Lecture 1

Ground field / $\mathbb{C}$.

Lemma 1: Let U be a unipotent gp acting on an affine alg. variety X. Then any orbit of U is Zariski closed.

Pf: $O \subset X$ a U-orbit; $\bar{O}$ = closure $\subset X$.

$\bar{O} \setminus O \subset \bar{O}$ a non-empty closed U-stable subset.

$I \subset \mathbb{C}[\bar{O}]$ be the ideal of $\bar{O} \setminus O$; I is U-stable sub-rep of $\mathbb{C}[\bar{O}]$.

Fact: Any alg. rep^n of U has a fixed point.

$\Rightarrow \exists\, 0 \neq f \in I$ fixed by U. (if $I \neq 0$)

$\Rightarrow f$ is constant $\Rightarrow\Leftarrow$
(since U acts transitively on O)

$\Rightarrow I = 0$ and $O = \bar{O}$.

$\square$

(2) Let P be a connected alg. group;
$\mathfrak{p} = \mathrm{Lie}\, P$

$P \longrightarrow GL(V)$ alg. rep^n

$E \subset V$ a P-stable sub-space; $v \in V$

$P \to V$; $p \mapsto p \cdot v$

$\mathfrak{p} \to V$; $X \mapsto Xv$

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Lemma 2: (i) $\mathfrak{p}\cdot v \in E \Leftrightarrow v+E$ is P-stable.

(ii) $\mathfrak{p}\cdot v = E \Leftrightarrow P\cdot v$ is Zariski open dense in $v+E$.

Pf: (i) $X \in \mathfrak{p} \rightsquigarrow$ vector field on V
For $e \in E$, the value at $v+e$ is $Xv + Xe \in E$ (assuming $\mathfrak{p}\cdot v \subset E$)

$\Rightarrow$ vector field is tangent to $V+E$ at every point.

P connected $\Rightarrow$ $v+E$ is P-stable.

(ii) $P \longrightarrow v+E$ (assuming (i))
$p \mapsto p\cdot v$

is a submersion $\Rightarrow$ its image is an open dense sub-scheme.

$\mathfrak{g}$: ss. Lie algebra $/\mathbb{C}$

G : Algebraic connected grp ; Lie $G = \mathfrak{g}$.

For $x \in \mathfrak{g}$; $\mathfrak{g}_x = \{ y \in \mathfrak{g} : [x,y]=0\}$

$G_x = \{ g \in G : \mathrm{Ad}_g(x) = x\}$

$\mathfrak{t} \subset \mathfrak{g}$ a Cartan sub-algebra.
$r = \mathrm{rank}\,\mathfrak{g} = \dim \mathfrak{t}$.

$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \text{ roots}} \mathfrak{g}_\alpha \right)$

$X \in \mathfrak{g}$ is called ss/nilpotent $\Leftrightarrow$ $ad(x)$ is ss/nilpotent.

- x is s.s $\Leftrightarrow$ x is $Ad(G)$-conj into $\mathfrak{t}$.

$X \in \mathfrak{g}$ is <u>regular</u> if $\dim \mathfrak{g}_x = r$

$$\mathfrak{g}^r \equiv \{x \in \mathfrak{g} : x \text{ regular}\}$$

- $x \in \mathfrak{t}$ is regular $\Leftrightarrow$ $\alpha(x) \neq 0$, $\forall$ roots α.

$$\mathfrak{g}^{rs} = \{x \in \mathfrak{g}^r : x \text{ is s.s.}\}$$

- $x \in \mathfrak{g}^{rs} \Leftrightarrow \mathfrak{g}_x$ is Cartan.

- $\mathfrak{g}^{rs} \subset \mathfrak{g}$ is <u>open dense</u>.

- $x \mapsto \dim \mathfrak{g}_x$ is lower-semi-continuous.

  $\Rightarrow \dim \mathfrak{g}_x \geq r$, $\forall x \in \mathfrak{g}$.

$B \subset G$ a Borel sub-gp

- B is connected
- $B = N_G(B)$
- $[B,B] = U$ unipotent radical; $B/U = H$ is a torus.

- Any character $B \to \mathbb{C}^\times$ factors through H.

$\underset{\text{Borel.}}{\mathfrak{b}} = \operatorname{Lie} B \supset \operatorname{Lie} U = \mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$

$\mathfrak{h} = \mathfrak{b}/\mathfrak{u}$.

Let's look at $Ad(B)$-action on $\mathfrak{g}$: this gives a B-stable ~~flag~~ maximal flag:

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^p = \mathfrak{b} \supset \cdots \supset \mathfrak{g}^{p+n} = \mathfrak{u} \supset \cdots \supset$$

s.t. $\dim \mathfrak{g}^i/\mathfrak{g}^{i+1} = 1$; and $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ gives a character for $\mathfrak{h}$: these are the $\underline{\text{roots of } \mathfrak{b}}$ $(i \geq p)$

$$\underset{\text{roots of } \mathfrak{b}}{\overset{\shortparallel}{R(\mathfrak{b})}} \subset X^*(\mathfrak{h}) = X^*(H)$$

$R(\mathfrak{g}/\mathfrak{b}) \subset X^*(\mathfrak{h})$ the $\underline{\text{other}}$ weights $(i<p)$

- B-action on $\mathfrak{b}/\mathfrak{u}$ is trivial.

$h \in \mathfrak{h}$ is $\underline{\text{regular}}$ if $\alpha(h) \neq 0$, $\forall \alpha \in R(\mathfrak{b}) \cup R(\mathfrak{g}/\mathfrak{b})$

Let $x \in \mathfrak{b}$ be regular s.s.

$$G_x = T \text{ is a max torus } \subset B$$

$$\mathfrak{g}_x = \mathfrak{t} \text{ is a } \underline{\text{Cartan}}.$$

$$\underbrace{\mathfrak{t} \hookrightarrow \mathfrak{b} \to \mathfrak{b}/\mathfrak{u}}_{\simeq} = \mathfrak{h}$$

"abstract" $\underline{\text{Cartans}}$.

- $B = T \cdot U$, $\quad \underbrace{T \hookrightarrow B \to B/U}_{\simeq} = H$

- Note that $R(\mathfrak{b}), R(\mathfrak{g}/\mathfrak{b}) \subset X^*(H)$ are $\underline{\text{entirely}}$ canonical w.r.t B.

Prop 1 (1) $x \in \mathfrak{b}$ is regular s.s. $\Leftrightarrow$ $x \pmod{\mathfrak{u}} \in \mathfrak{h}$ is regular.

(2) $x \in \mathfrak{b}$ is reg. s.s. $\Rightarrow$ $x + \mathfrak{u} \subset \mathfrak{h}$ is the closed B-orbit ($\mathcal{U}$-orbit) of x.

Pf: (1) "$\Rightarrow$" is clear from definition.

(2) $\mathfrak{g}_x = \mathfrak{t} \subset \mathfrak{b}$ is a ~~maximal torus~~ Cartan.

$G_x = T \subset B$ a max'l torus.

$B = \mathcal{U} \cdot T \Rightarrow Ad(B)(x) = Ad(\mathcal{U})(x)$.

Lemmas 1 + 2: says that $Ad(\mathcal{U})(x) = x + \mathcal{U}$ as long as $\mathfrak{u} \cdot x = \mathfrak{u}$

But $ad(x) : \mathfrak{u} \to \mathfrak{u}$ is invertible (since $\alpha(x) \neq 0$ $\forall \alpha$)

$\Rightarrow [\mathcal{U}, x] = \mathcal{U}$, which is what we wanted.

$\Rightarrow$ $\boxed{Ad(B)(x) = x + \mathfrak{u}}$

(1) "$\Leftarrow$" Suppose $x \pmod{\mathcal{U}}$ is regular; let $\mathfrak{t} \subset \mathfrak{b}$ be a Cartan and $x' \in \mathfrak{t}$ s.t. $x' \pmod{\mathfrak{u}} = x \pmod{\mathfrak{u}}$

But $Ad\,\mathcal{U}(x') = x' + \mathfrak{u}$. $\Rightarrow$ x is conjugate to x'

$\Rightarrow$ x is also reg. s.s.

$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{u} \supset \mathfrak{u}' = [\mathfrak{u}, \mathfrak{u}]$

Weights of H-action on $\mathfrak{u}/\mathfrak{u}' = \bar{\mathfrak{u}}$ are called simple roots; Δ = simple roots

- $\dim \bar{\mathfrak{u}} = r = \dim \mathfrak{h}$
- $\bar{\mathfrak{u}}$ has a canonical basis of wt. vectors. (up to scaling)
- Elements w. non-zero co-ordinates in this basis form an open dense orbit (under H), that is in fact an $\underline{H\text{-torsor}}$ $\bar{\mathfrak{u}}^{reg} \subset \bar{\mathfrak{u}}$.

$\mathfrak{u}^{reg} = \{x \in \mathfrak{u} : x (\mathrm{mod}\ \mathfrak{u}') \in \bar{\mathfrak{u}}^{reg}\}$

<u>Prop:</u> $\mathfrak{g}^r \cap \mathfrak{u} = \mathfrak{u}^{reg}$; this is a single $\underline{B\text{-orbit}}$.

We will require some <u>preparation</u>.

<u>Fix</u> $\mathfrak{t} \subset \mathfrak{b}$ <u>Cartan</u>, $\mathfrak{b} = \mathfrak{t} + \mathfrak{u}$, $\mathfrak{u} = \bigoplus_{\alpha \in R(\mathfrak{u})} \mathbb{C} \cdot e_\alpha$.

Δ = simple roots ; $e = \sum_{\alpha \in \Delta} e_\alpha$.

$\updownarrow$

vector $(1, 1, \ldots, 1) \in \bar{\mathfrak{u}}$

(in fact in $\bar{\mathfrak{u}}^{reg}$)

$\exists! \, h \in \mathfrak{t}$ s.t. $\alpha(h) = 2$, $\forall \alpha \in \Delta$.

$\Rightarrow \underline{[h, e] = 2e}$

By Jacobson-Morosov, we have an $\underline{sl(2)\text{-triple}}$.

$\{f, h, e\} \subset \mathfrak{g}$.

<u>In our case</u>, $\exists! f \in \sum_{\alpha\in\Delta} \mathfrak{g}_{-\alpha}$ s.t $[e,f] = h$, $[h,f] = -2f$.

Choose $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ s.t. $h_\alpha = [e_\alpha, e_{-\alpha}] \in \mathfrak{t}$.

$\Rightarrow$ we have $h = \sum c_\alpha \cdot h_\alpha$, $f = \sum c_\alpha \cdot e_{-\alpha}$

Indeed $[e,f] = [\sum e_\alpha, \sum c_\alpha \cdot e_{-\alpha}]$

$$= \sum_{\alpha,\beta\in\Delta} c_\beta [e_\alpha, e_{-\beta}]$$

$$= \sum_{\alpha\in\Delta} c_\alpha [e_\alpha, e_{-\alpha}]$$

$$= h.$$

$$\cancel{[h,f] = \sum_{\alpha,\beta\in\Delta} c_\alpha c_\beta [h_\alpha, e_{-\beta}]}$$

$$\cancel{= \sum_{\alpha,\beta\in\Delta} c_\alpha c_\beta (-\beta)(h_\alpha) e_{-\beta}}$$

$$\cancel{= -\sum_{\beta\in\Delta} c_\beta \Big(\sum_{\alpha\in\Delta} c_\alpha (\beta(h_\alpha))\Big) e_{-\beta}}$$

$[h,f] = [h, \sum_\beta c_\beta e_{-\beta}]$ (since $\beta(h) = 2$ $\forall \beta \in \Delta$)

$= -2f$

Ex: $\mathfrak{g} = sl_n(\mathbb{C})$

$x \in sl_n(\mathbb{C})$ regular s.s. $\Leftrightarrow$ $\det(\lambda - x)$ has n-distinct roots.

x is regular $\Leftrightarrow$ char pol = min. poly.

$\mathfrak{t} \subset \underset{\mathfrak{u}}{\overset{\cup}{\mathfrak{b}}}$ usual sub-algebras.

$$\mathfrak{u} = \begin{pmatrix} 0 & * & \\ & 0 & * \\ & & \ddots \\ 0 & & & 0 \end{pmatrix} \simeq \mathbb{C}^{n-1}$$

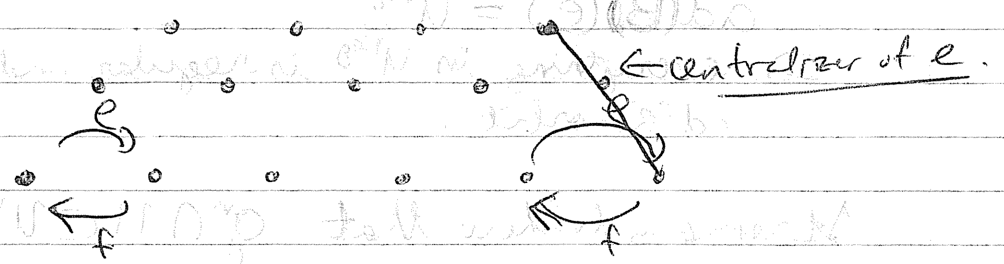
$$\alpha \in \Delta, \; e_\alpha = \begin{pmatrix} 0 & & \\ & \ddots & 1 \\ & & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

= nilpotent Jordan block of maximal size

$$h = \begin{pmatrix} n-1 & & & & 0 \\ & n-3 & & & \\ & & \ddots & & \\ & & & -(n-3) & \\ 0 & & & & -(n-1) \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & & & \\ n-1 & 0 & & \\ & 2(n-2) & \ddots & \\ & & \ddots & 0 \end{pmatrix}$$

$\langle e, h, f\rangle = sl_2 \hookrightarrow \mathfrak{g}$.



$\mathfrak{g}$

- $ad(h)$ has <u>even</u> eigenvalues.

- $\mathfrak{g}_h = t$ = zero wt space of h.

- $\mathfrak{g}_e$ = span of highest wt vectors in decomp of $\mathfrak{g}$ into irred. under $\langle e, h, f\rangle$.

- $\mathfrak{g}$ decomposes into exactly r irred. components

- $\dim \mathfrak{g}_e = r$ & $\mathfrak{g}_e$ is an r-dim h-stable sub-space. $\Rightarrow$ <u>e is regular</u>.

<u>Consider</u>:

$$ad(e) : \mathcal{U} \to [\mathcal{U}, \mathcal{U}] = \mathcal{U}'$$

$$\dim \mathcal{U}' = \dim \mathcal{U} - \dim \bar{\mathcal{U}} = \dim \mathcal{U} - \dim \mathfrak{g}_e$$
$$= \dim \mathcal{U} - \dim \ker ad(e)$$

$\Rightarrow ad(e)$ is <u>surjective</u>.

$\Rightarrow [\mathcal{U}, e] = \mathcal{U}'$

$\Rightarrow ad(\mathcal{U})(e) = e + \mathcal{U}'$ (from Lemmas 1 & 2)

Pf of Prop 2 : From above, it follows that

$$\mathrm{ad}(B)(e) = \mathcal{U}^{reg}$$

$\Rightarrow$ everything in $\mathcal{U}^{reg}$ is regular and this is an $\mathrm{ad}(B)$-orbit.

It remains to show that $\mathfrak{g}^r \cap \mathcal{U} \subset \mathcal{U}^{reg}$.

For this, if

LECTURE 2.

Weyl group.

(1) $T \subset G$ max. torus

$\mathfrak{t} = \mathrm{Lie}\, T \subset \mathfrak{g}$ a Cartan.

$\mathrm{Norm}(T)/T = W(T)$

$W(T)$ acts freely transitively on Borels $B \supset T$

(2) Fix a Borel $\mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$

$\mathfrak{h} = \mathfrak{b}/\mathfrak{u}$.

$\Delta \subset \mathfrak{h}^*$ canonically defined simple roots.

$\Delta \ni \alpha \rightsquigarrow$ simple reflection $s_\alpha : \mathfrak{h} \to \mathfrak{h}$.

$W =$ subgr of $GL(\mathfrak{h})$ generated by $\{s_\alpha, \alpha \in \Delta\}$

This is a Coxeter group w. distinguished set of generators.

If we fix $T \subset B$

$$\underbrace{\mathfrak{t} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{h}}_{\sim} \Rightarrow W(T) \xrightarrow{\sim} W.$$

endows $W(T)$ with a distinguished set of generators.

Relative position of Borels

$$\mathfrak{b} \subset \mathfrak{g}, \quad \mathfrak{u} = [\mathfrak{b}, \mathfrak{b}], \quad \underset{\substack{\text{another} \\ \text{Borel}}}{\mathfrak{b}' \subset \mathfrak{g}}, \quad \mathfrak{u}' = [\mathfrak{b}', \mathfrak{b}']$$

Facts: (1) $\exists$ a Cartan $\mathfrak{t} \subset \mathfrak{b} \cap \mathfrak{b}'$

(2) All Cartans in $\mathfrak{b} \cap \mathfrak{b}'$ are conjugate under the unipotent grp w. Lie algebra $\mathfrak{u} \cap \mathfrak{u}'$.

Fix $\mathfrak{t} \subset \mathfrak{b} \cap \mathfrak{b}'$.

$$\begin{tikzcd}
\mathfrak{t} \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] \arrow[dd, "\simeq"'] & & \mathfrak{b}/\mathfrak{u} \arrow[dl, "\simeq"] \\
& \frac{\mathfrak{b} \cap \mathfrak{b}'}{\mathfrak{u} \cap \mathfrak{u}'} \arrow[dl, "\simeq"] & \\
\frac{\mathfrak{b}'}{\mathfrak{u}'} & &
\end{tikzcd}$$

Using these identifications, we have

$$R(\mathfrak{b}) \subset \mathfrak{t}^* \supset R(\mathfrak{b}')$$

Cor: $\exists!\ w \in W(T)$ s.t. $R(\mathfrak{b}') = w(R(\mathfrak{b}))$

Fact (1) $\Rightarrow$ $w \in W$ (w.r.t $\mathfrak{b}$) is independent of choice of T.

We say that $\mathfrak{b}$ & $\mathfrak{b}'$ are in relative position w.

Fact(2) $\Rightarrow$ All Borels b' in relative position w w.r.t b are conjugate under B.

Flag variety

$\mathcal{B}$ = variety of all Borel subalg's in $\mathfrak{g}$.

$d = \dim(\text{Borel})$

$\mathcal{B} \subset \mathrm{Grass}_d(\mathfrak{g})$ a closed algebraic sub-variety.

$\Rightarrow \mathcal{B}$ is a projective alg. variety.

