). Hence

$$[\mathcal{O}_M] - [\mathcal{L}^{-1}] = [\mathcal{O}_M/\mathcal{I}_D] - [\mathcal{O}_M/\mathcal{I}_E] = [\mathcal{O}_D] - [\mathcal{O}_E] \tag{A.12}$$

holds in $G_0^{\mathrm{naive}}(M)$ up to the same ambiguity. Using the final claim of Theorem A.2.7, we find that that the image of $c_1(\mathcal{L}) = D - E$ under (A.10) is equal to $[\mathcal{O}_M] - [\mathcal{L}^{-1}]$. □

Proposition A.3.3. *As in Proposition A.1.3, let $\pi : M' \to M$ be a finite morphism of stacks with image of codimension r. Suppose in addition that M is a regular stack over $\mathbb{Z}$, and that for every prime p, there exists a quasi-projective scheme X over $\mathbb{Z}[1/p]$ equipped with the action of a finite group G such that*

$$[X/G] \cong M_{\mathbb{Z}[1/p]}.$$

For any line bundle $\mathcal{L} \in \mathrm{Pic}(M)$ we have

$$\pi_* c_1(\pi^*\mathcal{L}) = c_1(\mathcal{L}) \cdot [M'] \in \mathrm{CH}^{r+1}(M).$$

Proof. We claim first that for any finite subset $T \subset |M|$, there exists an $r \in \mathbb{Z}^+$ and a section $s \in H^0(M, \mathcal{L}^{\otimes r})$ whose vanishing locus is disjoint from T. For this, choose a prime p that does not divide the characteristics of $k(\xi)$ for any $\xi \in T$, and fix $[X/G] \cong M_{\mathbb{Z}[1/p]}$ as in the statement of the proposition. By [Liu02, Prop. 9.1.11] there is a section $\sigma \in H^0(X, \mathcal{L}|_X)$ whose vanishing locus is disjoint from the pre-image of T in X. If we write $G = \{g_1, \ldots, g_r\}$, then

$$g_1\sigma \otimes \cdots \otimes g_r\sigma \in H^0(X, \mathcal{L}^{\otimes r}|_X)^G = H^0(M_{\mathbb{Z}[1/p]}, \mathcal{L}^{\otimes r})$$

is a section whose vanishing locus in $M_{\mathbb{Z}[1/p]}$ is disjoint from T. Multiplying this section by a sufficiently large power of p provides us with the desired section $s \in H^0(M, \mathcal{L}^{\otimes r})$.

We apply the paragraph above with T equal to the image under π of the set of associated points[7] of M'. The Cartier divisor D of the resulting section $s \in H^0(M, \mathcal{L}^{\otimes r})$ then has the property that

$$D' \stackrel{\mathrm{def}}{=} D \times_M M'$$

is an effective Cartier divisor on M', and $\mathcal{L}^{\otimes r} \cong \mathcal{I}_D^{-1}$. Recalling that the Chow group $\mathrm{CH}^{r+1}(M)$ has rational coefficients, it suffices to prove the stated equality after replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes r}$. Thus we may ease notation by assuming $r = 1$.

The left hand side of the desired equality is now just the cycle class $[D']$ associated to the finite map $D' \to M$ by Definition A.1.4, so is represented in $\mathrm{Gr}_\gamma^{r+1} K_0(M)_{\mathbb{Q}}$ by the class $[\pi_*\mathcal{O}_{D'}]$.

On the other hand, the right hand side is represented by

$$[\mathcal{O}_D \otimes^{\mathbb{L}}_{\mathcal{O}_M} \pi_*\mathcal{O}_{M'}] = \sum_{i \geqslant 0} (-1)^i \cdot [\underline{\mathrm{Tor}}_i^{\mathcal{O}_M}(\mathcal{O}_D, \pi_*\mathcal{O}_{M'})].$$

Using the resolution

$$0 \to \mathcal{I}_D \to \mathcal{O}_M \to \mathcal{O}_D \to 0$$

of $\mathcal{O}_D$ by vector bundles on M, the Tor sheaves in the sum can be computed by taking the homology of the complex

$$\cdots \to 0 \to \mathcal{I}_D \otimes_{\mathcal{O}_M} \pi_*\mathcal{O}_{M'} \xrightarrow{f} \pi_*\mathcal{O}_{M'} \to 0,$$

