LOG p-DIVISIBLE GROUPS (*D'APRÉS KATO*)

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CONTENTS

1. FINE SATURATED LOG SCHEMES

We review here the theory of fs log schemes. The main references are [9], [10] and [17]. Since [10] is unpublished, we either give full proofs or give precise references to proofs in [17] of the results we need.

1.1. Preliminaries.

1.1.1. A **pre-log structure** on a scheme X is an étale sheaf of commutative monoids M over X equipped with a map $\alpha : M \to \mathscr{O}_X$ of sheaves of monoids. A **log structure** is a pre-log structure where α induces an isomorphism $\alpha^{-1}(\mathscr{O}_X^\times) \cong \mathscr{O}_X^\times$. The forgetful functor from the category of log structures to that of pre-log structures has a left adjoint that takes a pre-log structure (M, α) to its **associated log structure** (M^a, α^a).

Example 1.1.2. The main example to keep in mind is the log structure on $\operatorname{Spec} \mathbb{Z}[P]$ (where P is a monoid) induced by the pre-log structure $P \to \mathbb{Z}[P]$. In particular, we can consider $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t] = \operatorname{Spec} \mathbb{Z}[\mathbb{N}]$ with the log structure associated to the map $1 \mapsto t$ (in other words, the one associated to the origin).

Example 1.1.3. Another important example is the case where $X = \operatorname{Spec} \mathscr{O}$ with $\mathscr{O}$ a discrete valuation ring. For any choice of uniformizer π of $\mathscr{O}$, we can consider the log structure associated to the map $1 \mapsto \pi$, and we can show easily that this log structure is *independent* of the choice of uniformizer. We call this the **canonical log structure** of the discrete valuation ring $\mathscr{O}$.

A **log scheme** is a scheme X equipped with a log structure M. For any morphism $f : X \to Y$ of schemes, and any log structure N on Y, we denote by f^*N the log structure $(f^{-1}N)^a$ on X, and call it the **induced log structure** on X.

A morphism of $f : (X, M) \to (Y, N)$ of log schemes is a map $f : X \to Y$ of schemes equipped with a map $f^\sharp : f^{-1}N \to M$ making the following diagram commute:

$$\begin{tikzcd} f^{-1}N \arrow[r] \arrow[d] & f^{-1}\mathscr{O}_Y \arrow[d] \\ M \arrow[r] & \mathscr{O}_X . \end{tikzcd}$$

It is **strict** if the induced map $f^*N \to M$ is an isomorphism.

Note on Notation 1.1.4. From here on, we will consistently suppress the log structure, and conflate a log scheme (X, M) with the underlying scheme X, while calling its log structure M_X.

1.1.5. Before we proceed further, let us record some definitions related to the structure of monoids (always assumed commutative). The forgetful functor from the category of abelian groups to commutative monoids has a left adjoint, which we denote by $P \mapsto P^{\mathrm{gp}}$.

A sub-monoid $Q \subset P$ of a monoid is **saturated** if, for every $a \in P$ such that $a^n \in Q$, for some $n \geq 1$, a is already in Q.

A monoid P is **integral** if the canonical map $P \to P^{\mathrm{gp}}$ is injective; it is **saturated** if its image in P^{gp} is saturated in P^{gp}.

The forgetful functor from the category of integral monoids (resp. saturated, integral monoids) to the category of monoids (resp. integral monoids) has a left adjoint given by:

$$P^{\mathrm{int}} = \mathrm{im}(P \to P^{\mathrm{gp}});$$
$$(\text{resp. } P^{\mathrm{sat}} = \{x \in P^{\mathrm{gp}} : x^n \in P\}).$$

A finitely generated, integral, saturated monoid is called an **fs monoid** (short for fine, saturated). Such a monoid is **sharp** if the only invertible element in P is 1.

We will need one more definition from the world of monoids: A morphism $f : P \to Q$ of fs monoids is **Kummer** if it is injective, and if, for every $q \in Q$, there is $n \in \mathbb{N}$ such that q^n is in the image of P. Any Kummer map $f : P \to Q$ between fs monoids is automatically **exact**, that is, $(f^{\mathrm{gp}})^{-1}(Q) = P$.

Finally, we conclude with a couple of lemmas that will be used later.

Lemma 1.1.6. *Let $f : Q \to P$ be a surjection of integral monoids such that $\ker f^{\mathrm{gp}} \subset Q$. If $P^{\mathrm{gp}} \cong \mathbb{Z}^r$, for some $r \geq 0$, then f has a section.*

Proof. For (1), take any section $s : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ (which exists, since P^{gp} is free). Pick $p \in P$, and consider the image $s(p) \in Q^{\mathrm{gp}}$. Let $q \in Q$ be any element such that $f(q) = p$; then $s(p)q^{-1} \in \ker f^{\mathrm{gp}} \subset Q$, and so $s(p) \in Q$, which shows that $s(P) \subset Q$. □

Note on Notation 1.1.7. Let us make the following convention: for any monoid P, and $n \geq 1$, $P^{1/n}$ will be the monoid equipped with a map $P \to P^{1/n}$ isomorphic to $P \xrightarrow{\uparrow n} P$.

Lemma 1.1.8. *Let $f : P \to Q$ be a map of fs monoids, with P torsion-free. For every n, let Q_n be the fs monoid that makes the following diagram co-cartesian in the category of fs monoids:*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & Q \arrow[d] \\ P^{1/n} \arrow[r] & Q_n. \end{tikzcd}$$

Then f is Kummer if and only if the following condition holds: there exists an $n \geq 1$, such that the natural map $P^{1/n} \to Q_n$ induces an isomorphism of $P^{1/n}$ onto $Q_n/(Q_n)^{\mathrm{tor}}$.

Proof. The purported condition for Kummerness can be rephrased as follows: there exists an $n \geq 1$, such that, for every map $\psi : P \to M$ of fs monoids with M torsion-free, there is a unique map $\varphi : Q \to M$ such that $\varphi(f(p)) = \psi(p)^n$

First, suppose this condition holds, and consider the identity $e : P \to P$: to this there is associated a unique map $\varphi : Q \to P$ such that $\varphi(f(p)) = p^n$. In particular, f is injective, since $f(p) = f(p')$ implies $p^n = \varphi(f(p)) = \varphi(f(p')) = p'^n$, and P is torsion-free.

Let $\pi : Q \to Q/Q^{\mathrm{tor}}$ be the natural projection. Consider the map $\alpha = \pi \circ f : P \to Q/Q^{\mathrm{tor}}$. The unique map $\beta : Q \to Q/Q^{\mathrm{tor}}$ such that $\beta(f(p)) = \alpha(p)^n$ is simply $\beta : q \mapsto \pi(q)^n$. On the other hand, the map

$\beta' = \pi \circ \varphi : Q \to Q/Q^{\mathrm{tor}}$ also satisfies the condition $\beta'(f(p)) = \alpha(p)^n$. So we have the following commutative diagram:

$$\begin{tikzcd} Q \arrow[r, "\uparrow n"] \arrow[d, "\varphi"'] & Q \arrow[d, "\pi"] \\ P \arrow[r, "\alpha"] & Q/Q^{\mathrm{tor}} \end{tikzcd}$$

Let $m \in \mathbb{N}$ be a multiple of n such that Q^{tor} is killed by m. Then we see that the map $q \mapsto q^m$ will have to factor through f. In other words, f is Kummer.

Next, suppose f is Kummer; then there exists $n \geq 1$ such that, for all $q \in Q$, there is $p \in P$ such that $q^n = f(p)$. Let $\psi : P \to M$ be a map of torsion-free fs monoids. For $q \in Q$, let $\varphi(q) \in M$ be the unique element such that $\varphi(q)^n = \psi(p)^n$, where p is such that $f(p) = q^n$. It is easy to check that this is well-defined and gives us the map we need. □

1.1.9. We will work with fine, saturated log schemes X (fs log schemes for short): that is, log schemes, which étale locally have their log structures induced by a map (called a **chart**) $\alpha : P_X \to \mathscr{O}_X$, where P is an fs monoid, and P_X is the constant étale sheaf over X associated to P. Note that this is the same as saying that étale locally on X there is a map $X \to \operatorname{Spec} \mathbb{Z}[P]$ that induces the log structure on X. We will use the word 'chart' also to refer to this corresponding map.

Remark 1.1.10. Even though we allow P to be any fs monoid in the definition above, there will be no loss of generality in imposing the additional condition that P be torsion-free and sharp; that is, in requiring that P^{gp} be a free abelian group, and that the only invertible element in P be 1. Indeed, at any geometric point $\overline{x} \to X$ of an fs log scheme X, we find that $P = (M_X/\mathscr{O}_X^\times)_{\overline{x}}$ is a sharp and torsion-free fs monoid. To see this, let $\alpha : Q \to \mathscr{O}_X$ be any chart around $\overline{x}$; then we can identify P with $Q/\alpha^{-1}(\mathscr{O}_{X,\overline{x}}^\times)$, which is easily seen to have the claimed properties.