Fix Borel subgp $B_0 \subset G$, giving $b_0 \subset \mathfrak{g}$.

$$G \longrightarrow \mathrm{Grass}_d(\mathfrak{g})$$
$$g \mapsto ad(g)(b_0)$$

This gives a ~~isomorphism~~ bijection $G/B_0 \xrightarrow[\pi]{} \mathcal{B}$.

$$T_e(G/B_0) = \mathfrak{g}/b_0 \xrightarrow[d\pi]{\sim} T_{b_0}\mathcal{B} \leftarrow \text{Zariski tangent space.}$$

$\Rightarrow \pi$ is in fact an isomorphism (since $\mathcal{B}$ is smooth & hence normal)

There is a tautological vector bundle

$$0 \to \mathcal{U} \to b^{\checkmark} \to \mathfrak{h} \to 0$$
$$\downarrow$$
$$\mathcal{B}$$

(pull-back from tautological bundle on $\mathrm{Grass}_d(\mathfrak{g})$)

(a short exact seq. in fact)

Lemma: The vector bundle $\mathfrak{h}_\mathcal{B}$ is canonically trivialized. Moreover, under this trivialization, each root gives a constant section of $\mathfrak{h}_\mathcal{B}^*$.

Pf: Pick a Borel $\mathfrak{b}_0 \subset \mathfrak{g}$; for any $\mathfrak{b} \subset \mathfrak{g}$, $\exists g \in G$ s.t. $Ad g(\mathfrak{b}_0) = \mathfrak{b}$.

$$\Rightarrow Adg : \frac{\mathfrak{b}_0}{\mathfrak{u}_0} \xrightarrow{\sim} \frac{\mathfrak{b}}{\mathfrak{u}}.$$

Such a g is not unique; any other such g' is of the form gb, where $b \in B_0$.

But $Ad(b) : \frac{\mathfrak{b}_0}{\mathfrak{u}_0} \to \frac{\mathfrak{b}_0}{\mathfrak{u}_0}$ is the identity for any $b \in B_0$.

$$\Rightarrow \frac{\mathfrak{b}_0}{\mathfrak{u}_0} \xrightarrow{\sim} \frac{\mathfrak{b}}{\mathfrak{u}} \text{ canonically.}$$

Thus, there is the abstract Cartan $\mathfrak{h}_\mathcal{B}$ and the abstract Weyl group $W \subset GL(\mathfrak{h}_\mathcal{B})$.

The group G acts on $\mathcal{B} \Rightarrow G$ acts on $\mathcal{B} \times \mathcal{B}$.

Cor: G-diagonal orbits in $\mathcal{B} \times \mathcal{B} \simeq$ elements of W.

For each $w \in W$, $Y_w = \left\{ (b, b') \in \mathcal{B} \times \mathcal{B} \text{ where } b \,\&\, b' \text{ are in relative position } w \right\}$

Grothendieck-Springer resolution

Total space $\underline{\mathfrak{b}} = \{(b, x) \in \mathcal{B} \times \mathfrak{g} : x \in b\}$
$=: \tilde{\mathfrak{g}}$

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\nu"'] \arrow[d, "pr_{\mathcal{B}}"] \arrow[dr, "pr_{\mathfrak{g}} = \mu"] & \\ \mathfrak{h}. & \mathcal{B} & \mathfrak{g} \end{tikzcd}$$

ν is induced by the trivialization $\underline{\mathfrak{h}} \simeq \mathfrak{h} \times \mathcal{B}$

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[d] \arrow[dd, bend left, "\nu"] \\ \mathfrak{h} \times \mathcal{B} \arrow[d, "pr"] \\ \mathfrak{h} \end{tikzcd}$$

Now $\tilde{\mathfrak{g}}$ is a smooth alg. variety (a ~~affine~~ vector bundle over $\mathcal{B}$)

$$\begin{tikzcd} \tilde{\mathfrak{g}} = \underline{\mathfrak{b}} \arrow[d] \arrow[r, phantom, "="] & G \times^{B_0} \mathfrak{b}_0 \arrow[d] & G \times \mathfrak{b}_0 \arrow[l] \arrow[d] \\ \mathcal{B} \arrow[r, phantom, "="] & G/B_0 & G \arrow[l] \end{tikzcd}$$

Let $h \in \mathfrak{h}$; what is $\nu^{-1}(h) \subset \tilde{\mathfrak{g}}$?

Well, $\nu^{-1}(0) =$ total space of $\underline{\mathfrak{u}} = G \times^{B_0} \mathfrak{u}_0$

In general, $\nu^{-1}(h) = G \times^{B_0} (h + \mathfrak{u}_0)$: a torsor under $\underline{\mathfrak{u}}$

$\Rightarrow$ ν is an affine bundle and is thus smooth.

$\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$; $(b, x) \mapsto x$. (Grothendieck-Springer resolution)

μ is
- surjective
- proper. (projective in fact)
- G-equivariant.

Generic structure

$\mathfrak{g}^{rs} \subset \mathfrak{g}$ (reg. ss.)

$x \in \mathfrak{g}^{rs} \Rightarrow \mathfrak{g}_x \subset \mathfrak{g}$ is a Cartan
$G_x = T$ is a max. torus.

Now
- $Ad(G)(x) \simeq G/T$.
- $Ad(G)(x) \cap \mathfrak{t} = W(T) \cdot x$.
  ($ad(g) gxg^{-1} \in \mathfrak{t} \Rightarrow gTg^{-1} = T$).

$\mathfrak{h}^{reg} \subset \mathfrak{h}$ reg. elements.

and $\tilde{\mathfrak{g}}^{rs} = \mu^{-1}(\mathfrak{g}^{rs})$

Prop: (1) $\exists$ canonical W-action on $\tilde{\mathfrak{g}}^{rs}$ along fibers of μ. This action on $\mu^{-1}(x)$ is free and transitive, $\forall x \in \mathfrak{g}^{rs}$, so that
$\mu|_{\tilde{\mathfrak{g}}^{rs}} : \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is a Galois cover with Galois group W.

(2) $\tilde{\mathfrak{g}}^{rs} = \nu^{-1}(\mathfrak{h}^{reg})$ and ν is W-equivariant.

(3) Let $(b, x) \in \tilde{\mathfrak{g}}^{rs}$ and $h \equiv x \bmod [b, b]$
$= \nu((b, x)) \in \mathfrak{h}^{reg}$.

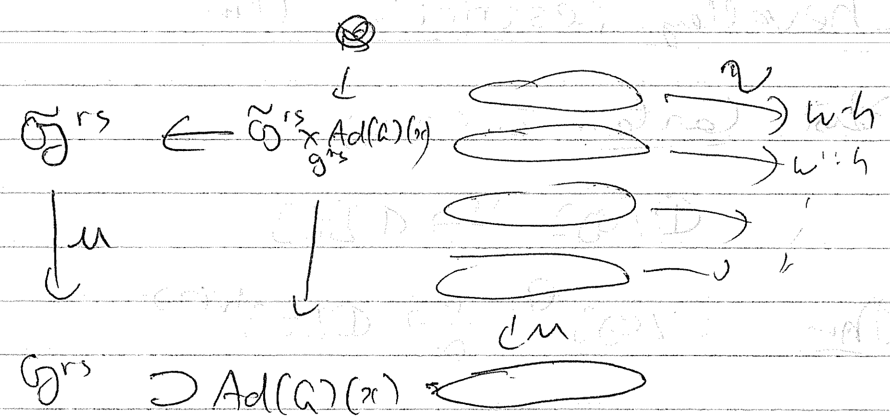
$$\begin{array}{c}(b,x)\\ \downarrow \nu\end{array}$$

For $h \in \mathfrak{h}^{reg}$, we have $G/G_x \xrightarrow{\sim} \nu^{-1}(h) \subset \tilde{\mathfrak{g}}$

$$\begin{tikzcd}
G/G_x \arrow[r, "\sim"] & \nu^{-1}(h) \subset \tilde{\mathfrak{g}} \arrow[dl, "pr_B"'] \arrow[d, "\mu", "\wr"'] \arrow[dr, bend left] & \\
G/B & Ad(G)(x) \simeq \frac{G}{G_x}. & \{h\} \in \mathfrak{h}^{reg}
\end{tikzcd}$$

$$\mu^{-1}(Ad(G)(x)) = \bigsqcup_{w \in W} \nu^{-1}(wh)$$

<u>Picture</u>



i.e. $\tilde{\mathfrak{g}}^{rs} \times_{\mathfrak{g}^{rs}} Ad(G)(x)$ is a <u>trivializable</u> <u>W-torsor</u>.
(<u>Note</u>: G/T is <u>simply connected</u>)

<u>Comments</u>: (i) each fiber $\nu^{-1}(h)$ is connected
$\Rightarrow$ $\underline{\mathfrak{g}^{rs} = \nu^{-1}(\mathfrak{h}^{reg}) \text{ is connected}}$.

(ii) $\dim \tilde{\mathfrak{g}} = \dim \mathfrak{g}$.
(but we knew this <u>anyway</u>)

<u>Pf</u>: (1) <u>Definition</u> of W-action on $\tilde{\mathfrak{g}}^{rs}$

Fix $(b,x) \in \mathfrak{g}^{rs}$, $w \in W$

$w \cdot (b,x) = (b', x)$; b' = <u>unique</u> Borel $b' \supset t = \mathfrak{g}_x$ in relative position w wrt b.

Take $h \in \mathfrak{h}^{res}$; fix $B_0 \supset U_0$. Then

$$\nu^{-1}(h) = G \times^{B_0} (h + \mathfrak{u}_0)$$

<u>Claim</u> $\nu^{-1}(h)$ is a single G-orbit $\simeq G/G_x$, where $x \pmod{\mathfrak{u}} = h$

We saw last time that $h + \mathfrak{u}_0$ is the <u>$Ad(B)$-orbit</u> of x. This shows everything.

<u>Chevalley restriction thm</u>

Fix <u>Cartan</u> $\mathfrak{t} \hookrightarrow \mathfrak{g}$.

$$\mathbb{C}[\mathfrak{g}] \xrightarrow{res} \mathbb{C}[\mathfrak{t}]$$

<u>Thm</u> $\mathbb{C}[\mathfrak{g}]^G \underset{\simeq}{\xrightarrow{res}} \mathbb{C}[\mathfrak{t}]^{W(T)}$

LECTURE 3

$\mathfrak{t} \subset \mathfrak{g}$ $\quad T \subset G$ as usual

Chevalley: $\mathrm{res}: \mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}]^{N(T)}$

$$f \mapsto f|_{\mathfrak{t}}$$

Pf: Injectivity: $f|_{\mathfrak{t}} = 0 \Rightarrow f|_{\mathfrak{g}^{ss}} = 0$

$$\Rightarrow \underline{f = 0}.$$

Surjectivity:

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} & & \mathfrak{h} \end{tikzcd}$$

G-S resolution

$$\mu^*: \mathbb{C}[\mathfrak{g}] \hookrightarrow \mathbb{C}[\tilde{\mathfrak{g}}] \xleftarrow{\nu^*} \mathbb{C}[\mathfrak{h}] : \circledast$$

Claim ("Abstract" Chevalley)

$$\mu^*(\mathbb{C}[\mathfrak{g}]^G) = \nu^*(\mathbb{C}[\mathfrak{h}]^W)$$

$\leftarrow$ abstract Weyl gr

Given this one needs to check compatibility w. original maps to finish.

Rem: $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{h}/W]$

nice alg. variety.
(smooth in fact)

$\mathfrak{g}/\!/G := \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G$

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ (*)\ \mathfrak{g} \arrow[dr] & & \mathfrak{h} \arrow[dl] \\ & \mathfrak{g}/\!/G \simeq \mathfrak{h}/W & \end{tikzcd}$$

commutative diagram (not Cartesian.)

abstract Chevalley.

<u>Pf of abstract Chevalley.</u>

$\forall f \in \mathbb{C}[\mathfrak{h}]^W \Rightarrow \exists! F \in \mathbb{C}[\mathfrak{g}]^G$ s.t. $\nu^* f = \mu^* F$.

For this, consider the diagram $(*)$ <u>generically</u>

$$\begin{tikzcd} & \tilde{\mathfrak{g}}^{rs} \arrow[dl, "\mu"', "\text{W-Galois cover}"] \arrow[dr, "\nu"] & \\ \mathfrak{g}^{rs} & & \mathfrak{h}^{reg} \end{tikzcd}$$

$\nu \leftarrow$ <u>fibers are G-orbits.</u>

So

$\nu^* f|_{\tilde{\mathfrak{g}}^{rs}}$ is $G \times W$-invariant

$\Rightarrow$ descends to a G-invariant function F on $\mathfrak{g}^{rs}$

Claim: F extends to $\mathfrak{g}$; i.e. F has <u>no</u> poles.
Use <u>properness</u> of μ.

Another point of view

Choose $B \subset G$ a Borel, $\mathfrak{b} \subset \mathfrak{g}$.

$$\mathfrak{g} \hookleftarrow \mathfrak{b} \xrightarrow{\nu_{\mathfrak{b}}} \mathfrak{h}_{\mathfrak{g}}.$$

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\mathrm{res}_{\mathfrak{b}}} \mathbb{C}[\mathfrak{b}]^B \xleftarrow[\nu_{\mathfrak{b}}^*]{} \mathbb{C}[\mathfrak{h}_{\mathfrak{g}}]$$

Prop: ① $\nu_{\mathfrak{b}}^* : \mathbb{C}[\mathfrak{h}_{\mathfrak{g}}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{b}]^B$

② $\nu_{\mathfrak{b}}^* : \mathbb{C}[\mathfrak{h}_{\mathfrak{g}}]^W \xrightarrow{\sim} \mathrm{res}_{\mathfrak{b}}\, \mathbb{C}[\mathfrak{g}]^G.$

Choose an invariant non-deg form $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

$$\mathfrak{b} \xrightarrow{\sim} \mathfrak{u}^{\perp} \qquad \mathfrak{h}_{\mathfrak{g}} = \mathfrak{b}/\mathfrak{u} \xrightarrow{\sim} \mathfrak{h}_{\mathfrak{g}}^*$$
$$\mathfrak{u} \xrightarrow{\sim} \mathfrak{b}^{\perp}$$
$$\mathfrak{g}/\mathfrak{u} \xrightarrow{\sim} \mathfrak{b}^*$$
$$\mathfrak{g}/\mathfrak{b} \xrightarrow{\sim} \mathfrak{u}^*$$

Then

$$\begin{array}{ccccc} \mathbb{C}[\mathfrak{g}]^G & \xrightarrow{\mathrm{res}_{\mathfrak{b}}} & \mathbb{C}[\mathfrak{b}]^B & \xleftarrow[\sim]{\nu_{\mathfrak{b}}^*} & \mathbb{C}[\mathfrak{h}_{\mathfrak{g}}] \\ \| & & & & \| \\ (\mathrm{Sym}\,\mathfrak{g})^G & \longrightarrow & (\mathrm{Sym}\,\mathfrak{g}/\mathfrak{u})^B & \longleftarrow & \mathrm{Sym}(\mathfrak{b}/\mathfrak{u}) = \mathrm{Sym}\,\mathfrak{h}_{\mathfrak{g}}. \end{array}$$

Why is $\nu_{\mathfrak{b}}^*$ an isomorphism?

Choose Cartan $\mathfrak{t} \subset \mathfrak{b}$

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}_-, \quad \mathfrak{g}/\mathfrak{u} = \mathfrak{t} \oplus \mathfrak{u}_-$$

$$\mathrm{Sym}(\mathfrak{g}/\mathfrak{u}) \simeq \mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{u}_-)$$

$$\Rightarrow \mathrm{Sym}(\mathfrak{g}/\mathfrak{u})^B \subset (\mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{u}_-))^T = \mathrm{Sym}(\mathfrak{t}).$$

Harish-Chandra isomorphism

$$U(\mathfrak{g}) = T(\mathfrak{g}) / (\cancel{x\otimes y} - y\otimes x - [x,y] : x, y \in \mathfrak{g})$$

$G \overset{ad}{\curvearrowright} \mathfrak{g} \Rightarrow G \curvearrowright U\mathfrak{g}$ by automorphism

Lemma: $U(\mathfrak{g})^G = Z(\mathfrak{g}) :=$ Center of $U(\mathfrak{g})$.

Pf: $a \in Z(\mathfrak{g}) \iff ax = xa, \ \forall x \in \mathfrak{g}$

$\iff ad(x)(a) = 0$

$\iff$ a is fixed by G

since G is connected.

Rem: $Z(\mathfrak{g})$ is a "quantum" version of $Sym(\mathfrak{g})^G$.

$= \mathbb{C}[\mathfrak{h}]^W$.

Fix Borel $\mathfrak{b} \supset \mathfrak{u}$

In $U(\mathfrak{b})$, $(U(\mathfrak{b}))\cdot \mathfrak{u} = \mathfrak{u}\cdot(U(\mathfrak{b}))$ is a two sided ideal in $U(\mathfrak{b})$, because $\mathfrak{u} \subset \mathfrak{b}$ is a Lie ideal.

$$U(\mathfrak{b}) / _{U(\mathfrak{b})\cdot \mathfrak{u}} = U(\mathfrak{b}/\mathfrak{u}) = U(\mathfrak{h}) \simeq Sym(\mathfrak{h})$$

Lemma: $\nu_{\mathfrak{b}}^{quant}: U(\mathfrak{h}) \xrightarrow{\sim} \left(U(\mathfrak{g}) / _{U(\mathfrak{g})\mathfrak{u}}\right)^B$ as vector spaces.

Pf: same as above.

$$\begin{tikzcd} U(\mathfrak{g})^G \arrow[r] \arrow[rr, bend right, "r"'] & \left(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{u}\right)^B \arrow[r, "(v_0^{\text{quant}})^{-1}"'] & U(\mathfrak{h}) \end{tikzcd}$$

$R(b)$: roots in b

$\rho = \frac{1}{2}\sum_{\alpha \in R(b)} \alpha \in \mathfrak{h}^*$

$W \times \mathfrak{h}^* \to \mathfrak{h}^*$

$w \cdot \lambda = w(\lambda+\rho) - \rho$

(Dot action).

$U(\mathfrak{h}) = \mathrm{Sym}\,\mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$ equipped with <u>dot</u> action of W.

<u>HC isom. thm</u>: ~~Inside $U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{u}$~~ We have

$$r(U(\mathfrak{g})^G) = U(\mathfrak{h})^W \text{ (dot action)}.$$

We have ~~HC~~ Ⓗ : $Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W$ (quantum analogue of Chev. thm)

as <u>algebras</u>.

<u>Pf</u>:

<u>Step 1</u>: ~~Ⓗ~~ Ⓗ : $Z(\mathfrak{g}) \to U(\mathfrak{h})$ is an algebra map

<u>Step 2</u>: Im Ⓗ $\subset$ dot-invariants

<u>Step 3</u>: Ⓗ is surjective.

Step 1: $U(\mathfrak{g})/(U(\mathfrak{g}))\cdot\mathfrak{u} =: V \leftarrow$ universal Verma module

$$\| $$

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{b}/\mathfrak{u})$$

$$\|$$

$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h}) \leftarrow$ ~~$U(\mathfrak{g})$~~ $(U(\mathfrak{g}), U(\mathfrak{h}))$-bimodule

The Lemma above V_b^{quant} above implies that
$\forall z \in Z(\mathfrak{g})$, $zV = V\Theta(z)$

$\Rightarrow$ it follows that Θ is an algebra map.