[7] An *associated point* of a stack Z is one that is the image of an associated prime of R for some étale map $\mathrm{Spec}(R) \to Z$.

where $f(a \otimes b) = ab$ is the multiplication map. Our assumption that D' is an effective Cartier divisor on M' guarantees that f is injective with image $\pi_* \mathcal{I}_{D'} \subset \pi_* \mathcal{O}_{M'}$. It follows that the $i = 0$ term in the sum is $[\pi_* \mathcal{O}_{D'}]$, while all terms with $i > 0$ vanish. □

A.4. **A generalized intersection pairing.** Throughout this subsection we assume that M is a regular stack. Our goal is to construct a refinement of the intersection pairing of Theorem A.2.6.

Analogously to the coniveau filtration (A.9) on $K^Z(M)_{\mathbb{Q}}$, for a finite morphism $Z \to M$ we define

$$F^d G_0(Z)_{\mathbb{Q}} = \bigcup_{\substack{Y \subset Z \\ \mathrm{codim}_M(Y) \geqslant d}} \mathrm{Image}\big(G_0(Y)_{\mathbb{Q}} \to G_0(Z)_{\mathbb{Q}}\big), \tag{A.13}$$

where the union is over all closed substacks $Y \subset Z$ whose image $\pi(Y) \subset M$ has codimension $\geqslant d$. This defines the *coniveau-in-M filtration* on $G_0(Z)_{\mathbb{Q}}$. Of course the filtration depends on the morphism $\pi : Z \to M$, but we suppress this from the notation as it will always be clear from context. It is clear that (A.8) restricts to a morphism

$$F^d G_0(Z)_{\mathbb{Q}} \to F^d K_0^Z(M)_{\mathbb{Q}}. \tag{A.14}$$

Suppose we are given finite morphisms

$$\begin{array}{ccccc} Z_1 & & & & Z_2 \\ & \searrow^{\pi_1} & & \swarrow^{\pi_2} & \\ & & M. & & \end{array}$$

The natural map $\pi : Z_1 \times_M Z_2 \to M$ is also finite, hence affine, and so

$$Z_1 \times_M Z_2 \cong \underline{\mathrm{Spec}}_{\mathcal{O}_M}(\pi_* \mathcal{O}_{Z_1 \times_M Z_2}). \tag{A.15}$$

Given coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on Z_1 and Z_2, respectively, we can form, for every $\ell \geqslant 0$, the coherent sheaf

$$\underline{\mathrm{Tor}}_\ell^{\mathcal{O}_M}(\pi_{1*}\mathcal{F}_1, \pi_{2*}\mathcal{F}_2) \tag{A.16}$$

on M. As the formation of Tor is functorial in both variables, (A.16) carries an action of the $\mathcal{O}_M$-algebra

$$\pi_* \mathcal{O}_{Z_1 \times_M Z_2} \cong \pi_{1*}\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_M} \pi_{2*}\mathcal{O}_{Z_2},$$

which determines a lift of (A.16) to a coherent sheaf on (A.15). This lift then determines a class

$$[\underline{\mathrm{Tor}}_\ell^{\mathcal{O}_M}(\pi_{1*}\mathcal{F}_1, \pi_{2*}\mathcal{F}_2)] \in G_0^{\mathrm{naive}}(Z_1 \times_M Z_2)_{\mathbb{Q}}$$

in the naive G-theory group of Remark A.2.1. In this way we obtain a bilinear pairing

$$G_0^{\mathrm{naive}}(Z_1)_{\mathbb{Q}} \otimes G_0^{\mathrm{naive}}(Z_2)_{\mathbb{Q}} \xrightarrow{\cap} G_0^{\mathrm{naive}}(Z_1 \times_M Z_2)_{\mathbb{Q}}$$

defined by

$$[\mathcal{F}_1] \cap [\mathcal{F}_2] = \sum_{\ell \geqslant 0} (-1)^\ell \cdot [\underline{\mathrm{Tor}}_\ell^{\mathcal{O}_M}(\pi_{1*}\mathcal{F}_1, \pi_{2*}\mathcal{F}_2)]. \tag{A.17}$$

Note that the sum on the right hand side is finite as M, being assumed regular, has finite Tor dimension.