By (1.1.6), the natural surjection $M_{X,\overline{x}} \to P$ has a section, which, since P is finitely presented, gives us a map $P \to M_X|_U$, for a suitable étale neighborhood U of $\overline{x}$, that induces an isomorphism $P_U \cong (M_X/\mathscr{O}_X^\times)|_U$. Given this, we find that, for every geometric point $\overline{y} \to U$, the isomorphism $P \cong (M_X/\mathscr{O}_X^\times)_{\overline{y}}$ induces an isomorphism $P \oplus \mathscr{O}_{X,\overline{y}}^\times \cong M_{X,\overline{y}}$, which shows that the composition $P \to M_X|_U \to \mathscr{O}_U$ is a chart in a neighborhood of $\overline{x}$.

Here is a useful lemma that gives a criterion for when a morphism of log schemes is strict.

Lemma 1.1.11. *Let $f : X \to Y$ be a morphism of fs log schemes such that the underlying map of schemes is open. Then f is strict if and only if the map*

$$f^\sharp : f^{-1}(M_Y/\mathscr{O}_Y^\times) \to M_X/\mathscr{O}_X^\times$$

is an isomorphism.

Proof. Choose a geometric point $\overline{x} \to X$, and let $P \to \mathscr{O}_Y$ be a local chart for Y in a neighborhood of $f(\overline{x})$ inducing an isomorphism $P \xrightarrow{\cong} (M_Y/\mathscr{O}_Y^\times)_{f(\overline{x})}$.

We will show that the induced map $P \to f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ is a local chart for X around $\overline{x}$. By replacing X with an étale neighborhood of $\overline{x}$ over which $\beta : P \to M_X/\mathscr{O}_X^\times$ is an isomorphism, and Y by the image of this neighborhood (which is open, by hypothesis), we are reduced to showing that, if $P_X \to M$ is a morphism such that $P \to M_X/\mathscr{O}_X^\times$ is an isomorphism in an étale neighborhood of $\overline{x}$, then the log structure induced from P is isomorphic to M. This was done towards the end of the remark above. □

1.1.12. A **local chart** around a point $x \in X$ for a map $f : X \to Y$ is a triple $(\alpha : P \to \mathscr{O}_Y|_U, \beta : Q \to \mathscr{O}_X|_V, g : P \to Q)$, where U and V are étale neighborhoods of $f(x)$ and x, respectively; α (resp. β) is a chart for Y (resp. X) in a neighborhood of $f(x)$ (resp. x); and g is a morphism of monoids. Together this triple satisfies the condition that the following diagram commutes in an étale neighborhood of x:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "g"'] & f^{-1}M_Y|_V \arrow[d, "f^\sharp"'] \\ Q \arrow[r] & M_X|_V. \end{tikzcd}$$

Every map $f : X \to Y$ of fs log schemes has local charts. To see this, it suffices to consider the case where Y and X are spectra of strictly local rings $\mathscr{O}_Y$ and $\mathscr{O}_X$, respectively, with log structures given by charts $\alpha : P \to \mathscr{O}_Y$ and $\beta : Q \to \mathscr{O}_X$, where P and Q are sharp fs monoids, so that $M_Y = P \oplus \mathscr{O}_Y^\times$ and $M_X = Q \oplus \mathscr{O}_X^\times$. Let $Q' \subset M_X$ be the saturation of the sub-monoid of M_X generated by Q and the image of P in M_X under $f^\sharp : f^{-1}M_Y \to M_X$. Then it is easy to see that $(P \to \mathscr{O}_Y, Q' \to \mathscr{O}_X, P \to Q')$ is a chart for f.

A morphism $f : X \to Y$ of fs log schemes is **log flat** (resp. **log étale**) if, in fppf local (resp. étale local) charts, it is given by a triple $(P \to \mathscr{O}_Y, Q \to \mathscr{O}_X, g : P \to Q)$ such that the map $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is injective (resp. an injective map with finite co-kernel of order invertible on X), and such that the induced map $X \to Y \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[Q]$ of schemes is strict and classically flat (resp. étale). The morphism f is **Kummer** if, for every geometric point $\overline{x} \to X$, the induced map $\left(M_Y/\mathscr{O}_Y^\times\right)_{f(\overline{x})} \to \left(M_X/\mathscr{O}_X^\times\right)_{\overline{x}}$ is Kummer.

Example 1.1.13. Let k be a field and let $X = \mathbb{A}_k^2 = \operatorname{Spec} k[\mathbb{N}^2]$, $Y = \mathbb{A}_k^1 = \operatorname{Spec} k[\mathbb{N}]$ with the natural log structures. Then the map $X \to Y$ given by $(x, y) \mapsto xy$ induces a log flat morphism of log schemes. Similarly the map $Y \to Y$ given by $x \mapsto x^n$ always induces a Kummer map of log schemes and induces a log étale morphism of log schemes as long as n is prime to the characteristic of k.

Example 1.1.14. More pertinently, let $\mathscr{O}$ be a discrete valuation ring with uniformizer π, and let Y be $\operatorname{Spec}\mathscr{O}$ equipped with its canonical log structure. Let $X = \operatorname{Spec}\frac{\mathscr{O}[t]}{(t^n - \pi)}$, where n is invertible in $\mathscr{O}$, equipped with the log structure induced by the map $1 \mapsto t$. Then we see immediately that X is log étale over Y.

1.1.15. Let us say a word about fiber products in the category of fs log schemes. It is given étale locally by the following construction: suppose morphisms $X \to Y$ and $Y' \to Y$ are given by charts $\mathscr{O}_Y \leftarrow P \to Q \to \mathscr{O}_X$ and $\mathscr{O}_Y \leftarrow P \to P' \to \mathscr{O}_{Y'}$. Take $X' = X \times_Y Y'$, and let M be the push-out of the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \\ P' & \end{tikzcd}$$

Then X' has a chart $M \to \mathscr{O}_{X'}$. Let $M' = \left(M^{\mathrm{int}}\right)^{\mathrm{sat}}$. Then the base change of X along $Y' \to Y$ is $X'' = X' \times_{\operatorname{Spec}\mathbb{Z}[M]} \operatorname{Spec}\mathbb{Z}[M']$, endowed with the log structure given by the natural chart $X'' \to \operatorname{Spec}\mathbb{Z}[M']$.

Example 1.1.16. One should think of saturation as a sort of normalization, which makes the objects involved better behaved.

For example, there are more sheaves (for example, see $\mathbb{G}_m^{\log}$ below), and more torsors available to us in the world of fs log schemes. Consider the following situation from [10]: Let $\mathscr{O}$ be a discrete valuation ring with uniformizer π, let $X = \operatorname{Spec}\mathscr{O}$ be equipped with its canonical log structure, and let n be an integer invertible in $\mathscr{O}$. We denote by $\mathbb{N} \to \mathbb{N}^{1/n}$ the map $n : \mathbb{N} \to \mathbb{N}$.

Let X' be the scheme $X \times_{\operatorname{Spec}\mathbb{Z}[\mathbb{N}]} \operatorname{Spec}\mathbb{Z}[\mathbb{N}^{1/n}]$, endowed with its natural log structure, and let $\mathbb{Z}/n\mathbb{Z}(1)$ be the group scheme corresponding to the sheaf of n^{th} roots of unity. This has a natural X-linear action on X'.

We claim that the group action map

$$\mathbb{Z}/n\mathbb{Z}(1) \times X' \to X' \times_X X',$$

is an isomorphism in the category of fs log schemes over X', but not in the category of fine log schemes over X'. We will see later that this implies that X' is a $\mathbb{Z}/n\mathbb{Z}(1)$-torsor over X in the log flat (and in fact the log étale) topology.

To show the claim, let M be the monoid that makes the following diagram co-cartesian in the category of integral monoids:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "1 \mapsto n"] \arrow[d, "1 \mapsto n"'] & \mathbb{N} \arrow[d] \\ \mathbb{N} \arrow[r] & M. \end{tikzcd}$$

It is easy to check that $M^{\mathrm{gp}} = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, inside which M sits as the sub-monoid

$$\{(a,i) \in \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} : 0 \leq i \leq n-1, a \geq i\}.$$

From this description, it is also easy to see that $M^{\mathrm{sat}} = \mathbb{N} \oplus \mathbb{Z}/n\mathbb{Z}$. Giving a map from M to another monoid P is the same as giving two elements $p_1, p_2 \in P$ such that $p_1^n = p_2^n$.

Now one sees that $X' \times_X X'$ in the category of fine log schemes has for its underlying scheme $X \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[M]$ with log structure induced by the chart $X'' \to \operatorname{Spec} \mathbb{Z}[M]$. This is certainly not isomorphic to $\mathbb{Z}/n\mathbb{Z}(1) \times X'$, since it is never normal.