Step 2 $\mathfrak{t} \subset \mathfrak{b}$ Cartan
$\alpha_1, \dots, \alpha_r$ simple roots $\in \mathfrak{h}^*$
$\alpha_1^\vee, \dots, \alpha_r^\vee$ " co-roots $\in \mathfrak{h}$
$s_i = s_{\alpha_i} \in W$

$$s_i \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha_i^\vee \rangle \alpha_i$$

For $i = 1, \dots, r$, $D_i = \{ \lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^* \rangle \in \mathbb{N} \cup \{0\} \}$

Let $z \in Z(\mathfrak{g})$, $\Theta(z) \in U(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$

Claim: $\Theta(z)$ is s_i-invariant, $\forall i = 1, \dots, r$.

i.e. $\Theta(z)(\lambda) = \Theta(z)(\lambda - \langle \lambda + \rho, \alpha_i^\vee \rangle \alpha_i)$, $\forall i, \forall \lambda$

Lemma: For $i \in 1, \dots, r$, $\lambda \in D_i$

$$\Theta(z)(\lambda) = \Theta(z)(\lambda - \langle \lambda+\rho, \alpha_i^\vee \rangle \alpha_i)$$

Pf: $\lambda \leftrightarrow \lambda : \mathcal{U}(\mathfrak{h}_g) \to \mathbb{C}$ (evaluation)
and let $\mathbb{C}_\lambda$ be the corresponding $\mathcal{U}(\mathfrak{h}_g)$-module.

Now, $V \otimes_{\mathcal{U}(\mathfrak{h}_g)} \mathbb{C}_\lambda \simeq \mathrm{ind}_{\mathcal{U}(\mathfrak{b})}^{\mathcal{U}(\mathfrak{g})} (\mathbb{C}_\lambda) =: V_\lambda$
as a $\mathcal{U}(\mathfrak{g})$-module.

Any $z \in Z(\mathfrak{g})$ acts on V_λ via $\Theta(z)(\lambda)$

Set $q := \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{N}$, so that $(\rho(\alpha_i^\vee) = 1, \forall i)$

$$s_i \cdot \lambda = \lambda - q\alpha_i - \langle \rho, \alpha_i^\vee \rangle \alpha_i = \lambda - (q+1)\alpha_i$$

Idea: Produce $0 \neq u \in V_\lambda$ s.t. $z u = \Theta(z)(\lambda - (q+1)\alpha_i) u$.

Construction of u: $\mathfrak{g} = \mathfrak{u} + \mathfrak{t} + \mathfrak{u}_-$
$e_1, \dots, e_r \in \mathfrak{u}$ root vectors corresponding to $\alpha_1, \dots, \alpha_r$.
$f_1, \dots, f_r \in \mathfrak{u}_-$ - - -

s.t. $[e_i, f_j] = \delta_{ij} h_i$,
$[h_i, e_i] = 2e_i$
$[h_i, f_i] = -2f_i$.

Now in $V_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ we have the vector $1 \otimes 1$ and $h_i(1\otimes 1) = \langle \lambda, \alpha_i^\vee \rangle \cdot (1 \otimes 1)$
$= q \cdot (1 \otimes 1)$

$ef^n = f^n e + n f^{n-1} h - n(n-1) f^{n}$

$u := f_i^q (1 \otimes 1) \quad (q \in \mathbb{N} \cup \{0\})$

Claim: $e_j u = 0, \ \forall j$.

If $j \neq i$, $e_j f_i^q (1 \otimes 1) = f_i^q e_j (1 \otimes 1)$
$= 0$, since $\lambda(e_j) = 0, \forall j$.

$\underline{j = i}$. $\cancel{e_i \ldots = \ldots}$

Consider $\mathbb{C}$-span of elements
$\{ v_\lambda = 1 \otimes 1, f_i v_\lambda, f_i^2 v_\lambda, f_i^3 v_\lambda, \ldots \} \subseteq V_\lambda$

This span is $\langle e_i, h_i, f_i \rangle$ - stable sub-space.

$h_i(f_i v_\lambda) = f_i(h_i v_\lambda) - 2 f_i(v_\lambda) = (q-2) f_i v_\lambda$

$\Rightarrow h_i(f_i^j v_\lambda) = (q - 2j) f_i^j v_\lambda \Rightarrow h_i(f_i^q v_\lambda) =$

$$[e, f^{n}] = (n-q-1) ef^{n-1} + (n-q) h.$$

$$e f^{q+1} = 0$$

$U_n \otimes 1$

U_n^{-1}

$$V_{\lambda - (q+1)\alpha_i} \hookrightarrow V_\lambda$$

LECTURE 4

Last time: $Z(\mathfrak{g})$ = Center of $\mathcal{U}(\mathfrak{g})$

$$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{t} + \mathfrak{u} \supset \mathfrak{t}$$

Proved: $\forall z \in Z(\mathfrak{g})$, $\exists!\ \Theta(z) \in \mathcal{U}(\mathfrak{t})$ s.t. $z - \Theta(z) \in \mathcal{U}(\mathfrak{g})\cdot \mathcal{U}$ $\quad (\mathcal{U} = \exp(\mathfrak{u}))$

$\Theta : Z(\mathfrak{g}) \mapsto \mathcal{U}(\mathfrak{t})$

The assignment $z \mapsto \Theta(z)$ is the H-C map. It is a map of algebras.

Lemma: Let V be a $Z(\mathfrak{g})$-module, $\gamma \in \mathfrak{t}^*$. Let $v \neq 0 \in V$ be s.t. $u \cdot v = 0$ and s.t. $h \cdot v = \gamma(h) v$, $\forall h \in \mathfrak{t}$. Then $zv = \Theta(z)(\gamma)\cdot v$.

Pf: $z - \Theta(z) \in \mathcal{U}(\mathfrak{g})\cdot\mathcal{U} \Rightarrow z\cdot v = \Theta(z)\cdot v$

$= \gamma(\Theta(z))\cdot v$

$(\because \Theta(z) \in \mathcal{U}(\mathfrak{t}))$

For any $\lambda \in \mathfrak{h}^*$, we defined a left $\mathcal{U}(\mathfrak{g})$-module V_λ s.t. $\exists v_\lambda \in V_\lambda$ s.t.
① $u \cdot v_\lambda = 0$, $\forall u \in \mathfrak{u}$
② $h \cdot v_\lambda = \lambda(h) v_\lambda$, $\forall h \in \mathfrak{t}$
③ $V_\lambda = \mathcal{U}(\mathfrak{g})\cdot v_\lambda$.

$\Rightarrow$ By Lemma above, $\forall z \in Z(\mathfrak{g})$, $z \cdot v_\lambda = \Theta(z)(\lambda)\cdot v_\lambda$.

Now, we choose a Zariski dense $D_i \subset \mathfrak{t}^*$; take $\lambda \in D_i$, $q = \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{N}$; let $u_\lambda = f^{q+1} \cdot v_\lambda$.

Proved: $u \cdot u_\lambda = 0$, $\forall u \in \mathfrak{u}$

We also have: ① $e_i f_i^n = f_i^n e_i + n f_i^{n-1} h_i - n(n-1) f_i^{n-1}$

② $\forall h \in \mathfrak{t}$, $h \cdot u_\lambda = (\lambda - (q+1)\alpha_i)(h)\, u_\lambda$.

By ① $e_i u_\lambda = e_i f_i^{q+1} v_\lambda$
$= (q+1) q f_i^{n-1} v_\lambda - (q+1) q f_i^{n-1} v_\lambda$
$= 0$

$\Rightarrow$ u_λ is annihilated by $\mathfrak{u}$; so we have

Lemma: $z \cdot u = \mathrm{H}(z)(\lambda - (q+1)\alpha_i) \cdot u$.

$\Rightarrow$ $\mathrm{H}(z)(\lambda - (q+1)\alpha_i) = \mathrm{H}(z)(\lambda)$, $\forall z \in Z(\mathfrak{g})$.

$\Rightarrow$ (by a density argument)

$\mathrm{H}(z) \in U(\mathfrak{t})^W$ (dot action invariants)

$\Rightarrow$ $\mathrm{H} : Z(\mathfrak{g}) \to U(\mathfrak{t})^W$.

Digression

Filtration $0 = A_{-1} \subseteq A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k \subseteq \cdots$

s.t. $A = \bigcup_{k \geq 0} A_k$. (in particular A is separated)

$gr(A) = \bigoplus_{k \geq 0} A_k / A_{k-1}$

Lemma 1 $f : A \to B$ a lin. map of filtered groups; then $gr f$ is surj./inj. $\Rightarrow$ f is surj./inj.

Let A be a filtered alg. so that ① $1 \in A_0$

② $A_k A_\ell \subseteq A_{k+\ell}$.

Lemma 2 • If $gr(A)$ has no zero-divisors $\Rightarrow$ A has none as well

• If $gr(A) = \mathbb{C}[\bar{x}_1, \ldots, \bar{x}_n]$ (homogeneous) and A is commutative,
$\Rightarrow$ $\underline{A \simeq \mathbb{C}[x_1, \ldots, x_n]}$.

$\mathfrak{g}$ a Lie algebra; $T^{\cdot}(\mathfrak{g}) \twoheadrightarrow \mathcal{U}(\mathfrak{g})$
tensor algebra

PBW filtration on $\mathcal{U}(\mathfrak{g})$: $\mathcal{U}_k(\mathfrak{g}) = \operatorname{Im}\left(\bigoplus_{h=0}^{k} T^h(\mathfrak{g})\right)$

We get a natural graded map

$$\mathrm{Sym}(\mathfrak{g}) \xrightarrow[\sim]{PBW} gr(\mathcal{U}(\mathfrak{g})).$$

i.e. $\forall k$, we have

$$0 \to \mathcal{U}_{k-1}(\mathfrak{g}) \to \mathcal{U}_k(\mathfrak{g}) \xrightarrow{\text{"}PBW^{-1}\text{"}} \mathrm{Sym}^k(\mathfrak{g}) \to 0$$

with $\sigma : \mathrm{Sym}^k(\mathfrak{g}) \dashrightarrow \mathcal{U}_k(\mathfrak{g})$ — symmetrization splitting (of v. spaces)

$$\text{symm}: \mathrm{Sym}^k(\mathfrak{g}) \to T^k(\mathfrak{g}) \twoheadrightarrow \mathcal{U}_k(\mathfrak{g})$$

splitting σ

Suppose $\mathfrak{g} = \mathrm{Lie}(G)$; then G acts on everything above, and all the maps are G-equivariant.

$\Rightarrow$ take G-invariants to get

$$0 \to \mathcal{U}_{k-1}(\mathfrak{g})^G \to \mathcal{U}_k(\mathfrak{g})^G \to \mathrm{Sym}^k(\mathfrak{g})^G \to 0$$

(since we have our splitting σ)

Equip $Z(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ with the induced filtration; then, observing that $Z(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^G$, we see:

$$gr(Z(\mathfrak{g})) \xrightarrow{\sim} \mathrm{Sym}(\mathfrak{g})^G \text{ (canonically)}$$

Now, let $\mathfrak{g}$ be a s.s. Lie algebra.

<u>Theorem</u> (H-S) The map $\Theta : Z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})^{W_\circ}$ is an isomorphism.

<u>Pf:</u> Θ respects filtrations $\Rightarrow$ we get

$$\begin{array}{ccc} gr(\Theta) : gr(Z(\mathfrak{g})) & \longrightarrow & gr(\mathcal{U}(\mathfrak{h})^{W_\circ}) \\ \uparrow\wr & & \downarrow\wr \\ Sym(\mathfrak{g})^G & \longrightarrow & Sym(\mathfrak{h})^W \end{array}$$

canonical action.

this composite is just the Chevalley restriction map (or its dual, at any rate.)

$\Rightarrow$ $gr(\Theta)$ is an isomorphism

$\Rightarrow$ Θ is an <u>isomorphism</u>. ▥

<u>Theorem</u> (Chevalley) $\Gamma \subset GL(V)$ <u>finite</u> <u>sub-grp</u> <u>generated</u> by <u>reflections</u>; then:

- $\mathbb{C}[V]^\Gamma \simeq \mathbb{C}[x_1, \ldots, x_n]$, where $n = \dim V$.
- $\mathbb{C}[V]$ is <u>free</u> over $\mathbb{C}[V]^\Gamma$.

<u>Example</u>: $V = \mathbb{C}^n$, $\Gamma = S_n$, $\mathbb{C}[V]^\Gamma$ = alg. of symmetric polys

<u>Cor</u>: $\mathbb{C}[\mathfrak{h}]^W \cong \mathbb{C}[\mathfrak{g}]^G \simeq Z(\mathfrak{g})$ are free poly. alg. on r generators.

Adjoint quotient.

$$\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$$

Thm (Kostant) $\mathbb{C}[\mathfrak{g}]$ is a free $\mathbb{C}[\mathfrak{g}]^G$-module. More precisely, $\exists$ a G-stable graded subspace $H \subset \mathbb{C}[\mathfrak{g}]$ s.t.

$$\mathbb{C}[\mathfrak{g}]^G \otimes_{\mathbb{C}} H \xrightarrow[\sim]{\text{mult}} \mathbb{C}[\mathfrak{g}]$$

of $\mathbb{C}[\mathfrak{g}]^G$-modules.

Rem: Identify $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$; then: $\mathrm{Sym}(\mathfrak{g})$ is a free $\mathrm{Sym}(\mathfrak{g})^G$-module.

Cor: $U(\mathfrak{g})$ is a free $Z(\mathfrak{g})$-module.

Pf: By filtered-to-graded reduction.

Pf of Thm: (1) algebraic (Bernstein-Lunts)

Fix a Cartan $\mathfrak{t} \subset \mathfrak{g}$; $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$.

We have
$$\begin{array}{ccc} \mathrm{Sym}(\mathfrak{g}) & \longrightarrow \mathrm{Sym}(\mathfrak{g}/\mathfrak{t}^\perp) \xrightarrow{\sim} & \mathrm{Sym}(\mathfrak{t}) \\ \cup & & \cup \\ (\mathrm{Sym}\,\mathfrak{g})^G & \xrightarrow{\quad\sim\quad} & \mathrm{Sym}(\mathfrak{t})^W \end{array}$$

Let $\bar{f}_1, \dots, \bar{f}_r$ be a free homogeneous basis of $\mathrm{Sym}(\mathfrak{t})$ as a $\mathrm{Sym}(\mathfrak{t})^W$-module, and $f_1, \dots, f_r$ be lifts of $\bar{f}_i$ to $\mathrm{Sym}(\mathfrak{g})$.

Claim: $\mathrm{Sym}(\mathfrak{g})$ is a free module over $\mathrm{Sym}(\mathfrak{g})^G \otimes_{\mathbb{C}} \mathrm{Sym}\,\mathfrak{t}^\perp$-module w. basis $f_1, \dots, f_r$

② geometric

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded, f.g. commutative $\mathbb{C}$-algebra, with $A_0 = \mathbb{C}$, $A_{>0} = \bigoplus_{i \geq 1} A_i$ (augmentation ideal).

Lemma: Let $M = \bigoplus_{i \geq 0} M_i$ be a graded A-module; then M is free over $A \iff M$ is flat.

Pf: Let $\overline{M} = M/_{A_{>0}M}$, and let $H \subseteq M$ be a graded lift of $\overline{M}$ (i.e. splitting $M \to \overline{M}$ as a graded v.s.)

We have an A-module map

$$A \otimes_{\mathbb{C}} H \to M$$

By graded Nakayama, this map is surjective. Let K be its kernel; then tensoring with $A/_{A_{>0}}$ gives:

$$0 \to K/_{A_{>0}K} \to \frac{A \otimes_{\mathbb{C}} H}{A_{>0}(A \otimes_{\mathbb{C}} H)} \xrightarrow{\sim} \overline{M} \to 0$$

$$\Rightarrow K/_{A_{>0}K} = 0 \Rightarrow K = 0.$$

$$\Rightarrow \boxed{A \otimes_{\mathbb{C}} H \xrightarrow{\sim} M}$$

∎

Cor: $\mathbb{C}[\mathfrak{g}]$ is free over $\mathbb{C}[\mathfrak{g}]^G \iff$ it is flat.

We will now show that $\mathbb{C}[\mathfrak{g}]$ is flat over $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$; in other words, we will show that the morphism

$$\begin{array}{ccc} \mathfrak{g} & & \\ \downarrow & \searrow & \\ \mathfrak{g}/\!/G & \xrightarrow{\sim} \mathfrak{h}/W & \underset{\text{by Chevalley}}{\cong} \mathbb{A}^r_{\mathbb{C}} \end{array}$$

is flat

We call the map $\pi: \mathfrak{g} \to \mathfrak{h}/W$ the adjoint quotient map.

Prop: π is surjective & all its fibers are irreducible of dimension $= \dim \mathfrak{g} - r$.

Cor: π is flat. (because $\mathfrak{h}/W$ is smooth).

Fibers of π

Let $\chi \in \mathfrak{g}/\!/G \longleftrightarrow$ max. ideal in $\mathbb{C}[\mathfrak{g}]^G$.

$\pi^{-1}(\chi)$: the fiber over χ (as a sub-scheme of $\mathfrak{g}$).

Jordan decomp: $\forall x \in \mathfrak{g}$, $x = s + n$, s s.s.
n nilpotent
(uniquely)
s.t. $[s,n]=0$.

Thm: For each $x \in (\mathfrak{g}/\!/G)(\mathbb{C})$

① $\exists!$ s.s. conj. class $\pi^{-1}(x)^{ss} \subset \pi^{-1}(x)$; this class underlies a closed sub-scheme of $\pi^{-1}(x)$, and is the class of minimal dimension in $\pi^{-1}(x)$.

② $\pi^{-1}(x) \underset{\uparrow}{=} \{x \in \mathfrak{g} : x = s+n,\ s \in \pi^{-1}(x)^{ss}\}$
set-theoretically

③ $\exists!$ regular conjugacy class $\pi^{-1}(x)^r = \mathfrak{g}^r \cap \pi^{-1}(x)$ in $\pi^{-1}(x)$ and it underlies a dense Zariski open of $\pi^{-1}(x)$.

Cor: ① $\pi^{-1}(x)$ is irreducible (by ③)

② $\dim \pi^{-1}(x) = \dim \pi^{-1}(x)^r$
$= \dim G - \dim G_x$ (for any $x \in \pi^{-1}(x)^r$)
$= \dim G - r$ (by defn of reg. elt).

③ π is flat

Rem As sets, we have bijections

$$\mathfrak{g}^{ss}/G \simeq \mathfrak{g}/\!/G \simeq \mathfrak{g}^r/G \text{ e.g.}$$

Fix $s \in \mathfrak{t} =$ Cartan $\subset \mathfrak{g}$.

$$R_s = \{\text{roots } \alpha \in R : \alpha(s) = 0\} \subseteq R.$$
s.s.