Lemma A.4.1. *There is a unique bilinear pairing*

$$G_0(Z_1)_{\mathbb{Q}} \otimes G_0(Z_2)_{\mathbb{Q}} \xrightarrow{\cap} G_0(Z_1 \times_M Z_2)_{\mathbb{Q}} \tag{A.18}$$

making the diagram

$$\begin{array}{ccc} G_0^{\mathrm{naive}}(Z_1)_{\mathbb{Q}} \otimes G_0^{\mathrm{naive}}(Z_2)_{\mathbb{Q}} & \xrightarrow{\cap} & G_0^{\mathrm{naive}}(Z_1 \times_M Z_2)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(Z_1)_{\mathbb{Q}} \otimes G_0(Z_2)_{\mathbb{Q}} & \xrightarrow{\cap} & G_0(Z_1 \times_M Z_2)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \pi_{1*}\otimes\pi_{2*}} & & \downarrow{\scriptstyle \pi_*} \\ K_0^{Z_1}(M)_{\mathbb{Q}} \otimes K_0^{Z_2}(M) & \xrightarrow{\otimes} & K_0^{Z_1 \times_M Z_2}(M)_{\mathbb{Q}} \end{array}$$

commute, where the top vertical arrows are the surjections of Remark A.2.3, and the bottom vertical arrows are those of Proposition A.2.5.

Proof. In the case of schemes, so that $G_0 = G_0^{\mathrm{naive}}$, this is clear from the definitions.

For the stack case, recall that (A.7) is surjective. In particular $G_0(Z_1)_{\mathbb{Q}} \otimes G_0(Z_2)_{\mathbb{Q}}$ is generated by elements of the form $[\mathcal{F}_1] \otimes [\mathcal{F}_2]$ for coherent sheaves $\mathcal{F}_i$ on Z_i, so there can be at most one pairing (A.18) making the top square of the diagram commute.

The cleanest way to the prove existence of (A.18) involves a little bit of derived algebraic geometry. Namely, the derived tensor product $\pi_{1*}\mathcal{F}_1 \otimes^{\mathbb{L}}_{\mathcal{O}_M} \pi_{2*}\mathcal{F}_2$ gives a coherent sheaf on the derived affine scheme over M with underlying structure sheaf $\pi_{1*}\mathcal{O}_{Z_1} \otimes^{\mathbb{L}}_{\mathcal{O}_M} \pi_{2*}\mathcal{O}_{Z_2}$. The underlying classical scheme here is just $Z_1 \times_M Z_2$. Therefore, using [Kha22, Corollary 3.4], this actually gives a global section of the sheaf $\mathbf{G}_{Z_1 \times_M Z_2}$, which can be identified explicitly with the right hand side of (A.17). □

Remark A.4.2. It is natural to expect that (A.18) restricts to

$$F^{d_1} G_0(Z_1)_{\mathbb{Q}} \otimes F^{d_2} G_0(Z_2)_{\mathbb{Q}} \xrightarrow{?} F^{d_1+d_2} G_0(Z_1 \times_M Z_2)_{\mathbb{Q}}. \tag{A.19}$$

If π_1 and π_2 are closed immersions of schemes this is clear from Theorem A.2.6 and the final claim of Proposition A.2.5. In general, even if one assume that π_1 and π_2 are finite morphisms of schemes, we are unable to provide a proof. If one attempts to imitate the proof of the analogous claim in Theorem A.2.6, one is immediately obstructed by the lack of Adams operators in this context.

To give a concrete sense of why finite maps are more difficult to deal with than closed immersions, let $C_1, \ldots, C_r$ be the connected components of $Z_1 \times_M Z_2$. Given a class

$$[\mathcal{F}_1] \otimes [\mathcal{F}_2] \in F^{d_1} G_0(Z_1)_{\mathbb{Q}} \otimes F^{d_2} G_0(Z_2)_{\mathbb{Q}},$$

we may decompose

$$[\mathcal{F}_1] \cap [\mathcal{F}_2] = c_1 + \cdots + c_r \in \bigoplus_{j=1}^{r} G_0(C_j)_{\mathbb{Q}} = G_0(Z_1 \times_M Z_2)_{\mathbb{Q}}.$$

The image of the sum $c_1 + \cdots + c_r$ in $K_0^{Z_1 \times_M Z_2}(M)_{\mathbb{Q}}$ lies in the $d_1 + d_2$ part of the coniveau filtration by Theorem A.2.6 and the commutativity of the diagram in Lemma A.4.1, but if (A.19) holds then the image of each *individual* c_j in $K_0^{Z_1 \times_M Z_2}(M)_{\mathbb{Q}}$ must also lie in the $d_1 + d_2$ part of the coniveau filtration. Even this weaker property seems quite subtle. (Note that the images of $C_1, \ldots, C_r$ in M may no longer be disjoint, leading to cancellation among the terms in $c_1 + \cdots + c_r$ after pushforward to M.)