On the other hand, the scheme underlying the fiber product of X' with itself over X in the category of fs log schemes is $X'' = X' \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec}[\mathbb{N} \oplus \mathbb{Z}/n\mathbb{Z}]$. We work out with a little thought that this is exactly $\mathbb{Z}/n\mathbb{Z}(1) \times X'$. The log structure is given by the chart

$$\mathbb{N} \oplus \mathbb{Z}/n\mathbb{Z} \to \mathscr{O}_{X''}$$

Since $\mathbb{Z}/n\mathbb{Z}$ is torsion, it follows that this log structure on $\mathbb{Z}/n\mathbb{Z}(1) \times X'$ is the same as the one induced from X', and our claim follows.

Remark 1.1.17. More generally, if the log structure on a log scheme X is given by a chart $P \to \mathscr{O}_X$, $P \to Q$ is a Kummer morphism of monoids, and if $X' = X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[Q]$, then we find that $X' \times_X X'$ is $X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec}[Q \oplus Q^{\mathrm{gp}}/P^{\mathrm{gp}}]$, which is naturally isomorphic to $\operatorname{Spec} \mathbb{Z}[Q^{\mathrm{gp}}/P^{\mathrm{gp}}] \times X'$.

Note on Notation 1.1.18. Let us establish some more notation: let Y be an fs log scheme equipped with a global chart $P \to \mathscr{O}_Y$, where P is a sharp fs monoid. For every $n \geq 1$, we set $Y_n = Y \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[P^{1/n}]$ with its log structure induced by the natural map $P^{1/n} \to \mathscr{O}_{Y_n}$.

Lemma 1.1.19. [17, Corollary 2.16]*Let Y be an fs log scheme equipped with a global chart $P \to \mathscr{O}_Y$, where P is a sharp fs monoid. If $X \to Y$ is log flat and Kummer, there exists $n \geq 1$ and a Kummer log flat cover $V \to X$ such that $V \times_Y Y_n \to Y_n$ is classically flat.*

Here is a nice criterion for Kummerness:

Proposition 1.1.20. *A morphism $f : X \to Y$ of fs log schemes is Kummer if and only if there is a log flat Kummer cover $Y' \to Y$ such that $X \times_Y Y' \to Y'$ is strict.*

Proof. Since Kummerness is local on X, there is no harm in assuming that X is quasi-compact. Working étale locally on Y, we can assume we have a chart $P \to \mathscr{O}_Y$ for Y. Suppose $Y' \to Y$ is a log flat Kummer cover such that $X \times_Y Y' \to Y'$ is strict. Let $V \to Y'$ be the Kummer log flat cover given to us by (1.1.19) such that $V \times_Y Y_n \to Y_n$ is classically flat, for some $n \geq 1$.

Then $X \times_Y (V \times_Y Y_n) \to V \times_Y Y_n$ is strict, and since $V \times_Y Y_n \to Y_n$ is strict, so is $X \times_Y Y_n \to Y_n$.

So it suffices to show: $f : X \to Y$ is Kummer if and only if there is $n \geq 1$ such that $X \times_Y Y_n \to Y_n$ is strict. This is clear from (1.1.8) once we note that, for any $x \in X$, we can find a chart $(P \to \mathscr{O}_Y, Q \to \mathscr{O}_X, g : P \to Q)$ for f in an étale neighborhood V of x such that f is Kummer on V if and only if g is Kummer. Indeed, let $Q' \to \mathscr{O}_X$ be any sharp, fs chart in an étale neighborhood V of x so that the induced maps $Q' \to (M/\mathscr{O}_X^{\times})_{\bar{y}}$ are isomorphisms for all geometric points $\bar{y} \to V$. Now take Q to be the saturated sub-monoid of M_X generated by the images of P and Q', and let $g : P \to Q$ be the obvious map. □

Remark 1.1.21. The proof shows that when f is log étale we can choose $Y' \to Y$ to also be log étale. So we can think of this result as a log version of Abhyankar's lemma.

1.2. **Certain Grothendieck Topologies.**

1.2.1. The **log flat site** $X^{\log}_{\mathrm{fl}}$ (resp. the **log étale site** $X^{\log}_{\mathrm{et}}$) over a fs log scheme X is the site whose underlying category consists of fs log schemes over X (resp. fs log schemes over X whose structure morphism is log étale), and whose coverings are (set-theoretically) surjective families of Kummer log flat (resp. Kummer log étale) morphisms, whose underlying maps of schemes are locally of finite presentation. That these definitions actually satisfy the axioms for a Grothendieck topology is shown in [16] and [17].

The classical flat (resp. étale) sites over the underlying scheme of X will be denoted by $X^{\mathrm{cl}}_{\mathrm{fl}}$ (resp. $X^{\mathrm{cl}}_{\mathrm{et}}$). Note that there is a natural map of sites $\varepsilon : X^{\log}_{\mathrm{fl}} \to X^{\mathrm{cl}}_{\mathrm{fl}}$ induced by the functor that takes a classical scheme Y over X and produces the log scheme with underlying scheme Y and the log structure induced by its structure morphism.

Let us note an important result about the openness of log flat morphisms.

Proposition 1.2.2. [10, Proposition 2.5] *Let $f : X \to Y$ be a Kummer log flat morphism whose underlying map of schemes is of finite presentation. Then f is an open map.*

Proof. By the openness of fppf morphisms, it suffices to show openness in the following situation: the log structure on Y is induced by a chart $P \to \mathscr{O}_Y$ and that on X is induced by an isomorphism $X \xrightarrow{\cong} Y \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[Q]$, where $P \to Q$ is a Kummer map. In this case, as we noted in (1.1.17), the natural group action map

$$m : \operatorname{Spec}\mathbb{Z}[Q^{\mathrm{gp}}/P^{\mathrm{gp}}] \times X \to X,$$

makes X into a G-torsor over Y, where G is the Cartier dual of the finite constant group scheme $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$.

We have the following cartesian diagram:

$$\begin{tikzcd} G \times X \arrow[r, "m"] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\pi : G \times X \to X$ is the projection onto X.

Since $X \to Y$ has finite underlying map of schemes, it is closed and surjective and is thus a quotient map of the underlying topological spaces. Let $U \subset X$ be an open subset; to show that $f(U)$ is open in Y, it suffices to show that $f^{-1}(f(U))$ is open. But $f^{-1}(f(U))$ is the union of the orbits of U under the action of G, and is thus open. $\square$

1.2.3. The pre-sheaves $Y \mapsto \Gamma(Y, \mathscr{O}_X)$ and $Y \mapsto \Gamma(Y, M_Y)$ are sheaves over the log flat site (and hence over the log étale site). In fact, they are representable, by $\mathbb{A}^1_X$ with the trivial log structure, and $\mathbb{A}^1_X$ with the natural log structure given by the identification $\mathbb{A}^1_X = X \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[\mathbb{N}]$, respectively. It is shown in [17, Theorem 2.20] (following [10]) that the canonical topology on the category of log schemes over X is finer than the log flat topology, and so every representable pre-sheaf is in fact a sheaf. Moreover, from this it follows that the sheaf M_X on $X^{\mathrm{cl}}_{\mathrm{et}}$ extends naturally to a sheaf on $X^{\mathrm{cl}}_{\mathrm{fl}}$ as the direct image under ε of the sheaf $Y \mapsto \Gamma(Y, M_Y)$ (strictly speaking, this fact is used in the proof of the cited result); we will denote this again by M_X.

1.2.4. We will use the symbol $\mathbb{G}^{\log}_m$ for the pre-sheaf on $X^{\log}_{\mathrm{fl}}$ given by the assignment $Y \mapsto \Gamma(Y, M^{\mathrm{gp}}_Y)$. That this is a sheaf follows from the fact that, classically étale locally, if M is associated to a chart $P \to \mathscr{O}_X$, then

$$\Gamma(Y, M^{\mathrm{gp}}_Y) = \lim_{a \in P} \Gamma(Y, a^{-1}M_Y).$$

Associated to this sheaf of groups, we have the **log Kummer sequence**:

$$0 \to \mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{G}^{\log}_m \xrightarrow{\uparrow n} \mathbb{G}^{\log}_m \to 0,$$

which is exact over the log flat (and in fact, the log étale site, when n is invertible on X). Taking the long exact sequence of cohomology associated to this short exact sequence, we get Kummer boundary maps

$$H^0(X^{\log}_{\mathrm{fl}}, \mathbb{G}^{\log}_m) \to H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1)).$$

Finally, we will denote the direct image $\varepsilon_*\mathbb{G}^{\log}_m$ on $X^{\mathrm{cl}}_{\mathrm{fl}}$ also by $\mathbb{G}^{\log}_m$.

1.3. **Descent of Objects.** Here we reproduce Kato's study of descent in the classical fppf topologies over an fs log scheme X. In general, descent of log objects does not work very well in the log flat topology. An exception, as we will see, is found in the case of finite log étale schemes.

For the rest of the section, we fix an fs log scheme X.

Theorem 1.3.1. [10, Theorem 8.1] *Let F be a sheaf on $X^{\mathrm{cl}}_{\mathrm{fl}}$, and suppose F is, classical fppf locally on X, representable by a Kummer fs log scheme, whose underlying structure morphism of schemes is affine. Then F is representable by a Kummer fs log scheme over X.*

Proof. By the classical theory of descent, if $Y \to X$ is an fppf covering, so that $F|_Y$ is representable by an affine, Kummer log scheme $S \to Y$, then we can descend S to an affine X-scheme T. It remains only to descend the log structure on S to T. We sketch a proof of this descent.