Consider the Levi $\mathfrak{g}_s \subset \mathfrak{g}$; $\mathfrak{g}_s = \mathfrak{g}(R_s) \oplus \mathfrak{z}(\mathfrak{g}_s)$ with $\mathfrak{z}(\mathfrak{g}_s) \subset \mathfrak{t}$.

Fact (easy): There are finitely many conj. classes of Levis

LECTURE 5

Lemma 1: The map $\mathfrak{g}^{ss}/G \xrightarrow{\sim} \mathfrak{g}/\!/G = \mathfrak{h}/W$

is a bijection of sets.

Pf: For $s \in \mathfrak{t}$, $Gs \cap \mathfrak{t} = Ws$

For $s \in \mathfrak{t}$, $C(s) := \left\{ \begin{array}{l} \text{elements of } \mathfrak{g} \text{ w. s.s. Jordan} \\ \text{component in } Ad(G)s \end{array} \right\}$

$$\mathfrak{g}_s = \text{Levi corresponding to } s$$

$$= \underset{\substack{\uparrow \\ \text{s.s. algebra}}}{\mathfrak{g}(R_s)} + Z(\mathfrak{g}_s)$$

$$R_s = \{\underset{\substack{\Downarrow \\ X^*(t)}}{\alpha} : \alpha(s) = 0\}$$

Lemma 2: (1) G_s is a connected reductive group.

$$Lie(G_s / Z(G_s)) = \mathfrak{g}(R_s)$$

(2) The assignment

$$Ad(G_s)n \longmapsto Ad(G)(s+n)$$

yields a bijection of sets

$$\mathfrak{g}(R_s)^{nil}/G_s \xrightarrow{\sim} C(s)/G$$

- This bijection preserves closure relations between conj. classes
- It takes regular classes to regular classes.

Pf: (2) $x_1, x_2 \in C(s)$; then up to conjugation by G, we can assume that $x_i^{s.s.} = s$, $i=1,2$.

$\Rightarrow x_i = s + n_i$, $i = 1,2$.

$\Rightarrow x_1$ is conjugate to $x_2 \iff n_1$ is conjugate to n_2 under

~~But~~ since $[n_i, s] = 0$, for $i=1,2$, we see that $n_i \in \mathfrak{g}(R_s)$. ∎

$\mathcal{N}(\mathfrak{g}) :=$ set of nilpotent elements of $\mathfrak{g}$ (this is of course a closed sub-variety of $\mathfrak{g}$)

$\cup$

$\mathcal{N}^r(\mathfrak{g}) :=$ ones that are regular.

Lemma 3: $\mathcal{N}^r(\mathfrak{g})$ is a single G-orbit and is Zariski dense in $\mathcal{N}(\mathfrak{g})$.

Pf of thm last time:

$$\begin{array}{c}\mathfrak{g}\\ \downarrow \pi \\ \mathfrak{g}/\!/G\end{array}$$

Claim:
For any $x \in \mathfrak{g}/\!/G$, $\pi^{-1}(x)^{reg}$ is a single G-orbit which is dense in the fiber.

Fix $s \in \mathfrak{t}$, $n \in \mathcal{N}^r(\mathfrak{g}(R_s))$; then, by Lemma 3, the G_s-orbit of n is dense in $\mathcal{N}(\mathfrak{g}(R_s))$ and equals $\mathcal{N}^r(\mathfrak{g}(R_s))$.

Then, by Lemma 2, $s+n \in \mathfrak{g}^r$ & $Ad(G)(s+n)$ is dense in $C(s)$.

$\Rightarrow$ Any element of $\mathbb{C}[\mathfrak{g}]^G$ is constant on $C(s)$

$\Rightarrow C(s) \subset \pi^{-1}(\pi(s))$.

Now, $\mathfrak{g} = \bigsqcup_{[s]\in\mathfrak{g}^{ss}/G} C(s)$, $\mathfrak{g} = \bigsqcup_{[s]\in\mathfrak{g}^{ss}/G} \pi^{-1}(\pi(s))$

(by Lemma 1)

$\Rightarrow$ it follows that $C(s) = \pi^{-1}(\pi(s))$

$$\begin{aligned}\Rightarrow \dim \pi^{-1}(\pi(s)) &= \dim C(s)\\ &= \dim \mathrm{Ad}(G)(s+n)\\ &= \dim G - \dim G_{s+n}\\ &= \dim G - r \quad (\text{since } s+n \text{ is reg.})\end{aligned}$$

$\square$

So, up to Lemma 3 & the algebro-geometric lemma on flatness, we have Kostant's theorem.

We return to the G-S resolution.

$$\begin{tikzcd}
 & \mu^{-1}(C(s)) \arrow[r, hook] \arrow[dl] & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[r, "\nu"] & \mathfrak{h} \arrow[dl, "\varpi"] \\
C(s) \arrow[r, hook] \arrow[d] & \mathfrak{g} \arrow[r] & \mathfrak{g}/G = \mathfrak{h}/W & \\
x \arrow[urr, hook, "\pi"'] & & &
\end{tikzcd}$$

- fibers of ϖ are W-orbits in $\mathfrak{h}$.

$s \in \mathfrak{t} \subset \mathfrak{b}$, $x = \pi(s) \in \mathfrak{h}/W$.

$$\begin{aligned}\text{Now, } \mu^{-1}(C(s)) &= \mu^{-1}(\pi^{-1}(x))\\ &= \nu^{-1}(\varpi^{-1}(x))\\ &= \bigsqcup_{s'\in Ws} \nu^{-1}(s')\\ &= \bigsqcup_{s'\in Ws} G\times^{B}(s'+\mathfrak{n}) \quad (\text{disjoint union of affine bundles over } G/B)\end{aligned}$$

Claim:

The map $\mu|_{\nu^{-1}(s')} : \nu^{-1}(s') \twoheadrightarrow \pi^{-1}(x) = C(s_x)$ is *proper*, *surjective* (in fact, it is a resolution of singularities)

In particular, for $x = 0$, $C(0) = \mathcal{N}(\mathfrak{g}) = \pi^{-1}(0)$.

$$\Rightarrow \nu^{-1}(0) = G \times^B \mathfrak{u} \twoheadrightarrow \mathcal{N}(\mathfrak{g})$$

$$\begin{array}{c} \| \\ \tilde{\mathcal{N}}(\mathfrak{g}) \end{array}$$

this is a resolution of singularities.

We will return to this claim later.

Pf of Lemma 3: $\mathfrak{b} \supset \mathfrak{u} \supset \mathfrak{u}^r$ ~~$= Ad(B)$~~

We saw that $\mathfrak{u}^r$ is an $Ad(B)$-orbit, dense is $\mathfrak{u}$. In fact $\mathfrak{u}^r = Ad(B)(e)$, $e = e_{\alpha_1} + \dots + e_{\alpha_r}$

$\alpha_1 \dots \alpha_r$ simple roots.

Consider the resolution

$$\mu : \tilde{\mathcal{N}}(\mathfrak{g}) \longrightarrow \mathcal{N}(\mathfrak{g}) : \text{this is clearly } \underline{\text{surjective}}$$

Then $\mu^{-1}(\mathcal{N}^{reg}(\mathfrak{g})) = G \times^B \mathfrak{u}^r = G \times^B Ad(B)(e)$

$\Rightarrow \mu^{-1}(\mathcal{N}^{reg}(\mathfrak{g}))$ is a dense G-orbit in $\tilde{\mathcal{N}}(\mathfrak{g})$

$\Rightarrow \mathcal{N}^{reg}(\mathfrak{g}) \subset \mathcal{N}(\mathfrak{g})$ is a *dense G-orbit*
(since μ is G-equivariant).

Ex. $\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+d=0 \right\}$

$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\Delta]$, where $\underline{\Delta = -\det}$.

$\Rightarrow$ $\pi : \mathfrak{g} \to \mathfrak{g}/\!/G$ is just $\det : \mathbb{C}^3 \to \mathbb{C}$

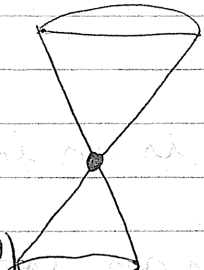
$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto a^2 + bc.$$

$\pi^{-1}(0)$

$\det = 0$

$\|$

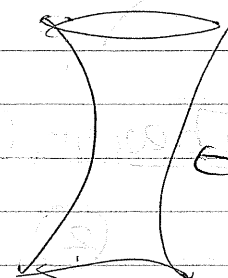
$\mathcal{N} = \{0\} \sqcup G\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$



$\pi^{-1}(a)$

$\det \neq 0$

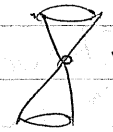
$a \neq 0$



$\leftarrow$ this quadric has two different pavings by lines, which corresponds to the two resolutions by affine bundles over $\mathbb{P}^1$. ($W = \mathbb{Z}/2\mathbb{Z}$ here)

$\mathcal{B} = \mathbb{P}^1$

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \supset & \tilde{\mathcal{N}} \\ \downarrow \mu & & \downarrow \\ \mathfrak{g} & \supset & \mathcal{N} \end{array}$$



$\leftarrow$ this is just the <u>blow up</u> of the singular point of $\mathcal{N}$.

Also $\det : \mathcal{N} \to \mathbb{C}$ is a <u>submersion</u> outside of 0.

We have:

$$\begin{tikzcd}
 & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[d, "\phi"] \arrow[dr, "\nu"] & \\
\mathfrak{g} \arrow[dr, "\pi"'] & X \arrow[l, "p_{\mathfrak{g}}"'] \arrow[r, "p_{\mathfrak{h}}"] & \mathfrak{h} \arrow[dl, "\varpi"] \\
 & \mathfrak{g}/\!/G &
\end{tikzcd}$$

$$X = \mathfrak{g} \times_{\mathfrak{g}/\!/G} \mathfrak{h}$$

$X = \{(x,h) \in \mathfrak{g} \times \mathfrak{h} : f(x) = (\operatorname{res} f)(h)\}$

$\mathbb{C}[X] = \mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{g}]^G} \mathbb{C}[\mathfrak{h}]$

For sl_2, $X \cong \{(x,y,z,h) : x^2+y^2+z^2 = h^2\}$

$X^r = p_{\mathfrak{g}}^{-1}(\mathfrak{g}^r)$.

Theorem (Kostant) (1) X is an integral scheme.

(2) $\tilde{\mathfrak{g}}^r \xrightarrow{\phi} X^r$ is an isomorphism.

(3) $\phi : \tilde{\mathfrak{g}} \to X$ is a resolution of singularities such that $\phi^*: \mathbb{C}[X] \to \mathbb{C}[\tilde{\mathfrak{g}}]$ is an isomorphism; i.e. $\mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_{\tilde{\mathfrak{g}}}$

Plan of proof:

Step 1: By Chevalley, $\mathbb{C}[\mathfrak{h}]$ is a free $\mathbb{C}[\mathfrak{h}]^W$-module $\Rightarrow$ X is ~~free over~~ finite free over $\mathfrak{g}$ of rank $\#W$.

Step 2: Generic analysis: Let $X^{rs} = p_{\mathfrak{g}}^{-1}(\mathfrak{g}^{rs})$

Then, just as in Step 1, $\begin{matrix} X^{rs} \\ \downarrow \\ \mathfrak{g}^{rs} \end{matrix}$ is a Galois cover with Galois group W.

But now we have

$$\begin{tikzcd} \cancel{\mathcal{X}}^{rs} & \tilde{\mathfrak{g}}^{rs} \arrow[l, "\phi"'] \arrow[dl, "u"] \\ \mathfrak{g}^{rs} \arrow[from=u, "p_{\mathfrak{g}}"'] & \end{tikzcd}$$

where u is again a Galois cover of the same rank as $\mathcal{X}^{rs}$

$$\Rightarrow \tilde{\mathfrak{g}}^{rs} \xrightarrow[\phi]{\sim} X^{rs}.$$

Step 3: By Step 1, X is Cohen-Macaulay, and by Step 2, X^{rs} is smooth.
$\Rightarrow X$ is generically reduced

So, CM + gen. reduced $\Rightarrow$ X is reduced.

In a more elementary fashion:

$$0 \to Nil(\mathbb{C}[X]) \to \mathbb{C}[X] \to \mathbb{C}[X]^{red} \to 0$$

is a sequence of f.g. $\mathbb{C}[\mathfrak{g}]$-modules with $\mathbb{C}[X]$ free over $\mathbb{C}[\mathfrak{g}]$.

$\Rightarrow Nil(\mathbb{C}[X])$ is torsion-free
But $Nil(\mathbb{C}[X])$ vanishes over X^{rs}, since X^{rs} is smooth $\Rightarrow$ $Nil(\mathbb{C}[X]) = 0$..

Step 4: It's clear for a number of reasons that X is irreducible, and so it follows that X is integral.

Step 5: We will define $\mathfrak{g}^{good} \subset \mathfrak{g}$ s.t.

$$\mathfrak{g}^{rs} \subset \mathfrak{g}^{good} \subset \mathfrak{g}^{r} \subset \mathfrak{g}$$

and s.t. if $\mathfrak{g}^{bad} = \mathfrak{g} \setminus \mathfrak{g}^{good}$, then $\operatorname{codim}(\mathfrak{g}^{bad}, \mathfrak{g}) \geq 2$.

and s.t. $X^{good} = p_{\mathfrak{g}}^{-1}(\mathfrak{g}^{good}) \subset X^{sm}$.

In particular, X is regular in codimension one and is thus normal (since it is C-M).

Again, more explicitly, we have two claims:

Claim 0: $\mathbb{C}[X] \xrightarrow{\sim} \mathbb{C}[X^{good}]$

Pf:
$$\begin{tikzcd} X & X^{good} \arrow[l, hook'] \arrow[d] \\ \mathfrak{g} \arrow[u, leftarrow] & \mathfrak{g}^{good} \arrow[l, hook'] \end{tikzcd} \Rightarrow \begin{tikzcd} \mathbb{C}[X] \arrow[r, hook, "\sim"'] & \mathbb{C}[X^{good}] \\ \mathbb{C}[\mathfrak{g}] \arrow[u] \arrow[r, hook] & \mathbb{C}[\mathfrak{g}^{good}] \arrow[u] \end{tikzcd}$$

$\cong \uparrow$ because $\operatorname{Codim}(\mathfrak{g}^{good}, \mathfrak{g}) \geq 2$.

LECTURE 6

Recall: $X = \mathfrak{g} \times_{\mathfrak{g}/\!/G} \mathfrak{h}$

$\phi : \tilde{\mathfrak{g}} \to X$.

Properties: (1) X is ~~reduced~~ integral.
(2) ϕ is birational & proper

From (2), it follows that $\pi_* \mathcal{O}_{\tilde{\mathfrak{g}}}$ is coherent and generically isomorphic to $\mathcal{O}_X$.

We showed last time, ~~under~~ assuming ~~a~~ the existence of $\mathfrak{g}^{good} \subset \mathfrak{g}$, that X is normal.

$\Rightarrow$ From (2), it follows that $\pi_* \mathcal{O}_{\tilde{\mathfrak{g}}} = \mathcal{O}_X$.

It remains to: (1) Construct $\mathfrak{g}^{good}$.
~~(2) Show that X^r is ~~ ~~the smooth locus~~
(2) Show that $\phi : \tilde{\mathfrak{g}}^r \xrightarrow{\sim} X^r$.

Lemma: $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$: for any $x \in \mathfrak{g}^r$, $\mu^{-1}(x)$ is finite.

Pf: Consider $\pi : \mathfrak{g} \to \mathfrak{g}/\!/G$; $x \in \mathfrak{g}/\!/G$.

We know that $\pi^{-1}(x)^r$ is a single open dense G-orbit in $\pi^{-1}(x)$.

$$\mu^{-1}(\pi^{-1}(x)) = \bigsqcup G \times^B (s+\mathfrak{u}) \twoheadrightarrow \pi^{-1}(x)$$

Now, $\dim \pi^{-1}(x)^r = \dim \mathfrak{g} - r = \dim G \times^B (s+\mathfrak{u})$

$\Rightarrow \dim(\text{fibers over } \pi^{-1}(x)^r) = 0$.

Pf of (2): $\phi : \tilde{\mathfrak{g}}^r \xrightarrow{\ \ } X^r = p_{\mathfrak{g}}^{-1}(\mathfrak{g}^r)$

$$\begin{tikzcd} \tilde{\mathfrak{g}}^r \arrow[r, "\phi"] \arrow[dr, "\text{quasi-finite}"'] & X^r \arrow[d, "\text{finite}"] \\ & \mathfrak{g}^r \end{tikzcd}$$

$\Rightarrow \phi$ is quasi-finite & proper

$\Rightarrow \phi$ is ~~proper~~ finite.

But $\phi_* \mathcal{O}_{\tilde{\mathfrak{g}}^r} = \mathcal{O}_{X^r} \Rightarrow$ it follows that $\tilde{\mathfrak{g}}^r \xrightarrow{\sim} X^r$.

Construction of $\mathfrak{g}^{good}$

$\mathfrak{h}^{good} = \{ h \in \mathfrak{h} ;\ \text{ann}(h) \subset R^+ \text{ has at most one element} \}$

$R = \{\text{roots}\}$

$\mathfrak{g}^{good} := \pi^{-1}(\mathfrak{h}^{good}/W) \cap \mathfrak{g}^r$

Fix Cartan $\mathfrak{t} \subset \mathfrak{g}$; then

$x \in \mathfrak{g}^{good} \Leftrightarrow x \in \mathfrak{g}^{rs}$ or x is conjugate to an element of the form $s + n$ where $s \in \mathfrak{t}^{good}$ & $n \in \mathfrak{g}_s^{reg}$.

Now, in the second case, $\mathfrak{g}_s = sl_2 \oplus \ker \alpha$ (for some root $\alpha \in R$)

Let's look at $\varpi : \mathfrak{t} \to \mathfrak{t}/W$ near s (mod W)

There will be $|W|/2$ sheets; each of which is the standard ramified double cover of the plane.

$$\begin{array}{c} X \\ \downarrow \\ \mathfrak{g} \text{ near } s+n \end{array}$$

has $|W|/2$ connected components.
and each connected piece looks like $X(sl_2)$

$$\begin{array}{c} \downarrow \\ sl_2 \text{ near } n \end{array}$$

and so is smooth.

Want to show:

(1) $X^{good} \subset X^{sm}$

(2) the complement of $\mathfrak{g}^{good}$ has codim ≥ 2 in $\mathfrak{g}$.

(1) is clear from above; and it is also clear that $\operatorname{codim}(\mathfrak{h}_{\mathfrak{g}} \setminus \mathfrak{h}_{\mathfrak{g}}^{good}, \mathfrak{h}_{\mathfrak{g}}) \geq 2 \Rightarrow \operatorname{codim}((\mathfrak{h}_{\mathfrak{g}} \setminus \mathfrak{h}_{\mathfrak{g}}^{good})/W, \mathfrak{h}_{\mathfrak{g}}/W) \geq 2$.