Remark A.4.3. One can define a coniveau-in-M filtration on $G_0^{\mathrm{naive}}(Z)_{\mathbb{Q}}$ in exactly the same way as (A.13), but it is dubious that one should expect the analogue of (A.19) to hold with this naive definition.

The following weaker version of (A.19) is enough for our applications.

Proposition A.4.4. *Suppose $Z_1 \to M$ and $Z_2 \to M$ are finite and unramified. For any $d \geqslant 0$, the pairing* (A.18) *restricts to*

$$F^d G_0(Z_1)_{\mathbb{Q}} \otimes F^1 G_0(Z_2)_{\mathbb{Q}} \xrightarrow{\cap} F^{d+1} G_0(Z_1 \times_M Z_2)_{\mathbb{Q}}.$$

Proof. Assume first that

$$\text{(A.20)} \qquad \mathrm{codim}_M(Z_1) \geqslant d, \quad \mathrm{codim}_M(Z_2) \geqslant 1.$$

If $\mathrm{codim}_M(Z_1 \times_M Z_2) \geqslant d+1$ then

$$F^{d+1} G_0(Z_1 \times_M Z_2) = G_0(Z_1 \times_M Z_2),$$

and there is nothing to prove. Thus we assume further that

$$d = \mathrm{codim}_M(Z_1) = \mathrm{codim}_M(Z_1 \times_M Z_2).$$

Lemma A.4.5. *Suppose $C \subset Z_1 \times_M Z_2$ is an irreducible component with* $\mathrm{codim}_M(C) = d$*, and with generic point η. For any pair of classes $(z_1, z_2) \in G_0(Z_1)_{\mathbb{Q}} \times G_0(Z_2)_{\mathbb{Q}}$, there is a Zariski open substack $U \subset Z_1 \times_M Z_2$ containing η for which*

$$z_1 \cap z_2 \in \ker\big(G_0(Z_1 \times_M Z_2) \to G_0(U)\big).$$

Proof. Let $\bar\eta \to Z_1 \times_M Z_2$ be a geometric point above η, and consider the commutative diagram of étale local rings

$$\begin{array}{ccccc} & & \mathcal{O}^{\mathrm{et}}_{Z_1\times_M Z_2,\bar\eta} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{O}^{\mathrm{et}}_{Z_1,\bar\eta} & & & & \mathcal{O}^{\mathrm{et}}_{Z_2,\bar\eta} \\ & \nwarrow & & \nearrow & \\ & & \mathcal{O}^{\mathrm{et}}_{M,\bar\eta} & & \end{array} \tag{A.21}$$

at $\bar\eta$. As both $Z_1 \to M$ and $Z_2 \to M$ are finite and unramified, all of the morphisms in (A.21) are surjective. For any one of these local rings R, we abbreviate $G_0(R) = G_0(\mathrm{Spec}(R))$ for the Grothendieck group of finitely generated R-modules. If $R \to S$ is any one of the four arrows in the above diagram, we similarly abbreviate

$$K_0^S(R) = K_0^{\mathrm{Spec}(S)}(\mathrm{Spec}(R)).$$

When $R = \mathcal{O}^{\mathrm{et}}_{M,\bar\eta}$, Proposition A.2.5 provides a canonical isomorphism

$$G_0(S)_{\mathbb{Q}} \cong K_0^S(R)_{\mathbb{Q}}.$$

Consider the commutative diagram

$$\begin{array}{ccc} G_0(Z_1)_{\mathbb{Q}} \otimes G_0(Z_2)_{\mathbb{Q}} & \xrightarrow{\cap} & G_0(Z_1 \times_M Z_2)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(\mathcal{O}^{\mathrm{et}}_{Z_1,\bar\eta})_{\mathbb{Q}} \otimes G_0(\mathcal{O}^{\mathrm{et}}_{Z_2,\bar\eta})_{\mathbb{Q}} & \longrightarrow & G_0(\mathcal{O}^{\mathrm{et}}_{Z_1\times_M Z_2,\bar\eta})_{\mathbb{Q}} \\ \cong\downarrow & & \cong\downarrow \\ K_0^{\mathcal{O}^{\mathrm{et}}_{Z_1,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}} \otimes K_0^{\mathcal{O}^{\mathrm{et}}_{Z_2,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}} & \longrightarrow & K_0^{\mathcal{O}^{\mathrm{et}}_{Z_1\times_M Z_2,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}} \end{array}$$

in which the middle arrow is defined in exactly the same way as the top pairing, and the bottom pairing is that of Theorem A.2.6.