Consider the T-schemes $\pi : S \to T$ and $\pi' : S' = S \times_T S \to T$: both of these are equipped with natural log structures so that S represents $F|_Y$ and S' represents $F|_{Y\times_X Y}$. Define M_T to be the equalizer of the diagram $(\pi_* M_S \rightrightarrows \pi'_* M_{S'})$, where the two arrows are given by the two choices of projection from S' to S. M_T is the log structure that we seek on T.

To see that this works, it is enough to show, by (1.1.11), that $\pi^{-1}(M_T/\mathscr{O}_T^\times) \to M_S/\mathscr{O}_S^\times$ is an isomorphism (note that $S \to T$ is fppf and hence open). For this, it will suffice to prove the following two assertions, where N is the equalizer of the diagram $\pi_*(M_S/\mathscr{O}_S^\times) \rightrightarrows \pi'_*(M_{S'}/\mathscr{O}_{S'}^\times)$:

(1) $f^{-1}(N) \cong M_S/\mathscr{O}_S^\times$.
(2) $M_T/\mathscr{O}_T \cong N$.

For (1), let $\overline{s} \to S$ be a geometric point and choose $a \in (M_S/\mathscr{O}_S^\times)_{\overline{s}}$. We want to show that a comes from an element in $N_{\pi(\overline{s})}$. This amounts to finding an étale neighborhood U of $\pi(\overline{s})$, and a section $a' \in \Gamma(\pi^{-1}(U), M_S/\mathscr{O}_S^\times)$ such that $\pi_1^* a' = \pi_2^* a'$, where $\pi_i : S' \to S$ is the projection onto the i^{th} factor, and such that a' agrees with a in $(M_S/\mathscr{O}_S^\times)_{\overline{s}}$.

Let $\overline{x} \to X$ be a geometric point under $\overline{s}$. Since $S \to X$ is Kummer, there is $n \geq 1$ such that a^n is the image of some $b \in (M_X/\mathscr{O}_X^\times)_{\overline{x}}$. Replace X with an étale neighborhood of $\overline{x}$ over which b lifts to a global section of $M_X/\mathscr{O}_X^\times$, and replace T, S, S' with the appropriate pre-images.

We claim that there is a section $a' \in \Gamma(S, M_S/\mathscr{O}_S^\times)$ such that $a'^n = b$. By the uniqueness of n^{th}-roots in $M_{S'}^{\mathrm{gp}}/\mathscr{O}_{S'}^\times$ (a torsion-free sheaf of groups), it follows that $\pi_1^* a' = \pi_2^* a'$.

So, to finish proving (1), it remains to show the existence of a'. For this, again, by the torsion-free nature of $M_S^{\mathrm{gp}}/\mathscr{O}_S^\times$, it suffices to show that, for every $u \in S$, there is an n^{th}-root of b in $(M_S/\mathscr{O}_S^\times)_{\overline{u}}$. For this, let $\overline{s'} \to S'$ be the geometric point $(\overline{s}, \overline{u})$. Then we get isomorphisms

$$(M_S/\mathscr{O}_S^\times)_{\overline{s}} \xrightarrow{\cong} (M_{S'}/\mathscr{O}_{S'}^\times)_{\overline{s'}} \xleftarrow{\cong} (M_S/\mathscr{O}_S^\times)_{\overline{u}}$$

The image of a in $(M_S/\mathscr{O}_S^\times)_{\overline{u}}$ does the job.

For (2), the main point is showing that $M_T \to N$ is surjective. Note that, for any section h of $M_S/\mathscr{O}_S^\times$, the pre-image of h under $M_S \to M_S/\mathscr{O}_S^\times$ is a $\mathbb{G}_{m,S}$-torsor. For any such h that lies in N, the corresponding $\mathbb{G}_{m,S}$-torsor comes equipped with descent data, and thus descends to a $\mathbb{G}_{m,T}$-torsor. This has a section over an étale cover of T, which means that there is a section of M_T over an étale cover of T that maps to h. $\square$

1.3.2. Now, we investigate more thoroughly the Kummer map

$$H^0(X^{\log}_{\mathrm{fl}}, \mathbb{G}_m^{\log}) \to H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1)).$$

This associates to every element $b \in \Gamma(X, M_X^{\mathrm{gp}})$, a $\mathbb{Z}/n\mathbb{Z}(1)$-torsor F_b over $X^{\log}_{\mathrm{fl}}$, and, for any fs log scheme T over X, we have $F_b(T) = \{a \in \Gamma(T, M_T^{\mathrm{gp}}) : a^n = b\}$. We claim that F_b is representable by a finite log flat scheme over X. By (1.3.1), it is enough to show this étale locally on X. So we can assume that there is a chart $P \to \mathscr{O}_X$ and an element $b' \in P$ that maps to $b \in \Gamma(X, M_X^{\mathrm{gp}})$. Let H be an infinite cyclic group with generator x, let L be the group $\frac{P^{\mathrm{gp}} \times H}{\langle b'^{-1} x^n \rangle}$, and let Q be the saturation of P in L. Let $Y = X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[Q]$ equipped with the log structure induced by the map $Q \to \mathscr{O}_Y$. Then it is easy to check that the map $h_Y \to F$ (where h_Y is the representable sheaf on $X^{\log}_{\mathrm{fl}}$ associated to Y) induced by $x \in F_b(Y)$ is an isomorphism of functors.

1.4. **The Log Fundamental Group.** Now, we will study the theory of the log fundamental group as exposed in [11].

1.4.1. We define a **log geometric point** $\overline{x}^{\log}$ of X to be a saturated log scheme $\overline{x}^{\log}$ equipped with a morphism $\overline{x}^{\log} \to X$ satisfying the following conditions:

(1) The scheme $\overline{x}$ underlying $\overline{x}^{\log}$ is the spectrum of a separably closed field.
(2) The group $\Gamma(\overline{x}, M^{\mathrm{gp}}_{\overline{x}^{\log}})$ is n-divisible for every n invertible in $\overline{x}$.

Lemma 1.4.2. [11, Lemma 4.3.3]*Let X be an fs monoid. For every point $x \to X$ we can find a log geometric point $\overline{x}^{\log}$ factoring through x, and satisfying the following properties:*

(1) *The scheme underlying $\overline{x}^{\log}$ is $\operatorname{Spec}\overline{k(x)}$, where $\overline{k(x)}$ is a separable closure of $k(x)$, the residue field at x.*
(2) *There is a natural isomorphism*
$$\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \left(M^{\mathrm{gp}}_{X,\overline{x}}/\mathscr{O}_{X,\overline{x}}\right) \cong M^{\mathrm{gp}}_{\overline{x}^{\log}}/\mathscr{O}^{\times}_{\overline{x}^{\log}},$$
where p is the residue characteristic at x.
(3) $\operatorname{Aut}(\overline{x}^{\log}/\overline{x})$ *is a pro-finite grou isomorphic to the projective limit*
$$\lim_{n} \operatorname{Hom}\left(M^{\mathrm{gp}}_{X,\overline{x}}/\mathscr{O}^{\times}_{X,\overline{x}}, \mathbb{Z}/n\mathbb{Z}(1)\right).$$

Proof. Suppose the log structure in a neighborhood of $\overline{x}$ is given by a chart $P \to \mathscr{O}_X$ that induces an isomorphism $P \to (M_X/\mathscr{O}_X^{\times})_{\overline{x}}$. Take Q to be the direct limit of monoids
$$\lim_{p \nmid n} P^{1/n},$$
where, $P^{1/n} = P$, and for $n|m$, the map from $P^{1/n} \to P^{1/m}$ is given my $a \mapsto a^{m/n}$ (if $p = 0$, we take the limit over all n). Then we find that Q is a saturated monoid, and that
$$Q^{\mathrm{gp}} = \lim_{p \nmid n} (P^{\mathrm{gp}})^{1/n} \cong \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} P.$$

Now, let $\overline{x}^{\log}$ be the log scheme whose underlying scheme is just $\operatorname{Spec} k(\overline{x})$ and whose log structure is induced from the trivial map $Q \to k(\overline{x})$ that takes everything to 1.

For (3), set $\overline{x}_n = \overline{x} \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{1/n}]$ equipped with the usual $P^{1/n}$-chart for its log structure. This, as we saw in, (1.1.17), is a $\operatorname{Hom}\big((P^{1/n})^{\mathrm{gp}}/P^{\mathrm{gp}}, \mathbb{Z}/n\mathbb{Z}(1)\big)$-torsor over $\overline{x}_n$. So (3) is simply the isomorphism of $\operatorname{Aut}(\overline{x}^{\log}/\overline{x})$ onto $\lim \operatorname{Aut}(\overline{x}_n/\overline{x})$. □

Let X be an fs log scheme. For any sheaf F over $X^{\log}_{\mathrm{et}}$, and any log geometric point $\overline{x}^{\log}$ of X, we define $F_{\overline{x}^{\log}}$ to be the direct limit
$$\lim_{\overline{x}^{\log} \to Y} F(Y)$$
over log étale covers Y of X through which $\overline{x}^{\log}$ factors.