(since $\mathfrak{h}_{\mathfrak{g}}^{good}$ is the complement of intersections of pairs of hyperplanes.)

Now, $\mathfrak{g} \setminus \mathfrak{g}^{good} = \pi^{-1}((\mathfrak{h}_{\mathfrak{g}} \setminus \mathfrak{h}_{\mathfrak{g}}^{good})/W) \cup (\pi^{-1}(\mathfrak{h}_{\mathfrak{g}}^{good}/W) \setminus \mathfrak{g}^{r})$

this is of codim ≥ 2 by a dimension count

So, we are reduced to showing: $\pi^{-1}(\mathfrak{h}_{\mathfrak{g}}^{good}/W) \setminus \mathfrak{g}^{r}$ has codim ≥ 2. We reduce again to the case of sl_2, where in fact codim $\geq$ ~~2~~.

Corollary: For any $\chi \in \mathfrak{g}//G$

(1) $u^*: \mathbb{C}[\pi^{-1}(\chi)] \xrightarrow{\sim} \mathbb{C}[\text{any connected component of } u^{-1}(\pi^{-1}(\chi))]$

(2) $\pi^{-1}(\chi)$ is <u>reduced</u> & <u>normal</u>.

(3) For any $x \in \pi^{-1}(\chi)^{reg}$, $\pi^{-1}(\chi)$ is smooth at x.

<u>Pf:</u> (1) Fix $h \in \mathfrak{h}$ s.t. $\varpi(h) = \chi$ and consider its residue field $k(h)$ $(\simeq \mathbb{C})$ and the residue field of χ, $k(\chi)$ $(\simeq \mathbb{C})$.

A conn. component of $u^{-1}(\pi^{-1}(\chi))$ is of the form $G \times_B (s + \mathfrak{u})$, for some s in the Weyl orbit of h.

$$\Rightarrow \mathbb{C}[\nu^{-1}(h)] \simeq \mathbb{C}[\tilde{\mathfrak{g}}] \otimes_{\mathbb{C}[\mathfrak{h}]} k(h) \overset{\text{thm}}{\simeq} \mathbb{C}[X] \otimes_{\mathbb{C}[\mathfrak{h}]} k(h) = \mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{g}//G]} k(\chi) = \mathbb{C}[\pi^{-1}(\chi)].$$

(2) Follows from (1), since $\nu^{-1}(h)$ is <u>smooth</u>.

(3)
$$\begin{tikzcd}
& \tilde{\mathfrak{g}}^r \arrow[d, "\phi", "\wr"'] \arrow[ddr, bend left, "\nu"] & \\
& X^r \arrow[dl] \arrow[dr, "p_{\mathfrak{h}}"] & \\
\mathfrak{g}^r \arrow[dr, "\pi"'] & & \mathfrak{h}^r \arrow[dl, "\varpi"] \\
& \mathfrak{h}/W &
\end{tikzcd}$$

Claim: $\pi|_{\mathfrak{g}^r}$ is smooth.

~~Since we know that π is flat, it suffices to show that all fibers of $\pi|_{\mathfrak{g}^r}$ are smooth.~~

Claim: $\Pi|_{\mathfrak{g}^r}$ is smooth;

By f. flat base change, it suffices to show that $P_{\mathfrak{g}}|_{X^r}$ is smooth. But this is isomorphic to $V|_{\tilde{\mathfrak{g}}^r}$, which is smooth by construction.

$\Rightarrow$ $\pi^{-1}(x)^r$ is smooth, for all $x \in \mathfrak{h}/W$.

Reformulation of (3): $f \in \mathbb{C}[\mathfrak{g}]$, $x \in \mathfrak{g}$.

$d_x f$: differential of f at x.

$$\begin{array}{c} \uparrow \\ \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g} \end{array} \qquad \left(df \in \mathfrak{g}^* \otimes \mathbb{C}[\mathfrak{g}] \xrightarrow{\sim} \mathfrak{g}^* \right)$$
$$df \mapsto d_x f$$

Remark: if $f \in \mathbb{C}[\mathfrak{g}]^G$, then $d_x f \in \mathfrak{g}_x$.

Now, by Chevalley, $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[p_1, \dots, p_r]$

$$x = (x_1, \dots, x_r) \in \mathfrak{g}/\!/G \simeq \mathbb{A}^r_{\mathbb{C}}$$

$$\pi^{-1}(x) = \{p_1 = x_1, \dots, p_r = x_r\}$$

So, the corollary says that, for any $x \in \pi^{-1}(x)^r$, $d_x p_1, d_x p_2, \dots, d_x p_r$ form a basis for $\mathfrak{g}_x$.

Example:

$\mathfrak{g} = \mathfrak{sl}_n$: Choose $p_i(x) = \mathrm{tr}(x^i)$; $i = 2, \dots, n$

$d_x p_i = i x^{i-1}$

$\Rightarrow$ $\mathfrak{g}_x = \mathrm{span}\{x^{i-1} : 2 \leq i \leq n\}$.

Alternative approach:

Fix $e \in \mathfrak{g}$ a regular nilpotent triple.

$\exists\ sl_2 = \langle e, h, f \rangle \subset \mathfrak{g}$.

Claim: $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{g}_f$.

$\dim \mathfrak{g}_f = \dim \mathfrak{g}_e = r = \dim \mathfrak{h}/W$.

Now, $T_e(G \cdot e) = [\mathfrak{g}, e]$

$e + \mathfrak{g}_f \subset \mathfrak{g}$.

Kostant or Slodowy slice.

Prop (Kostant) The map
$$\varphi_e : e + \mathfrak{g}_f \hookrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\!/G$$
is an isomorphism; i.e. we have
$$\varphi_e^* : \mathbb{C}[\mathfrak{h}]^W \xrightarrow{\sim} \mathbb{C}[e + \mathfrak{g}_f].$$

Corollary: (1) $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial algebra.

(2) $\pi^{-1}(\pi(e))^r$ is smooth at e.

Prop: For any $x \in \mathfrak{g}/\!/G$, $\pi^{-1}(x) \setminus \pi^{-1}(x)^r$ has codim ≥ 2 in $\pi^{-1}(x)$.

Pf:

Claim: For any $x \in \mathfrak{g} \setminus \mathfrak{g}^r$, $\dim \mu^{-1}(x) \geq 1$.

Given this claim, consider

$$\begin{array}{ccc} G \times^B (\mathfrak{s}+\mathfrak{u}) \simeq \left\{ \begin{matrix} \text{conn comp.} \\ \text{of } \mu^{-1}(\pi^{-1}(x)) \end{matrix} \right\} & \xrightarrow{\mu} & \pi^{-1}(x) \\ & & \cup \\ & & \pi^{-1}(x)^r \end{array}$$

If $\pi^{-1}(x) \setminus \pi^{-1}(x)^r$ has codim 1, then:

$\dim \mu^{-1}(\pi^{-1}(x) \setminus \pi^{-1}(x)^r)$

LECTURE 7.

$$\begin{array}{c} \tilde{\mathfrak{g}} \\ \downarrow \mu \\ \mathfrak{g} \end{array} \qquad \mathcal{B}_x = \mu^{-1}(x) = \{ \underbrace{b \subset \mathfrak{g}}_{\text{Borel}} : x \in b \}.$$

If x is <u>regular</u>, then $\mathcal{B}_x$ is finite.

If $x = 0$, then $\mathcal{B}_x = \mathcal{B}$

<u>Lemma</u>: $x \notin \mathfrak{g}^r \Rightarrow \dim \mathcal{B}_x \geq 1$.

<u>Pf</u>: Using Jordan decomposition, we can reduce to the case where x is <u>nilpotent</u>.

Fix $b \supset u \ni x$; so $x \in u \setminus u^r$.

For sl_n, $u = \left\{ \begin{pmatrix} 0 & a_1 & & \times \\ & 0 & a_2 & \\ & & \ddots & a_{n-1} \\ 0 & & & 0 \end{pmatrix} \right\}$

x not regular $\Rightarrow$ ~~...~~ $a_i = 0, \exists i$.

~~u_0 Set~~

In general, ~~...~~ $\exists$ a <u>simple</u> root α s.t. the e_α component of x is 0.

Set ~~u~~ $u_\alpha = \bigoplus_{\substack{\beta \in \Phi^+ \\ \beta \neq \alpha}} u_\beta$

Let $\langle e_\alpha, h_\alpha, f_\alpha \rangle = sl_2$ be ~~the~~ an sl_2-triple for e_α.

We get a map

$$\mathbb{P}^1 = \mathcal{B}(sl_2) \hookrightarrow \mathcal{B}(\mathfrak{g})$$
$$b' \longmapsto b' + u_\alpha$$

Now, the image of this map lies in $\mathcal{B}_x$
$\Rightarrow \underline{\dim \mathcal{B}_x \geq 1}$.

$R \subset \mathfrak{h}^*$ root system, $R^\vee \subset \mathfrak{h}$ co-roots.

Weight lattice: $P = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \forall \alpha^\vee \in R^\vee\}$

Root lattice $Q = \mathbb{Z}$-span of $R = \sum_{\alpha \in R} \mathbb{Z} \cdot \alpha$.

sc : simply connected
ad : adjoint.

$$\begin{tikzcd} 1 \arrow[r] & Z(G^{sc}) \arrow[r] & G^{sc} \arrow[r] & G^{ad} \arrow[r] & 1 \\ 1 \arrow[r] & Z(G^{x}) \arrow[r] \arrow[u, equal] & T^{sc} \arrow[r] \arrow[u, hook] & T^{ad} \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

& $Lie(G^{sc}) = Lie(G^{ad})$.

We have ~~an~~ ~~[illegible]~~

$$0 \to X^*(T^{ad}) \to X^*(T^{sc}) \to Z(G^{sc})^\vee \to 0$$
$$\| \qquad\qquad \| \qquad\qquad \|$$
$$0 \to Q \longrightarrow P \longrightarrow Z(G^{x})^\vee \to 0$$

$$\Rightarrow Z(G^{sc}) \xrightarrow{\sim} (P/Q)^\vee.$$

Fix $\mathfrak{t} \subset \mathfrak{g}$ Cartan; given $\mathfrak{g}$-rep V,

$V^{\mathfrak{t}=0}$ = zero wt space.

Lemma: Let V be a f.d. irrep of $\mathfrak{g}$, then the following are equivalent:

1. $V^{\mathfrak{t}} \neq 0$
2. $\mathrm{Weights}(V) \subset Q$
3. V can be exponentiated to a rep'n of G^{ad}.

(Clear for sl_2).

$G \curvearrowright X \qquad \mathrm{Map}(X, V) \cong \mathbb{C}[X] \otimes_{\mathbb{C}} V$

$x \in X$ gives a map $\mathrm{Map}(X,V)^G \longrightarrow V^{G_x} \subseteq V^{\mathrm{Lie}\, G_x}$

equality is G_x is connected.

$G = G^{ad} \curvearrowright \mathfrak{g}$.

Thm: $x \in \mathfrak{g}^r$, $\chi = \pi(x) \qquad \pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\!/G$.
V a f.d. rep^n of G^{ad} (so $\mathrm{Weights}(V) \subset Q$).

1. G^{ad}_x is connected.
2. $ev_x : \mathrm{Map}(\pi^{-1}(\chi), V)^G \xrightarrow{\simeq} V^{\mathfrak{g}_x} \; (= V^{G^{ad}_x})$
3. $\dim V^{\mathfrak{g}_x} = \dim V^{\mathfrak{t}}$

Pf: (1) is standard.
(2) $\pi^{-1}(\chi)^r$ is the conj. class of x. and
$\mathrm{codim}(\pi^{-1}(\chi) \setminus \pi^{-1}(\chi)^r, \pi^{-1}(\chi)) \geq 2$.
$\Rightarrow \mathrm{Map}(\pi^{-1}(\chi), V) \cong \mathrm{Map}(\pi^{-1}(\chi)^r, V)$
(since $\pi^{-1}(\chi)$ is normal)

$$\Rightarrow \mathrm{Map}(\pi^{-1}(x), V)^G = \mathrm{Map}(G/G_x, V)^G$$

$$\wr\!\!\downarrow$$

$$V^{\mathfrak{g}_x} = V^{G_x}$$

(3) $\mathbb{C}_x$: 1-dim $\mathbb{C}[\mathfrak{g}]^G$-module corresponding to $x : \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}$.

Kostant: $\mathbb{C}[\mathfrak{g}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \otimes_{\mathbb{C}} H$ ← v.s.
G-equivariant isom of $\mathbb{C}[\mathfrak{g}]^G$-modules.

$$\Rightarrow \mathbb{C}[\pi^{-1}(x)] \xrightarrow{\sim} \mathbb{C}_x \otimes_{\mathbb{C}} H$$

G-equivariant isom of $\mathbb{C}[\mathfrak{g}]^G$-modules.

$$\Rightarrow (\mathbb{C}[\pi^{-1}(x)] \otimes_{\mathbb{C}} V)^G \simeq (\mathbb{C}_x \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} H)^G$$

$$V^{\mathfrak{g}_x} \simeq (V \otimes_{\mathbb{C}} H)^G$$

does not depend on choice of $x \in \mathfrak{g}^r$

So, choose $x \in \mathfrak{t}$ regular semi-simple; then we obtain $V^{\mathfrak{t}} = (V \otimes_{\mathbb{C}} H)^G$. ∎

Corollary of proof: For any $V \in \mathrm{Irrep}(G^{ad})$,

Hom $(\mathbb{C}[\mathfrak{g}], V)^G$ is a free $\mathbb{C}[\mathfrak{g}]^G$-module of rank $= V^{\mathfrak{t}}$. (same dim at all fibers)

$$\mathbb{C}[\mathfrak{g}] \overset{G\text{-equiv}}{=} \bigoplus_{V \in \mathrm{Irrep}(G^{ad})} \mathbb{C}[\mathfrak{g}]^G \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V^{\mathfrak{t}}$$

$$
\begin{tikzcd}
 & \tilde{\mathcal N} \arrow[r] \arrow[dl] & \tilde{\mathfrak g} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathcal N \arrow[r, hook] \arrow[d] & \mathfrak g \arrow[dr, "\pi"] & & \mathfrak h \arrow[dl, "\varpi"] \\
0 \arrow[rr, hook] & & \mathfrak h/W &
\end{tikzcd}
$$

$\mathcal N := \pi^{-1}(0)$ is a reduced normal variety, the <u>nilpotent</u> variety.

Springer resolution

$\mu : \tilde{\mathcal N} \longrightarrow \mathcal N$ is a resolution of <u>singularities</u>

$\mu|_{\tilde{\mathcal N}^r} : \tilde{\mathcal N}^r \xrightarrow{\sim} \mathcal N^r$.

<u>Prop:</u> $\exists$ natural G-equivariant vector bundle isomorphism

$$
\begin{tikzcd}
G\times^B \mathfrak u = \tilde{\mathcal N} \arrow[rr, "\sim"] \arrow[dr, "p"'] & & T^*\mathcal B \arrow[dl] \\
 & \mathcal B &
\end{tikzcd}
$$

In particular, $\tilde{\mathcal N}$ is an alg. symplectic manifold; i.e. $\exists$ G-invariant non-degenerate symplectic two form $\omega \in \Lambda^2 \tilde{\mathcal N}$.

<u>Pf:</u>

$$T_{\mathfrak b}\mathcal B \simeq \mathfrak g/\mathfrak b \Rightarrow (T^*_{\mathfrak b}\mathcal B) \xrightarrow{\sim} (\mathfrak g/\mathfrak b)^\vee \xrightarrow[\text{under any } G\text{-invariant form on } \mathfrak g.]{\sim} \mathfrak u$$

Line bundles on $\mathcal{B}$

$B \subset G$ a Borel, $U \subset B$ its unipotent radical.

$H = B/U$, $\lambda \in X^*(H)$.

$\lambda \rightsquigarrow \mathbb{C}_{\lambda,B}$: 1-dim B-rep^n $(B \to H \xrightarrow{\lambda} \mathbb{C}^\times)$

$\mathcal{O}(\lambda) = G \times^B \mathbb{C}_{\lambda,B}$: line bundle over $\mathcal{B}$.

Let V_λ be the irred. f.d. $\mathfrak{g}$-rep with highest wt λ.

For every $B \subset G$, $\exists!\ l_B \subset V_\lambda$ w. $\dim l_B = 1$ and s.t. B acts on l_B via the character λ.

So we get a map $\varepsilon_\lambda : \mathcal{B} \to \mathbb{P}(V_\lambda)$.
$b \mapsto l_b$.

Then we see easily that $\varepsilon_\lambda^* \mathcal{O}(-1) \simeq \mathcal{O}(\lambda)$.

Def^n (1) $\lambda \in \mathfrak{h}^*$ is regular if $\langle \lambda, \alpha^\vee \rangle \neq 0$, $\forall \alpha^\vee \in R^\vee$.

(2) $P^{++} = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}, \forall \alpha^\vee \in R^\vee_+\}$

Suppose $\lambda \in -P^{++}$ is regular.

Then, $\forall B \in \mathcal{B}$, $\{g \in G : g l_B = l_B\} = B$, and we can see with some more work that
$\varepsilon_\lambda : \mathcal{B} \to \mathbb{P}(V_\lambda)$ is in fact a closed embedding.

Equivalently, $\mathcal{O}(\lambda) = \varepsilon_\lambda^* \mathcal{O}(1)$ is very ample.

$\Rightarrow H^i(\mathcal{B}, \mathcal{O}(\lambda)) = 0, \ \forall i > 0.$

<u>Thm</u>: If $\lambda \in -P^{++}$, then:

① for $i > 0$, $H^i(\mathcal{B}, \mathcal{O}(\lambda)) = 0$

$$H^i(\tilde{\mathcal{N}}, \underset{\substack{\| \\ p^*\mathcal{O}(\lambda)}}{\tilde{\mathcal{O}}(\lambda)}) = 0$$

Let Z be a smooth alg. variety and let ω_Z be its canonical bundle.

<u>Grauert-Riemenschneider vanishing</u>:

If $X \xrightarrow{f} Y$ is <u>proper</u> with X smooth, then

$$R^i f_* \omega_X = 0, \ \forall i > 0.$$

<u>Pf</u>:

We saw earlier that $\underline{\mathfrak{u}} \xrightarrow{\sim} \Omega^1_{\mathcal{B}}$ over $\mathcal{B}$.