The bottom pairing is multiplicative with respect to the coniveau filtration, but

$$F^d K_0^{\mathcal{O}^{\mathrm{et}}_{Z_1,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}} = K_0^{\mathcal{O}^{\mathrm{et}}_{Z_1,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}}$$
$$F^1 K_0^{\mathcal{O}^{\mathrm{et}}_{Z_2,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}} = K_0^{\mathcal{O}^{\mathrm{et}}_{Z_2,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}}$$

and, as $\dim(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta}) = d$ by hypothesis,

$$F^{d+1} K_0^{\mathcal{O}^{\mathrm{et}}_{Z_1\times_M Z_2,\bar\eta}}(\mathcal{O}^{\mathrm{et}}_{M,\bar\eta})_{\mathbb{Q}} = 0.$$

Thus the bottom horizontal arrow is trivial, and hence so is the composition

$$G_0(Z_1)_{\mathbb{Q}} \otimes G_0(Z_2)_{\mathbb{Q}} \xrightarrow{\cap} G_0(Z_1 \times_M Z_2)_{\mathbb{Q}} \to K_0(\mathcal{O}^{\mathrm{et}}_{Z_1\times_M Z_2,\bar\eta})_{\mathbb{Q}}.$$

As $\mathcal{O}^{\mathrm{et}}_{Z_1\times_M Z_2,\bar{\eta}}$ is an Artinian local ring, by dévissage (see [Gil84, Lemma 7.3]) and Remark A.2.2, we have

$$K_0(\mathcal{O}^{\mathrm{et}}_{Z_1\times_M Z_2,\bar{\eta}})_{\mathbb{Q}} \cong K_0(k(\bar{\eta}))_{\mathbb{Q}} \cong K_0(\eta)_{\mathbb{Q}}.$$

Therefore the composition

$$G_0(Z_1)_{\mathbb{Q}} \otimes G_0(Z_2)_{\mathbb{Q}} \xrightarrow{\cap} G_0(Z_1 \times_M Z_2)_{\mathbb{Q}} \to G_0(\eta)_{\mathbb{Q}} \cong K_0(\eta)_{\mathbb{Q}}.$$

is also trivial.

To finish, we only need to observe that

$$\mathrm{colim}_{\eta\in U} G_0(U)_{\mathbb{Q}} \cong G_0(\eta)_{\mathbb{Q}},$$

where on the left hand side the colimit is over pullbacks of inclusions of open neighborhoods of η in $Z_1 \times_M Z_2$. This once again be checked on the level of sheaves of rational spectra, where it comes down to the fact that the exact category of coherent sheaves over a point of a scheme is equivalent to the colimit of the exact categories of coherent sheaves over a system of affine neighborhoods of the point; see for instance [Gro66, §8.5]. □

We can now complete the proof of Proposition A.4.4 under the assumption (A.20). By Lemma A.4.5 there exists a Zariski open substack $U \subset Z_1 \times_M Z_2$ such that

$$\mathrm{codim}_M((Z_1 \times_M Z_2) \smallsetminus U) \geqslant d+1,$$

and such that $z_1 \cap z_2$ lies in the kernel of the second arrow in

$$G_0((Z_1 \times_M Z_2) \smallsetminus U)_{\mathbb{Q}} \to G_0(Z_1 \times_M Z_2)_{\mathbb{Q}} \to G_0(U)_{\mathbb{Q}}.$$

This sequence is exact [Gil84, Lemma 7.4], and so

$$\begin{aligned} z_1 \cap z_2 &\in \mathrm{Image}\big(G_0((Z_1 \times_M Z_2) \smallsetminus U)_{\mathbb{Q}} \to G_0(Z_1 \times_M Z_2)_{\mathbb{Q}}\big) \\ &\subset F^{d+1} G_0(Z_1 \times_M Z_2)_{\mathbb{Q}}. \end{aligned}$$