Proposition 1.4.3. [11, Lemma 4.3.5] *Let the notation be as above.*

(1) *The functor $F \mapsto F_{\overline{x}^{\log}}$ defines a* point *of the topos of sheaves over $X^{\log}_{\mathrm{et}}$.*
(2) *A map of sheaves $F \to G$ over $X^{\log}_{\mathrm{et}}$ is an isomorphism if and only if the maps $F_{\overline{x}^{\log}} \to G_{\overline{x}^{\log}}$ are isomorphisms for all log geometric point $\overline{x}^{\log}$ of X.*

Proof. Both assertions are immediate once we know the following fact: for any log étale cover $Y \to X$ and any log geometric point $\overline{x}^{\log}$ of X, we have a factoring
$$\begin{tikzcd} \overline{x}^{\log} \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$
For this, it suffices, by (1.1.20), to consider the case, where, around $\overline{x}$ the log structure is given by a chart $P \to \mathscr{O}_X$, with $P \cong (M_X/\mathscr{O}_X)_{\overline{x}}$, and where $Y = X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{1/n}]$, for n prime to the residue characteristic at x. But here it is obvious: a map $\overline{x}^{\log} \to Y$ can be defined by the natural inclusion of monoids $P^{1/n} \to Q$ (using the notation from (1.4.2), over any closed point of Y lying over x. □

1.4.4. For any site T, let $\mathrm{LC}(T)$ be the category of locally constant sheaves on T with finite fibers. For an fs log scheme X, we denote by $\mathrm{Et}^{\log} X$ the category of finite log étale covers of X.

Theorem 1.4.5. [10, Theorem 10.2] *Let X be an fs log scheme whose underlying scheme is locally Noetherian.*

(1) *For any object Y of $\mathrm{Et}^{\log}(X)$, the corresponding representable sheaf on $X^{\log}_{\mathrm{et}}$ is locally constant.*
(2) *There are equivalences of categories:*
$$\mathrm{Et}^{\log}(X) \xrightarrow{\cong} \mathrm{LC}(X^{\log}_{\mathrm{et}}) \xrightarrow{\cong} \mathrm{LC}(X^{\log}_{\mathrm{fl}}).$$
(3) *If the underlying scheme of X is $\operatorname{Spec}\mathscr{O}$ with $\mathscr{O}$ a strictly henselian local ring, then all the three categories in (2) are equivalent to the category of finite $\mathrm{Aut}(\overline{x}^{\log}/x)$-sets, where $x \to X$ is the closed point, and $\overline{x}^{\log}$ is a log geometric point lying above it.*

Proof. (1) follows because there is a log étale cover $X' \to X$ so that $Y \times_X X' \to X'$ is strict (see remark following (1.1.20)), and is hence classical finite étale.

For (2), note that the first arrow is fully faithful by Yoneda's lemma, so it suffices to show its essential surjectivity.

By (1.3.1), it is enough to consider the case where the underlying scheme of X is the spectrum of a strict local ring. In other words, it is enough to prove (3).

Let $x \to X$ be the closed point of X, let $\overline{x}^{\log}$ be a log geometric point lying over x and put $G = \mathrm{Aut}(\overline{x}^{\log}/x)$. We will show the following things:

- The functor from $\mathrm{Et}^{\log}(X)$ to finite G-sets given by $Y \mapsto \mathrm{Hom}_X(\overline{x}^{\log}, Y)$ is essentially surjective.
- For every connected sheaf F (that is, a sheaf that cannot be expressed as the disjoint sum of two non-empty sheaves) in $\mathrm{LC}(X^{\log}_{\mathrm{et}})$, $F_{\overline{x}^{\log}}$ is a transitive G-set.
- $\mathrm{LC}(X^{\log}_{\mathrm{et}})$ is a Galois category with fiber functor $\Gamma : F \mapsto F_{\overline{x}^{\log}}$.

Given these assertions, most of (3) follows immediately from [7, Corollaire 6.10].

Assume now that the log structure on X is given by a chart $P \to \mathscr{O}_X$ with P a sharp, torsion-free fs monoid, so that $P \xrightarrow{\cong} M_{X,\overline{x}}/\mathscr{O}^{\times}_{X,\overline{x}}$.

To show the first assertion, take any open sub-group $H \subset G$, and set $Q_H \subset Q$ to be the monoid defined in the following fashion: identify G with the group $\mathrm{Hom}(M_{\overline{x}^{\log}}/\mathscr{O}^{\times}_{\overline{x}}, \mathbb{Z}_{(p)}/\mathbb{Z})$ using (1.4.2), and let Q_H be the saturation of P in the sub-group of $M_{\overline{x}^{\log}}/\mathscr{O}^{\times}_{\overline{x}}$ given by the intersection $\cap_{\sigma\in H}\ker\sigma$. Now, define $X_H = X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[Q_H]$; one checks easily that $X_H(\overline{x}^{\log})$ is isomorphic to
$$G/H = \mathrm{Hom}(Q^{\mathrm{gp}}_H/P^{\mathrm{gp}}, \mathbb{Z}_{(p)}/\mathbb{Z})$$
as a G-set.

The second assertion follows from the fact that, for every $F \in \mathrm{LC}(X^{\log}_{\mathrm{et}})$, there is $H \subset G$ open such that $F|_{X_H}$ is *constant*. Since $X_H \to X$ is a G/H-torsor, the category of sheaves over X that are constant with finite fibers when restricted to Y is equivalent to the category of G/H-sets, and so F is connected if and only if G acts transitively on $F(X_H)$, which is isomorphic to $F(\overline{x}^{\log})$ once we have chosen a lift $\overline{x}^{\log} \to X_H$.

The third assertion is pretty much a direct check from the definitions. The only non-trivial thing to check is the connectedness of the final object in $\mathrm{LC}(X^{\log}_{\mathrm{et}})$, which follows, just as in the classical situation of the étale fundamental group, from the openness of log étale morphisms (1.2.2).

To finish, suppose $p > 0$ is the residue characteristic of X. Let $X_p \to X$ be as usual, and let x_p be the closed point of X_p, with $\overline{x_p}^{\log}$ a log geometric point lying over x_p and $\overline{x}^{\log}$. To show that the second arrow in (2) is an equivalence it now suffices (taking into account (1.3.1)) to show that $\mathrm{Aut}(\overline{x_p}^{\log}/x_p) \xrightarrow{\cong} G$. But this follows from the simple fact that the map
$$\mathrm{Hom}((P^{1/p})^{\mathrm{gp}}, \mathbb{Z}/n\mathbb{Z}(1)) \to \mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{Z}/n\mathbb{Z}(1))$$
is a bijection, for any n prime to p. ∎

2. Log Finite Flat Group Schemes

For this section, unless otherwise noted, X will be a log scheme with underlying scheme $\operatorname{Spec}\mathscr{O}$, where $\mathscr{O}$ is a henselian local ring, with log structure induced by a chart $P \to \mathscr{O}$, where P is a sharp, torsion-free fs monoid, so that $P \to \left(M_X/\mathscr{O}_X^\times\right)_{\overline{x}}$ is an isomorphism (here, $\overline{x}$ is a geometric point lying over the closed point of $\operatorname{Spec}\mathscr{O}$), and $P^{\mathrm{gp}} \cong M^{\mathrm{gp}}_{X,\overline{x}}/\mathscr{O}^\times_{X,\overline{x}} \cong \mathbb{Z}^r$, for some $r \geq 0$.

The results of this section are from [12].

2.1. **Definitions.**

2.1.1. We introduce four categories of sheaves of abelian groups over $X^{\log}_{\mathrm{fl}}$ *d'aprés* Kato. First, we remark that the **Cartier dual** of a sheaf of abelian groups over $X^{\log}_{\mathrm{fl}}$ is the sheaf of groups $\underline{\mathrm{Hom}}_{X^{\log}_{\mathrm{fl}}}(G, \mathbb{G}_m)$, where the homomorphisms are taken in the category of sheaves of abelian groups over $X^{\log}_{\mathrm{fl}}$.

Definition 2.1.2. fin^c_X is the full sub-category of the category of sheaves of finite abelian groups over $X^{\log}_{\mathrm{fl}}$ consisting of objects which are representable by a classical finite flat group scheme over X; or, more precisely, by a log finite flat group scheme over X whose structure morphism to X is strict.

fin^f_X is the full sub-category of the category of sheaves of finite abelian groups over $X^{\log}_{\mathrm{fl}}$ consisting of objects which, over a log flat cover of X, are representable by a classical finite flat group scheme.

fin^r_X is the full sub-category of fin^f_X consisting of objects which are representable by an fs log scheme over X.

fin^d_X is the full sub-category of fin^r_X consisting of objects whose Cartier duals are also in fin^r_X.