Let Z be the total space $\mathbb{V}(\Omega^1_{\mathcal{B}} \oplus \mathcal{O}(-\lambda))$; so that

$$\begin{tikzcd} Z \simeq G \times^B (\mathfrak{u} \oplus \mathbb{C}_{-\lambda, B}) \arrow[d, "q"] \arrow[dd, bend right, "\hat{p}"'] \\ \tilde{\mathcal{N}} = \mathbb{V}(\underline{\mathfrak{u}}) \arrow[d, "p"] \\ \mathcal{B} \end{tikzcd}$$

$$\omega_Z \cong q^*\omega_{\tilde{\mathcal{N}}} \otimes_{\mathcal{O}_Z} \hat{p}^*\mathcal{O}(\lambda)$$

$$\simeq q^* \underbrace{\omega_{\mathbb{V}(\Omega^1_{\mathcal{B}})}}_{\substack{\text{canonically} \\ \text{trivial}}} \otimes_{\mathcal{O}_Z} \hat{p}^*\mathcal{O}(\lambda)$$

$$\Rightarrow \omega_Z \simeq \hat{p}^*\mathcal{O}(\lambda)$$

Now, $p_* \tilde{\mathcal{O}}(\lambda) \simeq \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathrm{Sym}(\underline{U}^\vee)$

$$\simeq \bigoplus_{n \geq 0} \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathrm{Sym}^n(\underline{U}^\vee)$$

$$\hat{p}_* \omega_Z \simeq \bigoplus_{n \geq 0} \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathrm{Sym}^n(\underline{U}^\vee \oplus \mathcal{O}(\lambda)).$$

$$\simeq \bigoplus_{m,n \geq 0} \mathcal{O}((m+1)\lambda) \otimes_{\mathcal{O}_B} \mathrm{Sym}^n \underline{U}^\vee$$

LECTURE 8

04/23/10

Theorem (Grauert – Riemenschneider)
$f : X \to Y$ proper, surjective & generically finite

Then $R^i f_* \omega_X = 0$, for $i > 0$.

Recall: We have $Z = \mathbb{V}(\Omega^1_{\mathcal{B}} \oplus \mathcal{O}(\lambda))$
(λ : antidominant)

Let $V_{-\lambda}$ = irrep of highest wt $-\lambda$.

Since $\Omega^1_{\mathcal{B}} \simeq \underline{\mathfrak{u}}$, we see that the points of Z are triples $\{(b, n, v) : b$ Borel, $n \in \mathfrak{u}(b), v \in \mathbb{C}_{\lambda, B}\}$

So we have a map $Z \xrightarrow{\hat{\mu}} \mathcal{N} \times V_{-\lambda}$
$(b, n, v) \mapsto (n, v)$

Forgetting v gives the resolution $\tilde{\mathcal{N}} \to \mathcal{N}$ and from this it follows that $\hat{\mu}$ is proper and birational onto its image.

So, by GR above, we have $R^i \hat{\mu}_* \omega_Z = 0$.

Using the Leray spectral sequence, we find:

$$H^i(Z, \omega_Z) = H^i(\mathcal{N} \times V_{-\lambda}, \hat{\mu}_* \omega_Z) = 0, \text{ for } i > 0$$

(since $\mathcal{N} \times V_{-\lambda}$ is affine).

Now, if $\hat{p}: Z \to \mathcal{B}$, we have

$$\hat{p}_* \omega_Z = \bigoplus_{m,n \geq 0} \mathcal{O}((m+1)\lambda) \otimes \mathrm{Sym}^n \underline{u}^*$$

$$\Rightarrow H^i(Z, \omega_Z) = \bigoplus_{m,n \geq 0} H^i(\mathcal{B}, \mathcal{O}((m+1)\lambda \otimes \mathrm{Sym}^n \underline{u}^*))$$

Putting these together, we find:

$$H^i(\mathcal{B}, \mathcal{O}((m+1)\lambda) \otimes \mathrm{Sym}^n \underline{u}^*) = 0, \ \forall m,n \geq 0.$$
and $i > 0$.

In particular, if $m = 0$, we find:

$$H^i(\mathcal{B}, \mathrm{Sym}\, \underline{u}^* \otimes \mathcal{O}(\lambda)) = 0$$

$$\cancel{\Rightarrow H^i(\tilde{\mathcal{N}}, \omega_{\tilde{\mathcal{N}}}) = 0 \text{ for } i > 0.}$$

$$\Rightarrow H^i(\tilde{\mathcal{N}}, \tilde{\mathcal{O}}(\lambda)) = 0, \text{ for } i > 0$$

If $\begin{Bmatrix} m = 0 \\ n = 0 \end{Bmatrix}$, we find:

$$H^i(\mathcal{B}, \mathcal{O}(\lambda)) = 0, \ \forall i > 0.$$

Conclusion: For $i > 0$, and λ anti-dominant,

$$\underline{H^i(\tilde{\mathcal{N}}, \tilde{\mathcal{O}}(\lambda)) = H^i(\mathcal{B}, \mathcal{O}(\lambda)) = 0}$$

Atiyah-Bott-Lefschetz

A torus $T \circlearrowright X$, X smooth compact cplx manifold.

$\mathcal{E}$ a T-equivariant vector bundle over X.

For $t \in T$, we set

$$\chi^T(t, \mathcal{E}) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(t \mid H^i(X, \mathcal{E})).$$

This is a polynomial function on T.

Theorem (A-B): Assume that X^T: set of fixed points is finite. Then:

$$\chi^T(t, \mathcal{E}) = \sum_{x \in X^T} \frac{\operatorname{Tr}(t \mid \mathcal{E}_x)}{\det(\mathrm{Id} - t)\mid_{T_x X}}$$

Application: $X = \mathcal{B}$, $T \subset B \subset G$; then

$$X^T = \{w \cdot x : w \in W, \ x \text{ corresponds to } B\}.$$

$$\Rightarrow \chi^T(t, \mathcal{O}(\lambda)) = \sum_{w \in W} \frac{(w\lambda)(t)}{\prod_{\alpha \in \Phi^+}(1 - (w\alpha)(t)^{-1})}$$

$$\underset{\text{Character formula}}{\overset{\text{Weyl}}{=}} \operatorname{ch}(V_\lambda^*)$$

Borel-Weil: For $\lambda \in -\Phi^{++}$,

$$\Gamma(\mathcal{B}, \mathcal{O}(\lambda)) = V_\lambda^*, \text{ as a } G\text{-module}.$$

Pf: Follows easily from Vanishing + A-B.

~~But since this uses the~~

But we can in fact avoid the Weyl character formula.

Direct pf: (a) Γ is either simple or zero
(b) $\Gamma \neq 0$.

Vanishing + AB $\Rightarrow ch(\Gamma) \neq 0 \Rightarrow \Gamma \neq 0$; so (b) is fine
(look at asymptotics as $t \to 1$)

$B = TU$; we will show that dim $\Gamma^U \leq 1$.

By definition:

$$\Gamma = \{ s \in \mathbb{C}[G] : s(gb) = s(g) \cdot \lambda(b)^{-1},\ g \in G, b \in B \}$$

G-action:

$$(g \cdot s)(h) = s(g^{-1}h).$$

Let $Uw_0B \subset G$ be the ~~open~~ pullback of the open Bruhat cell.

Let $s \in \Gamma^U$, then $s(uw_0u_1t_1) = s(w_0)\lambda(t_1)^{-1}$

$\Rightarrow$ s is completely determined by $s(w_0)$!

$\Rightarrow \dim \Gamma^U \leq 1$

Now, for $s \in \Gamma^U$, $t \in T$, $(ts)(w_0) = s(t^{-1}w_0)$

$$= s(w_0 w_0^{-1} t^{-1} w_0)$$
$$= s(w_0)\lambda(w_0^{-1} t w_0)$$
$$= (w_0\lambda)(t)\, s(w_0)$$

So, Γ is irrep of highest wt $w_0\lambda$.

$\Rightarrow$ $\Gamma \simeq V_\lambda^*$, where V_λ : irrep with <u>lowest</u> wt λ.

<u>Brylinski - Kostant</u>

$e: V \to V$ <u>nilpotent</u> operator.

$F_i V := \ker(e^{i+1})$

$0 = F_{-1} \subset F_0 \subseteq \ldots \subseteq \ldots \subseteq V$.

$W \subset V$ subspace $\rightsquigarrow$ we have the induced filtration $F_i W = W \cap F_i V$.

Let $N(e) : \mathbb{G}_a \to GL(V)$ be the corresponding 1-parameter sub-group.

For $d := \dim W$, we get a map

$$\begin{aligned} \mathbb{G}_a &\to Gr(d, V) \\ t &\mapsto N(e)(t) \cdot [W]. \end{aligned}$$

This extends uniquely to a map

$$\begin{aligned} \mathbb{P}^1 &\to Gr(d, V) \\ \infty &\mapsto \lim_e W \end{aligned}$$: "the limit filtration".

If $v \in F_i V$, then $N(e)(t)(v) = \sum_{n=0}^{i} \frac{t^n}{n!} e^n v$

So $\lim_{t \to \infty} N(e)(t)(t^{-i} v) = \lim_{t \to \infty} \sum_{n=0}^{i} \frac{t^{n-i}}{n!} e^n v$

$$= \frac{e^i v}{i!}$$

So, we get ~~an injective~~ map

$$\lim : \bigoplus_{i=0}^{\infty} gr_i^F W \Longrightarrow \lim_e W$$

$$\oplus \bar{v}_i \mapsto \sum e^i v_i$$

Fix $sl_2 = \langle e, h, f \rangle \subset \mathfrak{g}$, e: a regular nilp.

$\mathfrak{t} = \mathfrak{g}_h$ a Cartan; V a f.d. $\mathfrak{g}$-rep^n.

Now, $e : V \to V$ is a nilpotent operator; fix a wt μ of $\mathfrak{t}$ in V, and let $W = V(\mu) \subset V$.

So, as above, we get a map

$$\lim : \oplus\, gr^F V(\mu) \hookrightarrow \lim_e V(\mu).$$

Lemma: (1) lim is an isomorphism.

(2) Suppose $\mu = 0$ & Weights$(V) \subset Q$.

Then: $\lim_e V^{\mathfrak{t}} = V^{\mathfrak{g}_e}$

Pf: (2) follows from (1) + $\dim V^{\mathfrak{t}} = \dim V^{\mathfrak{g}_e}$

For (1), if $v \in V(\mu)$, then $h \cdot v = \mu(h) v$

$$\Rightarrow h e^i v = (\mu(h) + 2i) v.$$

From this the linear independence of the $e^i v$ follows and hence the injectivity of the map $\underline{\lim}$.

It's clear that $\underline{\text{the image of } \lim \text{ lies in } V^{\mathfrak{g}_e}}$.

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \quad Q_+ = \sum_{\alpha \in \Phi^+} \mathbb{Z}_{\geq 0} \alpha$$

<u>Kostant partition function:</u>

$$\frac{1}{\prod_{\alpha \in \Phi^+} (1 - \alpha)} \overset{\text{by geometric series}}{=} \sum_{\mu \in Q_+} P(\mu) \cdot \mu$$

number of ways to write μ as a sum of positive roots.

If V_λ is an irrep w. highest wt λ, then

by WCF, we have: $\dim V_\lambda(\mu) = \sum_{w \in W} P(w(\lambda + \rho) - \mu - \rho)$

<u>Lusztig's q-analog:</u> $\displaystyle \frac{1}{\prod_{\alpha \in \Phi^+} (1 - q\alpha)} = \sum_{\mu \in Q_+} P_q(\mu) \cdot \mu$

a polynomial in q. $\in \mathbb{Z}_{\geq 0}[q]$.

Theorem (Kostant–Brylinski),

For $\mu, \lambda \in P^{++}$,

$$\sum_i q^i \dim \mathrm{gr}_i V_\lambda(\mu) = \sum_w \mathcal{P}_q(w(\lambda+\rho)-\mu-\rho)$$

Reformulation:

Consider $\chi^T(t, \mathcal{O}(-\mu) \otimes \mathrm{Sym}^i \underline{u}^*) = \sum_{w \in W} \underline{\text{traces}}$

Now $\sum q^i \, \mathrm{Tr}(t \mid \mathrm{Sym}^i u) = \dfrac{1}{\prod_{\alpha \in \Phi^+} (1-q\cdot\alpha(t))}$

So, one checks that the theorem is equivalent to showing:

$$\sum_{\lambda \in P^{++}} \dim \mathrm{gr}_i V_\lambda(\mu) \cdot \mathrm{ch}(t, V_\lambda^*) = \chi^T(t, \mathcal{O}(-\mu) \otimes \mathrm{Sym}^i \underline{u}^*)$$

$\Big\|$ Vanishing

$$\mathrm{Tr}(t \mid \Gamma(\mathcal{O}(-\mu) \otimes \mathrm{Sym}^i \underline{u}^*))$$

So it suffices to prove:

$$\underbrace{\Gamma(\mathcal{B}, \mathcal{O}(-\mu) \otimes \mathrm{Sym}^i \underline{u}^*)}_{\Gamma} \simeq \bigoplus_{\lambda \in P^{++}} \dim \mathrm{gr}\, V_\lambda(\mu) \cdot V_\lambda^*.$$

Or: that $\boxed{\mathrm{gr}_i V_\lambda(\mu) \simeq \mathrm{Hom}_G(V_\lambda^*, \Gamma)}$

Consider the special case where $u = 0$.

Then we want to show:

$$\mathrm{gr}_i V_\lambda^t \simeq \mathrm{Hom}_G(V_\lambda^*, \Gamma(\mathrm{Sym}^i \underline{u}^*))$$

Now, $\mathbb{C}[\mathfrak{g}] \simeq \bigoplus_\lambda \mathbb{C}[\mathfrak{g}]^G \otimes V_\lambda \otimes V_\lambda(0)$.

$$\Rightarrow V_\lambda^t = \mathrm{Hom}_G(V_\lambda, \mathbb{C}[G \cdot h])$$

Now, $\mathbb{C}[G \cdot h]$ is equipped with the filtration induced from the filtration on $\mathbb{C}[\mathfrak{g}]$ by degree.

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$$\begin{tikzcd} \mathfrak{g} \supset \nu^{-1}(h) \cong G\times_B (h+\mathfrak{u}) =: \mathcal{N}_h \arrow[d, "\nu"] \arrow[rr, phantom] & & \mathcal{N}_h \arrow[d, "p_h"] \\ \mathfrak{h}_{\mathfrak{g}} \ni h & & \mathcal{B} \ni \mathfrak{b} \end{tikzcd}$$

$$\Gamma_h(\mu) := \Gamma(\mathcal{N}_h, p_h^* \mathcal{O}(\mu)) \longrightarrow \Gamma(h+\mathfrak{u}, \mathbb{C}_{\mu,\mathfrak{b}})$$
$$\| $$
$$\mathrm{Map}(h+\mathfrak{u}, \mathbb{C}_{\mu,\mathfrak{b}})$$

equipped with a filtration by "degree".

$$\Gamma_h^{\leq i}(\mu) = \{ s \in \Gamma_h(\mu) : \forall \mathfrak{b} \in \mathcal{B},\ s|_{h+\mathfrak{u}} \text{ has deg} \leq i \}.$$

Assume

$G = G^{ad}$; ~~$\mathfrak{g}$~~ and V: f.d. G-rep^n

$$\Gamma_h(\mu, V) := \Gamma_h(\mu) \otimes V \xrightarrow[\text{for each } \mathfrak{b}\in\mathcal{B}]{} \mathrm{Map}(h+\mathfrak{u}, \mathbb{C}_{\mu,\mathfrak{b}} \otimes V)$$

Using these evaluation maps, we can define a filtration

$\Gamma_h^{\leq i}(\mu, V)$ on $\Gamma_h(\mu, V)$

G acts diagonally.

~~That~~ Now, suppose ~~$\mathfrak{g}$~~ h is part of a regular sl_2-triple $\langle e, h, f\rangle \subset \mathfrak{g}$.

[So we're choosing a Cartan $\mathfrak{g}_h = \underset{h}{\underset{\cup}{\mathfrak{t}}} \subset \mathfrak{b}$]

$ev_h : \Gamma(-\mu, V) \longrightarrow \mathbb{C}_{-\mu,B} \otimes V \qquad (B = \exp(\mathfrak{b})).$

<u>Lemma:</u> ① $ev_h : \Gamma(-\mu, V)^G \xrightarrow{\sim} (\mathbb{C}_{-\mu,B} \otimes V)^T \quad (T = \exp(\mathfrak{t}))$

$$\begin{tikzcd} & (\mathbb{C}_{-\mu,B} \otimes V)^T \arrow[d, equal] \\ \Gamma(-\mu,V)^G \arrow[r, "\simeq"] & V(\mu) \end{tikzcd}$$

② $ev_h : \Gamma^{\leq i}(-\mu, V)^G \xrightarrow{\sim} F_i V(\mu)$

(Brylinski filtration).

Pf: ① $(\mathfrak{b}, h) \in \mathcal{W}_h \subset \tilde{\mathfrak{g}}$

$$\begin{tikzcd} (\mathfrak{b},h) \in \mathcal{W}_h \arrow[d, "\simeq", "\mu"'] \arrow[r, phantom, "\subset"] & \tilde{\mathfrak{g}} \arrow[d, "\mu"] \\ G/T \simeq Ad(G)(h) \arrow[r, phantom, "\subset"] & \mathfrak{g}. \end{tikzcd}$$

From this, ① follows.

② We know: $Ad(B)(e) = \mathfrak{u}^{reg} \subset \mathfrak{u}$ is open dense.

$\Rightarrow$ lines of the form $Ad(b)(h + \mathbb{C}e) + c$ where $b \in B$, $c \in \mathfrak{u}$ form a dense subset in the set of affine lines in $h + \mathfrak{u}$.

$\Rightarrow$ for $s \in \Gamma(-\mu, V)^G$, $s|_{h+\mathfrak{u}}$ is a B-equivariant map $h + \mathfrak{u} \longrightarrow \mathbb{C}_{-\mu,B} \otimes V$

Also, $s \in \Gamma_h^{\leq i}(-\mu, V) \Longleftrightarrow s|_{c + b(h+\mathbb{C}e)}$ has deg $\leq i$, for $c \in \mathfrak{u}$, $b \in B$
(since s is G-invariant)

$\Longleftrightarrow s|_{h+\mathbb{C}e}$ has deg $\leq i$
($s|_{h+\mathfrak{u}}$ is B-inv.)

Note:

$$\exp(-te)h = h - t[e,h] + \frac{t^2}{2}[e,[e,h]] + \dots$$
$$= h + 2te$$

$$s(h+2te) = s(\exp(-te)h)$$
$$= \exp(-te)\,s(h)$$

Conclusion: $t \mapsto s(h+2te)$ has $\deg \leq i$
$\Leftrightarrow \exp(-te)^{s(h)}$ has $\deg \leq i$ in t
$\Leftrightarrow s(h) \in F_i V(\mu)$.