We now reduce the general case to the case just proved. Suppose we are given classes

$$z_1 \in F^d G_0(Z_1)_{\mathbb{Q}}, \qquad z_2 \in F^1 G_0(Z_2)_{\mathbb{Q}}.$$

By definition of the coniveau-in-M filtration, there are closed substacks $Y_1 \subset Z_1$ and $Y_2 \subset Z_2$ such that

$$\mathrm{codim}_M(Y_1) \geqslant d, \qquad \mathrm{codim}_M(Y_2) \geqslant 1,$$

and $z_1 \otimes z_2$ lies in the image of the left vertical arrow in the commutative diagram

$$\begin{array}{ccc} F^d G_0(Y_1)_{\mathbb{Q}} \otimes F^1 G_0(Y_2)_{\mathbb{Q}} & & \\ \| & & \\ G_0(Y_1)_{\mathbb{Q}} \otimes G_0(Y_2)_{\mathbb{Q}} & \xrightarrow{\quad\cap\quad} & G_0(Y_1 \times_M Y_2)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ F^d G_0(Z_1)_{\mathbb{Q}} \otimes F^1 G_0(Z_2)_{\mathbb{Q}} & \xrightarrow{\quad\cap\quad} & G_0(Z_1 \times_M Z_2)_{\mathbb{Q}}. \end{array}$$

The special case of the proposition proved above shows that the top horizontal arrow takes values in $F^{d+1}G_0(Y_1 \times_M Y_2)_{\mathbb{Q}}$, and Proposition A.4.4 follows immediately. □

Appendix B. Quadratic lattices

This appendix contains some technical results on the existence of isometric embeddings of quadratic lattices.

B.1. **Embeddings of hyperbolic planes.** Let L be a quadratic lattice over $\mathbb{Z}$. That is to say, a free $\mathbb{Z}$-module of finite rank endowed with a $\mathbb{Z}$-valued quadratic form such that $L \otimes \mathbb{Q}$ is nondegenerate.

Lemma B.1.1. *Suppose $L^\sharp$ is an indefinite quadratic lattice such that*

(1) for every prime p there exists an isometric embedding

$$\alpha_p : L \otimes \mathbb{Z}_p \to L^\sharp \otimes \mathbb{Z}_p,$$

(2) $\mathrm{rank}_{\mathbb{Z}}(L^\sharp) \geqslant \mathrm{rank}_{\mathbb{Z}}(L) + 4$.

If there exists an isometric embedding $a : L \otimes \mathbb{Q} \to L^\sharp \otimes \mathbb{Q}$ such that

$$a(L \otimes \mathbb{Z}_p) = \alpha_p(L \otimes \mathbb{Z}_p) \tag{B.1}$$

for all but finitely many primes p, then a can be chosen so that (B.1) *holds for every prime p.*

Proof. As all embeddings $L\otimes\mathbb{Q}_p \to L^\sharp\otimes\mathbb{Q}_p$ lie in a single $\mathrm{SO}(L^\sharp\otimes\mathbb{Q}_p)$-orbit, there exists a $g \in \mathrm{SO}(L^\sharp \otimes \mathbb{A}_f)$ such that

$$g_p \cdot a(L \otimes \mathbb{Z}_p) = \alpha_p(L^\sharp \otimes \mathbb{Z}_p). \tag{B.2}$$

By assumption, the orthogonal complement

$$W \stackrel{\mathrm{def}}{=} a(L \otimes \mathbb{Q})^\perp \subset L^\sharp \otimes \mathbb{Q}$$

has dimension $\geqslant 4$. As a quadratic space over $\mathbb{Q}_p$ of dimension $\geqslant 4$ represents every element of $\mathbb{Q}_p^\times$, the spinor norm

$$\mathrm{SO}(W \otimes \mathbb{Q}_p) \to \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$$

is surjective. Multiplying g by a suitable element of $\mathrm{SO}(W\otimes\mathbb{A}_f) \subset \mathrm{SO}(L^\sharp\otimes \mathbb{A}_f)$, which does not change the relation (B.2), we may assume that g has trivial spinor norm and fix a lift to $g \in \mathrm{Spin}(L^\sharp \otimes \mathbb{A}_f)$.