Remark 2.1.3. We note:

(1) Every finite flat group scheme and, more generally, any representable presheaf of groups on $X^{\log}_{\mathrm{fl}}$ is in fact a sheaf by [17, Theorem 2.20].
(2) By the argument in (1.1.20) and classical fppf descent, every object in fin^f_X becomes classical representable over a *finite* log flat cover of X.
(3) Though it may be more natural, in the log setting, to define the Cartier dual as $\underline{\mathrm{Hom}}(G, \mathbb{G}^{\log}_m)$, the torsion-free nature of $\mathbb{G}^{\log}_m/\mathbb{G}_m$ ensures that such a definition would agree with ours for sheaves of finite groups.
(4) By the stability of classical finite flat group schemes under Cartier duality, fin^f_X is also stable under Cartier duality.

2.1.4. Now, we prove a series of basic lemmas (all due to Kato) about log finite flat group schemes:

Lemma 2.1.5. *Any object of* fin^r_X *is Kummer over* X.

Proof. This follows immediately from (1.1.20). ☐

Lemma 2.1.6. *Let* G *be an object in* fin^f_X*; then there is a unique short exact sequence*

$$0 \to G^o \to G \to G^e \to 0,$$

such that over any finite log flat covering $Y \to X$*, over which* $G|_Y \in \mathrm{fin}^c_Y$*, this short exact sequence restricts to the classical connected-étale sequence*

Proof. This is evident: take any finite log flat cover $Y \to X$ over which G is classical; then, since the formation of the classical connected-étale sequence commutes with base change, it follows that this sequence for $G|_Y$ will descend to one for G over X. Uniqueness follows from that of the classical sequence. ☐

Proposition 2.1.7. *Let* G *be an object in* fin^r_X *whose underlying scheme is connected; then* G *belongs to* fin^c_X.

Proof. From (1.1.11), it follows that G has the induced log structure if and only if the natural map $\pi^\sharp : \pi^{-1}\left(M_X/\mathscr{O}_X^\times\right) \to M_G/\mathscr{O}_G^\times$ is an isomorphism of étale sheaves over the underlying scheme of G, where $\pi : G \to X$ is the structure morphism.

Let $e : X \to G$ be the identity section; then, since G is connected, any étale neighborhood of $e(X)$ has to be a covering of G; so we will be done if we know that $\pi^\sharp$ is an isomorphism at every geometric point of $e(X)$. But, at any geometric point $\overline{x} \to G$ factoring through e, the identity on $\left(M_X/\mathscr{O}_X^\times\right)_{\pi(\overline{x})}$ factors as

$$\left(M_X/\mathscr{O}_X^\times\right)_{\pi(\overline{x})} \xrightarrow{\pi^\sharp} \left(M_G/\mathscr{O}_G^\times\right)_{\overline{x}} \xrightarrow{e^\sharp} \left(M_X/\mathscr{O}_X^\times\right)_{\pi(\overline{x})}.$$

Since

$$\left(M_X^{\mathrm{gp}}/\mathscr{O}_X^{\times}\right)_{\pi(\overline{x})} \cong \mathbb{Z}^r \cong \left(M_G^{\mathrm{gp}}/\mathscr{O}_G^{\times}\right)_{\overline{x}},$$

and any surjection of free $\mathbb{Z}$-modules is an isomorphism, the result we need is a consequence of (2.1.5) and the following

Claim. Let $f : P \to Q$ be a Kummer map of integral, saturated monoids, such that f^{gp} is an isomorphism. Then f is an isomorphism.

This is an immediate consequence of the fact that a Kummer map between integral, saturated monoids is exact. ☐

Corollary 2.1.8. *Let G be a object in fin_X^r killed by a power of a prime p and representable by a log étale scheme over X. Suppose p is not invertible on X; then G belongs to $\mathrm{fin}_X^{\mathrm{c}}$ if and only if it belongs to fin_X^d.*

Proof. The non-trivial direction is showing that, if G is log étale over X, is killed by a power of p, and belongs to fin_X^d, then G belongs to $\mathrm{fin}_X^{\mathrm{c}}$. For this, it suffices to show that the Cartier dual of G is a connected object in fin_X^r, for then, (2.1.7) will show the dual, and hence G itself, to be classical. But, since this dual is classical multiplicative of p-power order after log flat base change–and therefore connected after log flat base change–it must be connected to begin with. ☐

2.2. **The First Higher Direct Image.** We give here the most important technical result towards the classification of finite log flat group schemes: the computation of $R^1\varepsilon_*G$ for a classical finite flat group scheme G. We also consider some of its immediate applications.

2.2.1. For now, let X be any fs log scheme. Given any sheaf of groups G on $X_{\mathrm{fl}}^{\log}$, and any $n \in \mathbb{N}$, we have a natural map

$$\underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes \left(\mathbb{G}_m^{\log}/\mathbb{G}_m\right) \to R^1\varepsilon_*G \tag{2.2.1.1}$$

defined in the following fashion: First, consider the boundary map $\mathbb{G}_m^{\log} \to R^1\varepsilon_*(\mathbb{Z}/n\mathbb{Z}(1))$ obtained from the Kummer short exact sequence (1.2.4). Since raising to the n^{th} power is a surjective map of flat sheaves for $\mathbb{G}_m$, this boundary map induces a map

$$\delta_n : \mathbb{G}_m^{\log}/\mathbb{G}_m \to R^1\varepsilon_*(\mathbb{Z}/n\mathbb{Z}(1)).$$

Then the map in (2.2.1.2) is given by:

$$\begin{aligned} \underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes \left(\mathbb{G}_m^{\log}/\mathbb{G}_m\right) &\to R^1\varepsilon_*G \\ \varphi \otimes a &\mapsto R^1\varepsilon_*\varphi(\delta_n(a)). \end{aligned}$$

Putting these maps together for varying n, we get a natural map

$$\gamma_G : \lim_n \underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes \left(\mathbb{G}_m^{\log}/\mathbb{G}_m\right) \to R^1\varepsilon_*G, \tag{2.2.1.2}$$

where the limit on the left hand side is that of the inductive system induced by the surjections $\mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{Z}/m\mathbb{Z}(1)$, for $m|n$.

The main result is the following:

Theorem 2.2.2. [17, Theorem 3.12] *Suppose the underlying scheme of X is locally Noetherian. Let G be an object of $\mathrm{fin}_X^{\mathrm{c}}$. Then γ_G is an isomorphism.*

2.2.3. *Representability of Torsors.* Now, suppose the underlying scheme of X is $\mathrm{Spec}\,\mathscr{O}$ with $\mathscr{O}$ a *strictly* henselian ring, equipped with its canonical log structure, so that $\mathbb{G}_m^{\log}/\mathbb{G}_m$ is a constant sheaf. Also, if $P \to \mathscr{O}_X$ is any chart such that $P \cong (M_X/\mathscr{O}_X^{\times})_x$, where x is the closed point of X, then we have an isomorphism $P^{\mathrm{gp}} \cong \mathbb{G}_m^{\log}/\mathbb{G}_m$. Fix such a chart for the rest of this paragraph.

For every n, the Kummer boundary map (since $\mathscr{O}_X^{\times}$ is n-divisible in the classical fppf topology) gives a distinguished section $\delta_n : \mathbb{Z} \to H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/n\mathbb{Z}(1)) \otimes P^*$, where $P^* = (P^{\mathrm{gp}})^*$ is the dual group of the free group P^{gp}.

Let $G \in \mathrm{fin}_X^c$ be a finite flat group scheme over $\mathscr{O}$. By (2.2.2), we have

$$R^1\varepsilon_*G \cong \lim_n \underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes P^{\mathrm{gp}}.$$

By the Leray spectral sequence associated to ε, we have a short exact sequence

$$0 \to H^1(X^{\mathrm{cl}}_{\mathrm{fl}}, G) \to H^1(X^{\log}_{\mathrm{fl}}, G) \to \lim_n \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes P^{\mathrm{gp}} \to 0.$$

This sequence has a splitting given by the maps

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes P^{\mathrm{gp}} \to \mathrm{Hom}(H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1)) \otimes P^*, H^1(X^{\log}_{\mathrm{fl}}, G)) \to H^1(X^{\log}_{\mathrm{fl}}, G),$$

where the map on the right is evaluation at the distinguished section δ_n.