Now, by the Lemma, we have an isomorphism

$$\mathrm{gr}_i^{\deg} \Gamma_h(-\mu, V)^{G} \xrightarrow[\sim]{ev_h} F_i V(\mu)/F_{i+1}V(\mu).$$

Last time, we were reduced to showing

$$[\Gamma(\mathcal{B}, \mathcal{O}(-\mu) \otimes \mathrm{Sym}^i \underline{u}^*) \otimes V]^G \simeq \frac{F_i V(\mu)}{F_{i+1}V(\mu)}$$

So it remains to show:

Prop: $\mathrm{gr}_i^{\deg} \Gamma_h(-\mu, V)^G \xrightarrow{\sim} [\Gamma(\mathcal{B}, \mathcal{O}(-\mu) \otimes \mathrm{Sym}^i \underline{u}^*) \otimes V]^G$

Pf: Consider $p_h : \mathcal{N}_h \longrightarrow \mathcal{B}$.

$$p_{h*}p_h^*\mathcal{O}(\mu) = \mathcal{O}(\mu) \otimes p_*\mathcal{O}_{\mathcal{N}_h}$$

Since $\mathcal{N}_h$ is an affine bundle over $\mathcal{B}$, again, $p_*\mathcal{O}_{\mathcal{N}_h}$, and hence $p_{h*}p_h^*\mathcal{O}(\mu)$, is equipped with a "degree" filtration.

$$\mathcal{F}^i(\mu) = \mathcal{O}(\mu) \otimes [p_{n*}\mathcal{O}_{W_n}]^{\leq i}$$

$$\Gamma_n(\mu) = \Gamma(W_n, p_n^* \mathcal{O}(\mu))$$

$$= \Gamma(\mathcal{B}, \mathcal{O}(\mu) \otimes (p_n)_* \mathcal{O}_{W_n})$$

So: $\Gamma_n^{\leq i}(\mu) = \Gamma(\mathcal{B}, \mathcal{F}^i(\mu))$.

Now, we have:

$$0 \to \mathcal{F}^{i-1}(\mu) \to \mathcal{F}^i(\mu) \to \mathcal{O}(\mu)\otimes \mathrm{Sym}^i \underline{\mathfrak{u}}^* \to 0$$

If $\mu \in P^{++}$, then $-\mu$ is anti-dominant, and so, by our vanishing theorems, we have

$$H^1(\mathcal{B}, \mathrm{Sym}^i \underline{\mathfrak{u}}^* \otimes \mathcal{O}(-\mu)) = 0.$$

$\Rightarrow$ we get: $H^1(\mathcal{F}^{i-1}(-\mu)) \twoheadrightarrow H^1(\mathcal{F}^i(\mu))$

Moreover $H^1(\mathcal{B}, \mathcal{F}^0(-\mu)) = H^1(\mathcal{B}, \mathcal{O}(-\mu)) = 0$
(again, by vanishing).

$\Rightarrow H^i(\mathcal{F}^i(-\mu)) = 0, \ \forall i.$

$\Rightarrow 0 \to \Gamma_n^{\leq i}(\mu) \hookrightarrow \Gamma_n^{\leq i}(\mu) \to \Gamma(\mathcal{B}, \mathcal{O}(-\mu) \otimes \mathrm{Sym}^i \underline{\mathfrak{u}}^*) \to 0$

taking H^0

Tensoring with V and taking G-invariants (exact process since G is reductive)

we get our proposition.

Differential Operators.

k: field of char 0.

A: f.s. commutative k-algebra.

$a \in A$, $m_a : A \to A$, $b \mapsto ab$ gives an embedding
$A \overset{m}{\hookrightarrow} \mathrm{End}_k(A)$.

For each $i = -1, 0, 1, 2, \ldots$, define

$$\mathcal{D}_i A \subset \mathrm{End}_k A \text{ by}$$

① $\mathcal{D}_{-1} A = 0$

② $\mathcal{D}_i A = \{ u \in \mathrm{End}_k A : [u, m_a] \in \mathcal{D}_{i-1} A \ \forall a \in A \}$

Rmks: (a) $\mathcal{D}_0 A \xrightarrow{\sim} A$
$\quad u \mapsto u(1)$

(b) $u \in \mathcal{D}_i A,\ v \in \mathcal{D}_j A \Rightarrow uv, vu \in \mathcal{D}_{i+j} A$.

(c) $u \in \mathcal{D}_i A,\ v \in \mathcal{D}_j A \Rightarrow [u,v] \in \mathcal{D}_{i+j-1} A$.

Note: $[m_a, uv] = [m_a, u]v + u[m_a, v]$

We have: $A = \mathcal{D}_0 A \subset \mathcal{D}_1 A \subset \mathcal{D}_2 A \subset \cdots$

$$\mathcal{D} A = \bigcup_{i \geq 0} \mathcal{D}_i A : \text{ a } \underline{\text{filtered}}, \underline{\text{associative}} \text{ sub-algebra of } \mathrm{End}_k(A).$$

By (c) above, $\mathrm{gr}\,\mathcal{D}A$ is <u>commutative</u>.

Jets: $A \otimes_k A \supset I = (a \otimes 1 - 1 \otimes a)$ the ideal of the diagonal.

$$J^n A := \frac{A \otimes_k A}{I^{n+1}}$$: functions on the n^{th} infinitesimal nbd of diagonal.

$\uparrow$ an A bi-module.

Now

$$\mathrm{Hom}_k(A,A) \simeq \mathrm{Hom}_A(A \otimes_k A, A)$$
$$u \mapsto \tilde{u}$$

For $a \in A$, what is $\tilde{m}_a$?

We have: $\widetilde{[m_a, u]} = \tilde{u} \circ (m_a \otimes 1 - 1 \otimes m_a)$

Indeed: $\widetilde{[m_a, u]}(f \otimes g) = fau(g) - fu(ag)$
$= \tilde{u}(af \otimes g - f \otimes ag)$

So we see:

$$u \in \mathcal{D}_n A \iff \tilde{u}|_{I^{n+1}} = 0$$

i.e. $\tilde{u} \in \mathrm{Hom}_A(J^n A, A)$

$$\mathcal{D}_n A \xrightarrow{\sim} \mathrm{Hom}_A(J^n A, A)$$ as A-bi-modules.

Now, we set $A = k[X]$, where X is an affine variety.

$$\mathcal{D}_i(X) = \mathcal{D}_i A, \quad J^i_X = J^i A.$$

$$I/I^2 = \Omega^1_{X/k}; \quad \mathrm{Hom}_A(I/I^2, A) = \mathrm{Der}_k A =: T_{X/k}$$

Dualizing $0 \to \Omega^1_{X/k} \to J^1_X \to \mathcal{O}_X \to 0$, we get:

$$0 \to \mathcal{O}_X \to \mathcal{D}_1(X) \to T_{X/k} \to 0$$
$$u \mapsto u - m_{u(1)}$$

In generality, we have, for any multiplicative subset $S \subset A$, we have

$$S^{-1}A \otimes_A \mathcal{J}^i(A) \xrightarrow{\sim} \mathcal{J}^i(S^{-1}A) \xrightarrow{\sim} \mathcal{J}^i(A) \otimes_A S^{-1}A.$$

And so we get:

$$\mathcal{D}_i A \otimes_A S^{-1}A \xrightarrow{\sim} \mathcal{D}_i(S^{-1}A) \xrightarrow{\sim} S^{-1}A \otimes_A \mathcal{D}_i A.$$

Now, suppose X is smooth: then $\Omega^1_{X/k}$ is locally free of finite rank, and we have:

$$\mathrm{Sym}^n(I/I^2) \xrightarrow{\sim} I^n/I^{n+1}.$$

So we have:

$$0 \to \mathrm{Sym}^n \Omega^1_{X/k} \to \mathcal{J}^n_X \to \mathcal{J}^{n-1}_X \to 0$$

a short exact seq. of locally frees.

$\Rightarrow$ Dualizing, we get:

$$0 \to \mathcal{D}_{n-1}(X) \to \mathcal{D}_n(X) \xrightarrow[\text{principal symbol map.}]{} \mathrm{Sym}^n T_{X/k} \to 0$$

$$\Rightarrow \operatorname{gr} \mathcal{D}(X) \xrightarrow{\sim} \mathrm{Sym}^{\bullet} T_{X/k} = \mathcal{O}_{\mathbb{V}(\Omega^1_{X/k})}$$

Prop: $\underline{U(\mathcal{D}_1(X))} \xrightarrow{\cong} \mathcal{D}(X)$

$\left(\begin{array}{c} f\otimes u - fu,\ \forall f\in \mathcal{D}_0, u\in \mathcal{D}_1 \\ 1_u - 1_A \end{array}\right)$

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A smooth

$$\tilde{\mathcal{D}} := \frac{\mathcal{U}(\mathcal{D}_1)}{\langle f\otimes u - fu,\ 1_u - 1_A \rangle} \xrightarrow[\sim]{F} \mathcal{D}(A)$$

Pf: This is a map of A-algebras.

Let $\mathcal{U}_{\leq i}(\mathcal{D}_1)$ be the PBW filtration on $\mathcal{U}(\mathcal{D}_1)$. and set

$$F_i\tilde{\mathcal{D}} = A\cdot \operatorname{Im} \mathcal{U}_{\leq i}(\mathcal{D}_1).$$

The map $\operatorname{Sym}(\mathcal{D}_1) \twoheadrightarrow \operatorname{gr}\mathcal{U}(\mathcal{D}_1)$ descends to a map

$$\begin{tikzcd}
\operatorname{Sym}_A T^!_X \arrow[r, "\cong"] \arrow[ddrr, "\text{is the identity}"'] & \operatorname{Sym}_A(\mathcal{D}_1/\mathcal{D}_0) \arrow[r, two heads] & \operatorname{gr}^F\tilde{\mathcal{D}} \arrow[d] \\
& & \operatorname{gr}\mathcal{D}(A) \arrow[d, "S\ \text{principal symbol}"] \\
& & \operatorname{Sym}_A T^!_X
\end{tikzcd}$$

$\Rightarrow$ $\operatorname{gr} F$ is an isomorphism

$\Rightarrow$ F is an isomorphism.

Corollary: The principal symbol map is an algebra isomorphism and its inverse is given by

$$\operatorname{Sym}_A T^!_X \xrightarrow{\sim} \operatorname{Sym}_A(\mathcal{D}_1/\mathcal{D}_0) \twoheadrightarrow \operatorname{gr}\mathcal{D}(A)$$

induced by $T^!_X \longrightarrow \operatorname{gr}\mathcal{D}(A)$.

Let $p: X \to \mathbb{A}^n$ be an étale map, and let $t_1, \dots, t_n$ be the co-ordinates on $\mathbb{A}^n$.

Set $\partial_i := p^* \frac{\partial}{\partial t_i}$: they give a global trivialization for $T^!_X$.

<u>Prop</u>: $\mathcal{D}_X = \mathcal{O}_X\{\partial_1, \dots, \partial_n\} = \{\sum f_{k_1 \dots k_n} \partial_1^{k_1} \cdots \partial_n^{k_n} : f_{k_1 \dots k_n} \in \mathcal{O}_X\}$

<u>Pf:</u> As before, reduced to checking isomorphism on the associated graded level.

Let X be a <u>smooth</u>, not necessarily affine alg. variety. By the localization property of the algebra of differential operators; i.e. the fact that if we have $\operatorname{Spec} A[f^{-1}] \hookrightarrow \operatorname{Spec} A$, then $D_{A[f^{-1}]} \simeq D_A \otimes_A A[f^{-1}]$ canonically, we have a <u>sheaf</u> D_X of diff. operators on X.

Moreover, we get the principal symbol map

$$\operatorname{gr} \mathcal{D}_X \xrightarrow{\sim} \operatorname{Sym}_{\mathcal{O}_X} T_X' \quad \left(\begin{array}{l}\text{i.e. the filtrations}\\ \text{also localize nicely}\end{array}\right)$$

If $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$, then we get an <u>injection</u>

$$\operatorname{gr} \mathcal{D}(X) \hookrightarrow \Gamma(X, \operatorname{Sym}_{\mathcal{O}_X} T_X') = \mathbb{C}[V(\Omega_X^1)]$$

G: a connected, linear alg. group.

$G \curvearrowright X$; then $\mathcal{D}_X$ is a G-equivariant ~~sheaf~~ filtered sheaf of $\mathcal{O}_X$-modules.

$\mathcal{D}(X)^G \subset \mathcal{D}(X)$ is the sub-algebra of G-invariant operators and we have

$$\operatorname{gr} \mathcal{D}(X)^G \hookrightarrow \mathbb{C}[\mathbb{V}(\Omega^1_X)]^G$$

$$\mathfrak{g} := \operatorname{Lie}(G) \xrightarrow{a} \Gamma(X, T^1_X) \text{ extends to}$$

$$\mathcal{U}(\mathfrak{g}) \xrightarrow{a} \mathcal{D}(X) \quad G\text{-equivariant.}$$

$$a(\operatorname{Ad}(g)u) = g_*(a(u))$$

In particular, we have

$$Z(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^G \xrightarrow{a} \mathcal{D}(X)^G.$$

Also, a takes the PBW filtration to the order filtration and so gives a map

$$\begin{array}{ccc} \operatorname{gr} \mathcal{U}(\mathfrak{g}) & \xrightarrow{\operatorname{gr}(a)} & \operatorname{gr} \mathcal{D}(X) \\ \text{PBW} \downarrow \wr & & \downarrow \\ \operatorname{Sym}(\mathfrak{g}) & & \\ \| & & \\ \mathbb{C}[\mathfrak{g}^*] & \longrightarrow & \mathbb{C}[\mathbb{V}(\Omega^1_X)] \end{array}$$

Lemma: The map $\operatorname{gr}(a): \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathbb{V}(\Omega^1_X)]$ above is induced by a map of algebraic varieties
$\mathbb{V}(\Omega^1_X) \xrightarrow{\mu} \mathfrak{g}^*$ (The Moment Map)

given, for $x \in X$, $\phi_x \in \Omega^1_X \otimes k(x)$, by the formula:

$$\mu(\phi_x) : \mathfrak{g} \longrightarrow \mathbb{C}$$

$$V \mapsto \langle a(V)_x, \phi_x \rangle \in \mathbb{C}.$$

Pf: Essentially we're just using the fact that the differentials on $\mathfrak{g}^*$ are canonically trivialized.

Let $H = \mathbb{G}_m^r$, $\mathfrak{h} = \mathrm{Lie}\, H$, $q : \tilde{X} \longrightarrow X$ be an H-torsor. We get:

$$0 \to \mathfrak{h} \otimes \mathcal{O}_{\tilde{X}} \to T^1_{\tilde{X}} \to q^* T^1_X \to 0 :$$

Then $\mathfrak{h} \otimes \mathcal{O}_X \subset Z((q_* T^1_{\tilde{X}})^H)$ (Lie algebra center).

Now, $T^1_X = (q_* q^* T^1_X)^H$ $[= (T^1_X \otimes q_* \mathcal{O}_{\tilde{X}})^H \simeq T^1_X]$.

By the exact sequence above, we see that we have:

$$T^1_X = (q_* T^1_{\tilde{X}} / \mathfrak{h})^H \text{ (by reductivity of } H).$$

In general, we have

$$a : \mathfrak{h} \hookrightarrow Z((q_* \mathcal{D}_X)^H) \text{ and}$$

$$\mathcal{D}_X \simeq (q_* \mathcal{D}_{\tilde{X}})^H / a(\mathfrak{h}) (q_* \mathcal{D}_{\tilde{X}})^H \subset \text{right ideal}.$$

holds without any condition on H. $\quad \simeq (q_* \mathcal{D}_{\tilde{X}} / a(\mathfrak{h}) q_* \mathcal{D}_{\tilde{X}})^H$

Geometrically: Consider the moment map

$$\mathbb{V}(\Omega^1_{\tilde{X}}) \xrightarrow{\mu} \mathfrak{h}^*.$$

Since H is abelian, we get:

$$\mathbb{V}(\Omega^1_{\tilde{X}})/H \xrightarrow{\bar{\mu}} \mathfrak{h}^*.$$

(In general, we would get $\mathbb{V}(\Omega^1_{\tilde{X}})/H \xrightarrow{\mu} \mathfrak{h}^*/_{Ad(H)}$.)

We have: $\bar{\mu}^{-1}(0) = \mu^{-1}(0)/_H$ (since H is reductive)

Claim: $\bar{\mu}^{-1}(0) \simeq \mathbb{V}(\Omega^1_X)$.

If X is affine, then so is $\tilde{X}$.

$$\mathbb{C}[\mu^{-1}(0)] = \frac{\mathbb{C}[\mathbb{V}(\Omega^1_{\tilde{X}})]}{\mathbb{C}[\mathbb{V}(\Omega^1_{\tilde{X}})]\mathfrak{h}} = \frac{(\operatorname{Sym} T^1_{\tilde{X}})}{(\operatorname{Sym} T^1_{\tilde{X}})\mathfrak{h}}.$$

$$\mathbb{C}[\mu^{-1}(0)/H] = \operatorname{Sym}(T^1_{\tilde{X}})^H/_{\operatorname{Sym}(T^1_{\tilde{X}})^H\mathfrak{h}}.$$

So, what we have above with $\mathcal{D}_{\tilde{X}}$ & $\mathcal{D}_X$ is a quantization of this picture.

Now, $G \curvearrowright G$ by left translation; then we get a map

$$a_\ell : \mathfrak{g} \to \Gamma(G, T^1_G)^{G_r} \quad \text{((right translation invariants)}$$

Prop: We have the following isom:

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \arrow[r, "\sim"', "a"] & \mathcal{D}(G)^{G_r} \\ Z(\mathfrak{g}) \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & \mathcal{D}(G)^{G_\ell \times G_r} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

(so $\mathcal{U}(\mathfrak{g})$ can be imbedded as either left or right invariant diff. operators)

Pf: As always, we look at the map on the level of associated gradeds.

$$\begin{tikzcd} \mathrm{Sym}_{\mathcal{O}_G}(T^1_G)^{G_r} \simeq \mathrm{Sym}_{\mathcal{O}_G}(\mathfrak{g}\otimes \mathcal{O}_G)^{G_r} \arrow[d, "\simeq"'] & \\ & \simeq \mathrm{Sym}\,\mathfrak{g}. \arrow[dll, equal] \arrow[d, "\simeq"] \\ \mathrm{Sym}(\mathfrak{g}) & \mathrm{gr}\,\mathcal{D}(G)^{G_r} \arrow[l, "\simeq"'] & \mathrm{gr}(\mathcal{U}(\mathfrak{g})) \arrow[l, "\mathrm{gr}(a)"'] \end{tikzcd}$$

Cor: $\mathrm{gr}\,\mathcal{D}(G)^{G_r} \simeq \mathrm{Sym}(\mathfrak{g})$

PBW! (Used that $\mathfrak{g}$ can be integrated to G).

Fix G-invariant isom $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ and thus a G-invariant isom. $\mathrm{Sym}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]$.

depends on identification $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

Lemma $G \circlearrowright \mathcal{B}$

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[r, "\sim"] \arrow[d, "\mu"', "\text{Springer resolution}"] & \mathbb{V}(\Omega^1_{\mathcal{B}}) \arrow[d, "\mu", "\text{moment map}"'] \\ \mathcal{N} \arrow[r, hook] & \mathfrak{g} = \mathfrak{g}^* \end{tikzcd}$$

$\mathcal{U}(\mathfrak{g}) \supset \mathfrak{g}\cdot\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})\mathfrak{g}$ augmentation ideal.