Using strong approximation for the (simply connected) spin group we may replace this lift by a $g \in \mathrm{Spin}(L^\sharp \otimes \mathbb{Q})$ in such a way that (B.2) still holds, and the resulting embedding $ga : L \otimes \mathbb{Q} \to L^\sharp \otimes \mathbb{Q}$ has the desired properties. □

Let H be the hyperbolic plane over $\mathbb{Z}$. In other words, $H = \mathbb{Z}\ell \oplus \mathbb{Z}\ell_*$ where ℓ and ℓ_* are isotropic vectors with $[\ell, \ell_*] = 1$. The following result was used in the proof of Proposition 3.1.3.

Proposition B.1.2. *Let $\gamma \geqslant 0$ be the minimal number of elements needed to generate the finite abelian group $L^\vee/L$. If L is indefinite with*

$$\mathrm{rank}_{\mathbb{Z}}(L) \geqslant 2\gamma + 6,$$

then there exists an isometric embedding $H \to L$.

Proof. Using Lemma B.1.1 and the Hasse-Minkowski theorem, we are reduced to proving the existence of an isometric embedding $H \otimes \mathbb{Z}_p \to L \otimes \mathbb{Z}_p$ for every prime p.

Using the classification of quadratic lattices over $\mathbb{Z}_p$, one can find an orthogonal decomposition

$$L \otimes \mathbb{Z}_p \cong J_1 \oplus \cdots \oplus J_t$$

in such a way that each J_i has $\mathbb{Z}_p$-rank either 1 or 2. Each summand satisfies $J_i \subset J_i^\vee$, and we collect together into one self-dual $\mathbb{Z}_p$-quadratic space K those summands for which equality holds. This gives a decomposition

$$L \otimes \mathbb{Z}_p \cong J_1 \oplus \cdots \oplus J_s \oplus K$$

in such a way that $J_i \subsetneq J_i^\vee$ and $K = K^\vee$.

Equating the $\mathbb{Z}_p$-ranks of both sides shows that

$$\mathrm{rank}_{\mathbb{Z}}(L) = \mathrm{rank}_{\mathbb{Z}_p}(J_1 \oplus \cdots \oplus J_s) + \mathrm{rank}_{\mathbb{Z}_p}(K) \leqslant 2s + \mathrm{rank}_{\mathbb{Z}_p}(K).$$

On the other hand, the definition of γ implies the existence of a surjective $\mathbb{Z}_p$-module map

$$\mathbb{Z}_p^\gamma \to (L^\vee/L) \otimes \mathbb{Z}_p \cong \bigoplus_{i=1}^{s} J_i^\vee/J_i,$$

which in turn implies $s \leqslant \gamma$. Combining these gives the second inequality in

$$2\gamma + 6 \leqslant \mathrm{rank}_{\mathbb{Z}}(L) \leqslant 2\gamma + \mathrm{rank}_{\mathbb{Z}_p}(K),$$

and so $\mathrm{rank}_{\mathbb{Z}_p}(K) \geqslant 6$.

As every quadratic space over $\mathbb{Q}_p$ of dimension at least 5 contains an isotropic vector, there exists an isometric embedding

$$H \otimes \mathbb{Q}_p \to K \otimes \mathbb{Q}_p.$$

Certainly the image of $H \otimes \mathbb{Z}_p$ is contained in some maximal lattice (in the sense of Definition 2.2.1) in $K \otimes \mathbb{Q}_p$, and it is a theorem of Eichler that all maximal lattices in $K \otimes \mathbb{Q}_p$ are isometric. Thus $H \otimes \mathbb{Z}_p$ can be embedded isometrically into *any* maximal lattice in $K \otimes \mathbb{Q}_p$, including K itself (which is self-dual, hence maximal). In particular, $H \otimes \mathbb{Z}_p$ embeds isometrically into $L \otimes \mathbb{Z}_p$. □

B.2. **Embeddings into self-dual lattices.** As above, let H be the hyperbolic plane over $\mathbb{Z}$.

Lemma B.2.1. *If $r, s \in \mathbb{Z}_{\geqslant 0}$ satisfy $r \equiv s \pmod 8$, then there exists a quadratic space V over $\mathbb{Q}$ of signature (r,s) such that*

$$V \otimes \mathbb{Q}_p \cong (H \otimes \mathbb{Q}_p)^{\frac{r+s}{2}}$$

for every prime p.

Proof. This is an application of the classification of quadratic forms over $\mathbb{Q}$, as found in [Shi10, Theorem 28.9]. □

The following result is needed to make sense of Definition 3.1.1.