So we have

$$H^1(X^{\mathrm{cl}}_{\mathrm{fl}}, G) \oplus \lim_n \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes P^{\mathrm{gp}} \xrightarrow{\cong} H^1(X^{\log}_{\mathrm{fl}}, G). \tag{2.2.3.1}$$

Now, $H^1(X^{\log}_{\mathrm{fl}}, G)$ classifies G-torsors over X. We can ask if we can show every such G-torsor to be representable. The answer, as it turns out, is: yes, we can. Let $H^1_r(X^{\log}_{\mathrm{fl}}, G)$ be the sub-set of $H^1(X^{\log}_{\mathrm{fl}}, G)$ consisting of co-cycles that correspond to representable G-torsors. By classical fppf descent $H^1(X^{\mathrm{cl}}_{\mathrm{fl}}, G) \subset H^1_r(X^{\log}_{\mathrm{fl}}, G)$. Now, consider the following claims:

(1) If $\alpha \in H^1(X^{\mathrm{cl}}_{\mathrm{fl}}, G)$ and $\beta \in H^1_r(X^{\log}_{\mathrm{fl}}, G)$, then $\alpha + \beta \in H^1_r(X^{\log}_{\mathrm{fl}}, G)$.
(2) If $G' \subset G$ is a finite flat sub-group scheme, then $H^1_r(X^{\log}_{\mathrm{fl}}, G')$ maps into $H^1_r(X^{\log}_{\mathrm{fl}}, G)$.
(3) For every $n \geq 1$, the image of $\mathrm{End}(\mathbb{Z}/n\mathbb{Z}(1)) \otimes P^{\mathrm{gp}}$ is contained in $H^1_r(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1))$.

With all these claims in hand it is not hard to show that $H^1_r(X^{\log}_{\mathrm{fl}}, G)$ is all of $H^1(X^{\log}_{\mathrm{fl}}, G)$. Given claim (1), it suffices to show that the image of $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}(1), G)$ lies in $H^1_r(X^{\log}_{\mathrm{fl}}, G)$, for every n. But the image factors through $H^1(X^{\log}_{\mathrm{fl}}, G^{\mathrm{mult}})$, where G^{mult} is the largest multiplicative sub-group of G. But G^{mult} is a direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}(1)$; so we're now done by claims (2) and (3).

To prove claim (1), just note that α dies classical fppf locally, and so $\alpha + \beta$ is classical fppf locally representable and hence representable by (1.3.1). For (2), if F is a G'-torsor corresponding to a co-cycle in $H^1_r(X^{\log}_{\mathrm{fl}}, G')$, the corresponding G-torsor is the log flat quotient F' of $G \times F$ by the action of G' given by $g'(g, f) = (gg'^{-1}, g'f)$. Consider the following cartesian diagram:

$$\begin{tikzcd} G \times F \arrow[r] \arrow[d] & F' \arrow[d] \\ G \arrow[r] & G' \setminus G, \end{tikzcd}$$

where $G' \setminus G$ is the quotient of G by G'. Since G is finite flat over $G' \setminus G$ and $G \times F$ is representable over G, it follows by (1.3.1) that F' is also representable. Claim (3) was essentially shown in (1.3.2).

Let us record this result in the following

Theorem 2.2.4. [10, Theorem 9.1] *Let X be an fs log scheme, and let G be an object in fin^c_X; then every G-torsor over $X^{\log}_{\mathrm{fl}}$ is representable.*

Proof. As we have seen, this theorem is already true in the strictly henselian situation. Now, just apply (1.3.1) to conclude. □

2.2.5. *Stability Under Extensions.* We now study the stability of the category of log finite flat group schemes under extensions. X will again be $\mathrm{Spec}\, \mathscr{O}$, with $\mathscr{O}$ a henselian local ring.

Lemma 2.2.6. *The sub-categories fin^r_X and fin^d_X of fin^f_X consist precisely of those objects G for which G^o is an object of fin^c_X.*

Proof. First, note that, for any $G \in \mathrm{fin}^f_X$, G^e is representable by (1.4.5). Since G^o is the kernel of the surjection $G \to G^e$, it is representable whenever G is representable. So, if G is in fin^r_X, then so is G^o, which is then represented by a connected fs log scheme over X, and is thus in fin^c_X by (2.1.7) above. Conversely, suppose G^o is in fin^c_X, then G is representable by (2.2.4), in whose statement we take $G = G^o$ and $X = G^e$. □

Proposition 2.2.7. *The categories* fin^r_X *and* fin^d_X *are stable under extensions in the category of sheaves of abelian groups over* $X^{\log}_{\mathrm{fl}}$.

Proof. Let
$$0 \to G' \to G \to G'' \to 0$$
be a short exact sequence of sheaves of abelian groups over $X^{\log}_{\mathrm{fl}}$, where G' and G'' are objects in fin^r_X. To show that G is also in fin^r_X, it suffices, by (2.2.6), to show that G^o is in fin^c_X. Since it is enough to do this étale locally, we can assume that X is strictly henselian with a chart $P \to \mathscr{O}_X$ inducing an isomorphism $P \xrightarrow{\cong} (M_X/\mathscr{O}_X^\times)_{\overline{x}}$ over the closed point x of X. We have a short exact sequence
$$0 \to (G')^o \to G^o \to (G'')^o \to 0$$
on $X^{\log}_{\mathrm{fl}}$, which gives us an exact sequence
$$0 \to (G')^o \to \varepsilon_* G^o \to (G'')^o \to R^1\varepsilon_*(G')^o$$
on $X^{\mathrm{cl}}_{\mathrm{fl}}$. If we know that the boundary map $\delta : (G'')^o \to R^1\varepsilon_*(G')^o$ is 0, then we can conclude that $\varepsilon_* G^o$ is representable over $X^{\mathrm{cl}}_{\mathrm{fl}}$. But, for any connected group $H \in \mathrm{fin}^c_X$, we have, by (2.2.2),
$$R^1\varepsilon_* H \cong \lim_{n\geq 1} \underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}(1), H) \otimes P^{\mathrm{gp}} \cong \lim_{n\geq 1} \underline{\mathrm{Hom}}(H^*, \mathbb{Z}/n\mathbb{Z}) \otimes P^{\mathrm{gp}},$$
where H^* is the Cartier dual of H. This last sheaf is clearly represented by an étale scheme over $\mathscr{O}$, and so the boundary map δ must be 0.

It now remains to show the following fact: if G is in fin^f_X and $\varepsilon_* G$ is representable by a finite flat group scheme G', then G is isomorphic to G'. There is a natural map $G' \to G$ induced by the canonical map $\varepsilon^*\varepsilon_* G \to G$ and the isomorphism $G' \xrightarrow{\cong} \varepsilon^*\varepsilon_* G$. This natural map is log flat locally an isomorphism (since G is classical over some big enough log flat cover) and is thus an isomorphism over X.

The statement for fin^d_X follows immediately from that for fin^r_X. □

Corollary 2.2.8. *Suppose* $\mathscr{O}$ *has finite residue characteristic* p. *The sub-category of* fin^d_X *of objects of* p*-power torsion consists precisely of those* p*-power torsion objects of* fin^r_X *whose maximal log étale quotients belong to* fin^c_X.

Proof. From (2.2.7), it follows that G is in fin^d_X if and only if G^o and G^e are. But, if G is in fin^r_X, then G^o is classical and therefore has a representable Cartier dual. So, G is in fin^d_X if and only if G^e is in fin^d_X, which, by (2.1.8), can be true if and only if G^e is in fin^c_X. □

2.2.9. *Some Examples.* We will use the above to construct some examples, taken from [12], of log finite flat group schemes over $X = \mathrm{Spec}\,\mathscr{O}$, with $\mathscr{O}$ a strictly henselian discrete valuation ring, equipped with its canonical log structure, and with residue characteristic $p > 0$.

We have, by (2.2.3.1):
$$H^1(X^{\mathrm{cl}}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1)) \oplus \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cong} H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1)),$$
where the map from $\mathbb{Z}/n\mathbb{Z} \to H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1))$ is induced by the Kummer boundary map and a choice of identification of $\mathbb{G}_m^{\log}/\mathbb{G}_m$ with $\mathbb{Z}$ (which amounts to a choice of uniformizer in $\mathscr{O}$.)

Example 2.2.10. The group $H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1))$ classifies extensions of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}(1)$ in the category of log group schemes (by (2.2.4)). If we pick $a \in \mathbb{Z}$ prime to n, that will give us an element of $H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n\mathbb{Z}(1))$ corresponding to an object of fin^r_X that is not in fin^c_X.

Example 2.2.11. Suppose $\mathscr{O}$ is an $\mathbb{F}_p$-algebra, and let α_p be the closed sub-group of $\mathbb{G}_a$ consisting of points x such that $x^p = 0$. Then α_p is connected and $\mathrm{Aut}(\alpha_p) = \mathbb{G}_m$. Using the inclusion $\mathbb{Z}/p\mathbb{Z}(1) \to \mathbb{G}_m$, we get an inclusion
$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/p\mathbb{Z}(1)) \hookrightarrow H^1(X^{\log}_{\mathrm{fl}}, \mathrm{Aut}(\alpha_p))$$
So, for every non-zero element $a \in \mathbb{Z}/p\mathbb{Z}$, we obtain a twist of $(\alpha_p)_a$ of α_p that is not classical and is therefore, by (2.1.7), is not in fin^r_X.