Proposition B.2.2. *Let L be a quadratic lattice over $\mathbb{Z}$ of signature (n,m), with $m > 0$. There exist an integer $r \geqslant 1$, a self-dual quadratic lattice $L^\sharp$ of signature $(n+r,m)$, and an isometric embedding $L \to L^\sharp$ identifying L with a $\mathbb{Z}$-module direct summand of $L^\sharp$.*

Proof. First, we claim that for every prime p and every quadratic lattice J over $\mathbb{Z}_p$ there is an isometric embedding

$$J \to (H \otimes \mathbb{Z}_p)^{\mathrm{rank}_{\mathbb{Z}_p}(J)}$$

realizing the source as a $\mathbb{Z}_p$-module direct summand of the target. As in the proof of Proposition B.1.2, one can write J as an orthogonal direct sum of quadratic lattices of rank $\leqslant 2$, so we may assume that $\mathrm{rank}_{\mathbb{Z}_p}(J) \leqslant 2$. For rank 1 lattices, this just amounts to the fact that, for every $m \in \mathbb{Z}_p$, there exists a basis $v, w \in H \otimes \mathbb{Z}_p$ with $Q(v) = m$. If $\mathrm{rank}_{\mathbb{Z}+p}(J) = 2$ and J is diagonalizable (which is always the case if $p > 2$), we are immediately reduced to the rank one case. This leaves us with the case where $p = 2$ and J is non-diagonalizable of rank 2. In this case there is a basis $v, w \in J$ such that

$$Q(v) = 2^k a, \quad Q(w) = 2^k b, \quad [v,w] = 2^k c$$

for some $a, b, c \in \mathbb{Z}_2^\times$ and $k \geqslant 0$. Suppose that e_1, f_1, e_2, f_2 is a standard hyperbolic basis for $(H \otimes \mathbb{Z}_2)^2$, and set

$$v' = e_1 + 2^k a f_1 \quad \text{and} \quad w' = a^{-1} c e_1 + e_2 + 2^k b f_2.$$

One can easily check that $v \mapsto v'$ and $w \mapsto w'$ defines an isometric embedding $J \to (H \otimes \mathbb{Z}_2)^2$ onto a direct summand.

By the paragraph above, for every prime p and every $r \geqslant 0$, there exists an isometric embedding

$$L \otimes \mathbb{Z}_p \to (H \otimes \mathbb{Z}_p)^{m+n+r} \tag{B.3}$$

realizing the source as a $\mathbb{Z}_p$-module direct summand of the target. Choosing $r \geqslant 4$ so that $n + r \equiv m \pmod 8$, Lemma B.2.1 allows us to choose a quadratic space $V^\sharp$ over $\mathbb{Q}$ of signature $(n+r,m)$ such that

$$V^\sharp \otimes \mathbb{Q}_p \cong (H \otimes \mathbb{Q}_p)^{m+n+r}$$

for every prime p. By the Hasse-Minkowski theorem, there exists an isometric embedding $a : L \otimes \mathbb{Q} \to V^\sharp$.

Let $L^\sharp \subset V^\sharp$ be any maximal lattice containing $a(L)$. By Eichler's theorem that all maximal lattices in a $\mathbb{Q}_p$-quadratic space are isometric,

$$L^\sharp \otimes \mathbb{Z}_p \cong (H \otimes \mathbb{Z}_p)^{m+n+r}$$

for every prime p. In particular, $L^\sharp$ is self-dual. For all but finitely many primes p, the embedding

$$\alpha_p : L \otimes \mathbb{Z}_p \to L^\sharp \otimes \mathbb{Z}_p$$

induced by a realizes the source as a $\mathbb{Z}_p$-module direct summand of the target. For the remaining primes, we take α_p to be the composition

$$L \otimes \mathbb{Z}_p \xrightarrow{\text{(B.3)}} (H \otimes \mathbb{Z}_p)^{m+n+r} \cong L^\sharp \otimes \mathbb{Z}_p.$$

By Lemma B.1.1, there exists an isometric embedding $b : L \to L^\sharp$ such that $b(L \otimes \mathbb{Z}_p) = \alpha_p(L \otimes \mathbb{Z}_p)$ for all p, from which it follows that $b(L)$ is a $\mathbb{Z}$-module direct summand of $L^\sharp$. $\square$

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