Example 2.2.12. Now, suppose also that $\mathscr{O}$ contains a primitive $(p-1)^{th}$ root of unity ϖ. Then we have inclusions:

$$\mathbb{Z}/(p-1)\mathbb{Z} \hookrightarrow H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/(p-1)\mathbb{Z}(1)) \cong H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/(p-1)\mathbb{Z}) \cong H^1(X_{\mathrm{fl}}^{\log}, \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})).$$

So, every non-zero element $a \in \mathbb{Z}/(p-1)\mathbb{Z}$ gives us a twist $(\mathbb{Z}/p\mathbb{Z})_a$ of the constant sheaf $\mathbb{Z}/p\mathbb{Z}$ that is representable by a log étale scheme, but is not classical. Therefore, by (2.1.8), its Cartier dual (which is a twist of $\mathbb{Z}/p\mathbb{Z}(1)$) is not representable.

2.3. **The Classification of Log Finite Flat Group Schemes.** Suppose that X is the spectrum of a henselian local ring $\mathscr{O}$ with residue characteristic $p > 0$, equipped with a global chart $P \to \mathscr{O}$ that induces an isomorphism $P \xrightarrow{\cong} (M_X/\mathscr{O}_X^\times)_{\overline{x}}$ over the closed point x of X.

Let G be an object in fin_X^f. If G is killed by n, where n is relatively prime to p, then, by (1.4.5), G is representable by a log étale group scheme over X.

For p-power torsion sheaves, on the other hand, we will now give Kato's classification of those which are in fin_X^d in terms of objects in fin_X^c with an extra 'monodromy' structure.

Let G and H be objects in fin_X^c, and let $\mathfrak{Ext}_{X_{\mathrm{fl}}^{\log}}(G,H)$ (resp. $\mathfrak{Ext}_{X_{\mathrm{fl}}^{\mathrm{cl}}}(G,H)$) be the category of extensions of G by H in fin_X^r (resp. fin_X^c). Let n be an integer that kills G, and let $G(1) = \mathbb{Z}/n\mathbb{Z}(1) \otimes_{\mathbb{Z}/n\mathbb{Z}} G$. Let $\mathfrak{Hom}(G(1),H) \otimes P^{\mathrm{gp}}$ be the discrete category associated to the set $\mathrm{Hom}(G(1),H) \otimes P^{\mathrm{gp}}$. Then there is a natural functor

$$\Phi : \mathfrak{Ext}_{X_{\mathrm{fl}}^{\mathrm{cl}}}(G,H) \times \mathfrak{Hom}(G(1),H) \otimes P^{\mathrm{gp}} \to \mathfrak{Ext}_{X_{\mathrm{fl}}^{\log}}(G,H) \tag{2.3.0.1}$$

defined in the following fashion:

For every element $a \in P^{\mathrm{gp}}$, let $T_{n,a}$ be the sheaf of groups making the diagram

$$\begin{tikzcd} T_{n,a} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "a"] \\ \mathbb{G}_m^{\log} \arrow[r, "\uparrow n"] & \mathbb{G}_m^{\log}. \end{tikzcd}$$

cartesian. Then, tensoring with G gives a short exact sequence

$$0 \to G(1) \to T_{n,a} \otimes G \to G \to 0.$$

Given an element $\beta = \sum_i N_i \otimes a_i \in \mathrm{Hom}(G(1),H) \otimes P^{\mathrm{gp}}$, we obtain an object $(G,H)_\beta$ of $\mathfrak{Ext}_{X_{\mathrm{fl}}^{\log}}(G,H)$ as follows: For each i, let $(G,H)_{N_i \otimes a_i}$ be the base change of $T_{n,a} \otimes G$ along the map $N_i : G(1) \to H$. Set $(G,H)_\beta$ to be the Baer sum (over i) of the extensions $(G,H)_{N_i \otimes a_i}$.

Now, the functor Φ is defined by setting, for any L in $\mathfrak{Ext}_{X_{\mathrm{fl}}^{\mathrm{cl}}}(G,H)$, $\Phi(L,\beta)$ to be the Baer sum of L and $(G,H)_\beta$.

Theorem 2.3.1. *The functor Φ in (2.3.0.1) is an equivalence of categories.*

Proof. We will define a candidate Ψ for the quasi-inverse for Φ, and leave to the reader the formal exercise of checking that it actually does its job. Given an extension

$$0 \to H \to L \to G \to 0$$

in $X_{\mathrm{fl}}^{\log}$, we get an exact sequence

$$0 \to H \to \varepsilon_* L \to G \to R^1\varepsilon_* H.$$

Since, by (2.2.2),

$$R^1\varepsilon_* H \cong \lim_{m \geq 1} \underline{\mathrm{Hom}}(\mathbb{Z}/m\mathbb{Z}(1), H) \otimes P^{\mathrm{gp}},$$

we find that, to every extension L, we can associate an element

$$\beta(L) \in \mathrm{Hom}(G, \lim_{m \geq 1} \underline{\mathrm{Hom}}(\mathbb{Z}/m\mathbb{Z}(1), H)) \otimes P^{\mathrm{gp}} = \mathrm{Hom}(G(1), H) \otimes P^{\mathrm{gp}}.$$

Note that L is a classical extension if and only if $\beta(L) = 0$.

We define $\Psi(L)$ to be the pair $(L', \beta(L))$, where L' is the Baer difference between L and $(G,H)_{\beta(L)}$. One checks easily that $\beta(L') = 0$, and so L' is indeed in $\mathfrak{Ext}_{X^{\mathrm{cl}}_{\mathrm{fl}}}(G,H)$. $\square$

Let $\mathrm{fin}_X^{c,N}$ be the category whose objects are pairs (G,N), where G is an object in fin_X^c killed by a power of p, and $N : G^e(1) \to G^o \otimes \mathbb{G}_m^{\log}/\mathbb{G}_m$ is a homomorphism, and let $\mathrm{fin}_X^{d,p}$ (resp. $\mathrm{fin}_X^{r,p}$) be the full sub-category of fin_X^d (resp. fin_X^r) consisting of p-power torsion objects (resp. p-power torsion objects with classical maximal log étale quotients).

Corollary 2.3.2. *For every choice of chart $\alpha : P \to \mathscr{O}$ for X with P sharp and torsion-free, there is a natural equivalence of categories:*

$$\Lambda_\alpha : \mathrm{fin}_X^{c,N} \xrightarrow{\cong} \mathrm{fin}_X^{r,p} = \mathrm{fin}_X^{d,p}.$$

Proof. Follows immediately from (2.3.1), (2.2.7), and (2.2.8). $\square$

Remark 2.3.3. Giving such a chart is equivalent to choosing a section for the surjection $M^{\mathrm{gp}}_{X,\overline{x}} \to M^{\mathrm{gp}}_{X,\overline{x}}/\mathscr{O}^\times_{X,\overline{x}}$.

2.3.4. We define a **log Barsotti-Tate group** or a **log p-divisible group** over X to be a sheaf of abelian groups G over $X^{\log}_{\mathrm{fl}}$ satisfying the following properties, where we denote by $G[p^i] \subset G$ the sub-group of p^i-torsion:

(1) $G = \bigcup_{i\geq 1} G[p^i]$.
(2) $p : G \to G$ is surjective.
(3) $G[p]$ is in fin_X^r.

We denote the category of log p-divisible groups over X by lpdiv_X. The full sub-category of lpdiv_X consisting of objects G, for which $G[p]$ is in fin_X^d (resp. $G[p^i]$ is in fin_X^c, for all $i \geq 1$) will be denoted by lpdiv_X^d (resp. pdiv_X).

Remark 2.3.5. By (2.2.7), it follows that if G is in lpdiv_X (resp. lpdiv_X^d), then $G[p^i]$ is in fin_X^r (resp. fin_X^d), for all $i \geq 1$.

For every log p-divisible group G, we define $G^o = \bigcup_i G[p^i]^o$, $G^e = \bigcup_i G[p^i]^e$ and $G(1) = \bigcup_i G[p^i](1)$. A log p-divisible group is **étale** (resp. **connected**) if $G = G^e$ (resp. $G = G^o$). We have a short exact sequence

$$0 \to G^o \to G \to G^e \to 0$$

of log p-divisible groups.

We denote by pdiv_X^N the category of pairs (G,N), where G is in pdiv_X and $N : G^e(1) \to G^o$ is a homomorphism of p-divisible groups.

Theorem 2.3.6. *For every choice of chart $\alpha : P \to \mathscr{O}$ for X, with P sharp and torsion-free, there is a natural exact equivalence*

$$\Lambda_\alpha : \mathrm{pdiv}_X^N \xrightarrow{\cong} \mathrm{lpdiv}_X^d.$$

Proof. Immediate from (2.3.2). $\square$

Now, we specialize to the case where $\mathscr{O}$ is a discrete valuation ring. Let X to be the fs log scheme whose underlying scheme is $\mathrm{Spec}\,\mathscr{O}$ equipped with its canonical log structure.

Corollary 2.3.7. *For every choice of uniformizer π of $\mathscr{O}$, we have a natural exact equivalence of categories:*

$$\Lambda_\pi : \mathrm{pdiv}_X^N \xrightarrow{\cong} \mathrm{lpdiv}_X^d.$$

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