$\cup$

$Z(\mathfrak{g}) \supset Z_+ := Z(\mathfrak{g}) \cap \mathfrak{g}\cdot\mathcal{U}(\mathfrak{g})$.

$\langle Z_+ \rangle \subset \mathcal{U}(\mathfrak{g})$: ideal generated by Z_+.

$$a : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}(\mathcal{B}).$$

Prop: (1) $a(Z_+) = 0$

(2) $a : \dfrac{\mathcal{U}(\mathfrak{g})}{\langle Z_+ \rangle} \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$.

With parameters

Fix $G \supset B \supset U$, $B/U = H$.

Enhanced flag varieties

$$\tilde{\mathcal{B}} := G/U \xrightarrow{\pi} \mathcal{B} = G/B$$

principal H-bundle.

$$\Omega^1_{\tilde{\mathcal{B}}} = G \times^U (\mathfrak{g}/\mathfrak{u})^* \simeq G \times^U \mathfrak{b}.$$

$$G \curvearrowright \tilde{\mathcal{B}} \curvearrowleft H$$

$$\mathbb{V}(\Omega^1_{\tilde{\mathcal{B}}})/H = (G \times^U \mathfrak{b})/H = G \times^B \mathfrak{b} \simeq \tilde{\mathfrak{g}}.$$

$$
\begin{tikzcd}
 & & T^*\tilde{\mathcal{B}} \arrow[r, "\mu_H"] \arrow[d] & \mathfrak{h}^* = \mathfrak{h} \\
\bar{\mu}_H^{-1}(0) \arrow[r, hook] \arrow[d, equal] & \tilde{\mathfrak{g}} \arrow[r, phantom, "="] \arrow[urr, bend left=60, "\nu"] \arrow[dd, "\mu"] & T^*\tilde{\mathcal{B}}/H \arrow[ur, "\bar{\mu}_H"] \arrow[dd, "\mu_G"] & \\
\tilde{\mathcal{N}} \arrow[d] & & & \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g} \arrow[r, phantom, "="] & \mathfrak{g}^* &
\end{tikzcd}
$$

So, μ & ν are simply moment maps corresponding to G & H actions on $\tilde{\mathcal{B}}$.

Now, we have:

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}) \xrightarrow{a} \mathcal{D}(\tilde{\mathcal{B}})^H$$

And the H-C isom:

$$\Theta : Z(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{h})^{\cdot W}$$

Prop: $\forall z \in Z(\mathfrak{g})$

$$a(z \otimes 1) = a(1 \otimes \Theta(z)).$$

Now, consider the following object:

$$\tilde{\mathcal{U}}(\mathfrak{g}) := \mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{h}).$$

Then, $Z(\tilde{\mathcal{U}}(\mathfrak{g})) \simeq \mathcal{U}(\mathfrak{h})$.

<u>Kostant</u> $\Rightarrow$ $\tilde{\mathcal{U}}(\mathfrak{g})$ is <u>free</u> over $\mathcal{U}(\mathfrak{h})$, and thus over $Z(\mathfrak{g})$.

We have $W \curvearrowright \tilde{\mathcal{U}}(\mathfrak{g})$ via its action on $\mathfrak{h}$, and

$$\tilde{\mathcal{U}}(\mathfrak{g})^W = \mathcal{U}(\mathfrak{g}) \otimes 1$$

so: $\tilde{\mathcal{U}}(\mathfrak{g}) \supset \mathcal{U}(\mathfrak{g})$ is a "Galois extension"

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$$a : \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}) \xrightarrow{a} \mathcal{D}(\mathcal{B})^H$$

a filtration respecting "action" map.

$$\begin{tikzcd}
\mathrm{gr}(a) : \mathrm{gr}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h})) \arrow[r] \arrow[d, equal, "\mathrm{PBW}"] & \mathrm{gr}(\mathcal{D}(\mathcal{B})^H) \arrow[d, leftrightarrow] \\
\mathrm{Sym}\,\mathfrak{g} \otimes \mathrm{Sym}\,\mathfrak{h} \arrow[r] \arrow[d, equal] & \mathbb{C}[V(\Omega^1_{\mathcal{B}})]^H \arrow[d, equal] \\
\mathbb{C}[\mathfrak{g}^*] \otimes \mathbb{C}[\mathfrak{h}^*] \arrow[r, "\mu^*_{G\times H}"'] & \mathbb{C}[V(\Omega^1_{\mathcal{B}})/H]
\end{tikzcd}$$

moment map

$$\begin{tikzcd}
\mathfrak{g}^* \times \mathfrak{h}^* \arrow[d, "\sim"] & V(\Omega^1_{\mathcal{B}})/H \arrow[l, "\mu_{G\times H}"'] \arrow[d, "\sim"] \\
\mathfrak{g} \times \mathfrak{h} & \tilde{\mathfrak{g}} \arrow[l, "\mu \times \nu"'] \arrow[dl] \\
\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} = X \arrow[u] &
\end{tikzcd}$$

(9) {11}

So, we have:

$$\begin{tikzcd}
\mathrm{gr}(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{h})) \arrow[r, "\mathrm{gr}(a)"] \arrow[d, "\sim"] & \mathrm{gr}(\mathcal{D}(\tilde{\mathcal{B}})^H) \arrow[d] \\
\mathbb{C}[\mathfrak{g} \times \mathfrak{h}] \arrow[r, "\mu^b \times \nu^*"] \arrow[dr, two heads] & \mathbb{C}[\mathfrak{g}] \\
& \mathbb{C}[X] \arrow[u, "\simeq\ (\text{Kostant})"']
\end{tikzcd}$$

$\Rightarrow$ It follows that: (1) $\mathrm{gr}(a)$ is <u>surjective</u>, and

$$(2)\ \mathrm{gr}(\mathcal{D}(\tilde{\mathcal{B}})^H) \xrightarrow[\substack{\text{principal} \\ \text{symbol} \\ \text{map}}]{\sim} \mathbb{C}[V(\Omega'_{\tilde{\mathcal{B}}})/H].$$

$$(3)\ \mathbb{C}[\mathfrak{h}] \xrightarrow{\sim} \mathbb{C}[X]^G \xrightarrow{\sim} (\mathrm{gr}\,\mathcal{D}(\tilde{\mathcal{B}})^H)^G \xrightarrow{\sim} \mathrm{gr}(\mathcal{D}(\tilde{\mathcal{B}})^{G\times H})$$

<u>Corollary:</u> a is surjective and induces an <u>isomorphism</u>

$$1 \otimes \mathcal{U}(\mathfrak{h}) \xrightarrow{\sim} \mathcal{D}(\tilde{\mathcal{B}})^{G\times H}$$

In particular, this gives a map

$$\Theta' : Z(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$$

s.t.

$$a(z \otimes 1) = a(1 \otimes \Theta'(z)).$$

<u>Prop:</u> $\Theta'(z) = w_0\, \Theta(z)$, $\forall\, z \in Z(\mathfrak{g})$, where $w_0 \in W$ is the longest element.

Before we prove this statement, we have the following discussion:

For any $\lambda \in X^*(H)$, we have the line bundle $\mathcal{O}(\lambda)$ over $\mathcal{B}$.

$$\Gamma(\mathcal{B}; \mathcal{O}(\lambda)) = \{ f \in \mathbb{C}[G] : f(gb) = f(g)\cdot\lambda(b)^{-1}, \forall g \in G, b \in B \}$$

$$\|$$

$$\mathbb{C}[\tilde{\mathcal{B}}](\lambda) := \{ f \in \mathbb{C}[\tilde{\mathcal{B}}] : f(gh) = f(g)\lambda(h)^{-1}, \forall g \in G, h \in H \}$$

Namely: $\mathbb{C}[\tilde{\mathcal{B}}]$ is an H-rep^n and $\Gamma(\mathcal{B}, \mathcal{O}(\lambda))$ is just the λ wt. space in this rep^n.

$$\Rightarrow \mathbb{C}[\tilde{\mathcal{B}}] = \bigoplus_{\lambda \in P} \mathbb{C}[\tilde{\mathcal{B}}](\lambda) = \bigoplus_{\lambda \in P} \Gamma(\mathcal{B}, \mathcal{O}(\lambda)).$$

Pf. of proposition:

$$p \in \mathbb{C}[\mathfrak{h}^*] = \mathcal{U}(\mathfrak{h})$$

$\alpha(1 \otimes p)$ acts on $\mathbb{C}[\tilde{\mathcal{B}}]$ as a differential operator and on $\mathbb{C}[\tilde{\mathcal{B}}](\lambda)$, it acts by scalar multiplication by $p(\lambda)$.

Borel-Weil: if λ is anti-dominant, then

$\Gamma(\mathcal{B}, \mathcal{O}(\lambda)) = \mathbb{C}[\tilde{\mathcal{B}}](\lambda)$ is the highest wt rep^n of highest wt $w_0 \lambda$.

Θ is defined in the following way:

For $z \in Z(\mathfrak{g})$, $\Theta(z) - z \in U(\mathfrak{g})\mathfrak{u}$.

~~But if $v \in \mathbb{C}[\tilde{\mathcal{B}}](\lambda)$ is a highest wt vector~~, then

$\Rightarrow$ z acts on $\mathbb{C}[\tilde{\mathcal{B}}](\lambda)$ by $\Theta(z)(w_0\lambda)$.
$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{V_{w_0\lambda}}$

$\Rightarrow$ $\Theta'(z)(\lambda) = \Theta(z)(w_0\lambda)$, $\forall$ anti-dominant λ

$\Rightarrow$ By a density argument:

$$\Theta'(z) = w_0\,\Theta(z) \text{ (recall that } w_0^2 = 1).$$

Theorem: $a : U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{h}_0) \to \mathcal{D}(\tilde{\mathcal{B}})^H$ induces an isomorphism

$$\tilde{U}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}) \to \Theta' = w_0\Theta} U(\mathfrak{h}_0) \xrightarrow{\sim} \mathcal{D}(\tilde{\mathcal{B}})^H$$

Pf: It factors through this quotient by the Prop. above, and it's an isomorphism by the usual associated graded argument.

$$\operatorname{gr}(U(\mathfrak{g})) \otimes_{\operatorname{gr}(Z(\mathfrak{g}))} \operatorname{gr}(U(\mathfrak{h}_0)) \xrightarrow{\sim} \operatorname{gr}(\mathcal{D}(\tilde{\mathcal{B}})^H)$$

by what we saw earlier

so this is an isom! $\rightarrow$ $\downarrow$

$$\operatorname{gr}(U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{h}_0)) \xrightarrow{\operatorname{gr}(a)} \operatorname{gr}(\mathcal{D}(\tilde{\mathcal{B}})^H)$$

$\uparrow$

and this is an isomorphism

Remark: There is a puzzle!: W acts on $\tilde{U}(\mathfrak{g})$; what is the explanation of this action on $\mathcal{D}(\tilde{\mathcal{B}})^H$??

Sheaf theoretic version

$$q: \tilde{\mathcal{B}} \longrightarrow \mathcal{B}, \quad \tilde{T} = (q_* T_{\tilde{\mathcal{B}}})^H$$

We have:

$$0 \to \mathcal{O}_{\mathcal{B}} \otimes \mathfrak{h} \to \tilde{T} \to T_{\mathcal{B}} \to 0$$

$$\tilde{\mathcal{D}} := (q_* \mathcal{D}_{\tilde{\mathcal{B}}})^H \; ; \; U(\mathfrak{h}) \hookrightarrow Z(\tilde{\mathcal{D}})$$

For an open affine $S \subset \mathcal{B}$ s.t. q trivializes over S; then $\Gamma(S, \tilde{\mathcal{D}}) = \mathcal{D}_{\mathcal{B}}(S) \otimes U(\mathfrak{h})$..

$$\mathrm{gr}\, \tilde{\mathcal{D}} = p_* \mathcal{O}_{\tilde{\mathfrak{g}}}, \quad p: \tilde{\mathfrak{g}} \to \mathcal{B}.$$

Thm: $\tilde{U}(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathcal{B}, \tilde{\mathcal{D}})$.

Helpful Table

Commutative		Non-Commutative
$\mathbb{C}[\mathfrak{g}]$	$\xleftarrow{\mathrm{gr}}$	$U(\mathfrak{g})$
$\mathbb{C}[X]$		$\tilde{U}(\mathfrak{g})$
$p_* \mathcal{O}_{\tilde{\mathfrak{g}}}$ global sections		$\tilde{\mathcal{D}}$ global sections

Interpolation: For any Lie algebra $\mathfrak{g}$,

Consider
$$U_t \mathfrak{g} = \frac{\mathbb{C}[t] \otimes T(\mathfrak{g})}{\langle 1 \otimes (x \otimes y - y \otimes x) - t \otimes [x,y] \rangle}$$

Then: $\mathcal{U}_0 \mathfrak{g} = \mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}(\mathfrak{g})$

$\mathcal{U}_1 \mathfrak{g} = \mathcal{U}(\mathfrak{g})$

If $\mathfrak{g}$ is semi-simple, $\mathfrak{g} = \mathrm{Lie}(G)$,

$$\tilde{\mathcal{D}}_t := \frac{U_t(\mathcal{O}_{\mathcal{B}} + \tilde{T})}{\left\langle \begin{matrix} f \otimes u - fu,\ f \in \mathcal{O}_{\mathcal{B}}, u \in \mathcal{U}_t(-), \\ 1_u = 1 \end{matrix} \right\rangle}$$

Then $\tilde{\mathcal{D}}_0 = p_* \mathcal{O}_{\tilde{\mathfrak{g}}}$, $\tilde{\mathcal{D}}_1 = \tilde{\mathcal{D}}$.

Now, we can specialize all our objects at a point $\lambda \in \mathfrak{h}_{\mathfrak{g}}^*$:

$$\lambda : \mathcal{U}(\mathfrak{h}_{\mathfrak{g}}) \longrightarrow \mathbb{C} \quad \leftrightarrow \quad \mathbb{C}_\lambda : \text{1-dim } \mathcal{U}(\mathfrak{h}_{\mathfrak{g}})\text{-module.}$$

$$\mathcal{D}_{\mathcal{B}}^{\lambda} := \tilde{\mathcal{D}} \otimes_{\mathcal{U}(\mathfrak{h}_{\mathfrak{g}})} \mathbb{C}_\lambda = \tilde{\mathcal{D}} / (h - \lambda(h) : h \in \mathfrak{h}_{\mathfrak{g}}).$$

$\uparrow$

λ-twisted differential operators (TDOs).

When $\lambda = 0$, we just get $\mathcal{D}_{\mathcal{B}}$.

When $\lambda \in X^*(H)$, then $\mathcal{D}_{\mathcal{B}}^{\lambda} = \mathcal{D}_{\mathcal{B}}(\mathcal{O}(\lambda)) = \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(-\lambda)$

$$\tilde{\mathcal{D}}_t^\lambda = \tilde{\mathcal{D}}_t \otimes_{U(\mathfrak{h}_{\mathfrak{g}})} \mathbb{C}_\lambda .$$

$$\chi_\lambda : Z(\mathfrak{g}) \xrightarrow{\Theta'} U(\mathfrak{h}_{\mathfrak{g}}) \xrightarrow{\lambda} \mathbb{C}.$$

$\mathbb{C}_{\chi_\lambda}$: 1-dim $Z(\mathfrak{g})$-module.

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{h}_{\mathfrak{g}})} \mathbb{C}_\lambda = (U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{h}_{\mathfrak{g}})) \otimes_{U(\mathfrak{h}_{\mathfrak{g}})} \mathbb{C}_\lambda$$
$$= U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathbb{C}_{\chi_\lambda}$$
$$=: U^{\chi_\lambda}(\mathfrak{g}).$$

<u>Theorem</u>: $\forall \lambda \in \mathfrak{h}_{\mathfrak{g}}^*$, a induces an isomorphism:

$$a : U^{\chi_\lambda}(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}}^\lambda).$$

<u>Pf</u>: Usual <u>associated graded argument</u>.

On both sides we set $\mathbb{C}[\mathfrak{g}^*] \otimes_{\mathbb{C}[\mathfrak{h}^*/W]} \mathbb{C}_\lambda$ $= \mathbb{C}[\pi^{-1}(\lambda)]$

Poisson geometry

Def^n A: comm k-alg., is a Poisson algebra if:

we have $\{,\}: A \wedge A \to A$. a Lie bracket s.t.

s.t. $\{ab, c\} = \{a, c\}b + a\{b, c\}$, $\forall a, b, c \in A$.

Poisson center $Z(A) = \{z \in A : \{z, a\} = 0, \forall a \in A\}$.

If A is a Poisson algebra, then we say that $\operatorname{Spec} A$ is a Poisson variety.

Moreover, every fiber of $\operatorname{Spec} A \to \operatorname{Spec} Z(A)$ inherits a Poisson structure

- If Y is an alg. symplectic manifold, then $\mathcal{O}_Y$ is a sheaf of Poisson algebras.

- Let A be an associative algebra over $\mathbb{C}[t]$ s.t.

(i) A is flat, i.e. $t \notin$ Zerodivisors(A).
(ii) $A|_{t=0}$ is commutative.

$$A|_{t=0} = A \otimes_{\mathbb{C}[t]} \mathbb{C} = A/(t).$$

This gives us a Poisson structure on $A|_0$ in the following way:

$\forall a, b \in A$, $ab - ba = tc$, $\exists! c \in A$.
So, for $\bar{a}, \bar{b} \in A|_{t=0}$, we set $\{\bar{a}, \bar{b}\} = \bar{c}$.

Def^n: A quantization of a Poisson variety $(Y, \{\ \})$ is a sheaf A on Y of flat $\mathbb{C}[[t]]$-algebras s.t. $A|_{t=0} = \mathcal{O}_Y$ and the Poisson structure of $\mathcal{O}_Y$ is the one induced from A.

Thm (Bezrukavnikov, Kaledin)
Let (Y, ω) be a smooth alg. symplectic variety s.t.

$$H^i(Y, \mathcal{O}_Y) = 0, \text{ for } i = 1, 2.$$

Then, quantizations of $\mathcal{O}_Y$ are parametrized by

$$H^2(Y, \mathbb{C}) \times \mathbb{C}.$$

~~i.e.~~

i.e. there is a universal flat $\mathbb{C}[H^2(Y,\mathbb{C}) \times \mathbb{C}]$-algebra A over $\mathcal{O}_Y$ s.t. (i) $A|_{c=t=0} = \mathcal{O}_Y$ and s.t.

~~every quantization~~

(ii) $A|_{t=0}$ is a flat commutative $\mathbb{C}[H^2(Y,\mathbb{C})]$-al

(iii) For $c \in H^2(Y, \mathbb{C})$, we get a Poisson variety $Y_c \leftrightarrow A|_{t=0, c}$.

Period map: (Y_c, ω_c) is symplectic $\leadsto [\omega_c] \in H^2(Y_c, \mathbb{C})$.

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(Y, ω): smooth symplectic alg. variety
s.t. $H^i(\mathcal{O}_Y) = 0$, for $i = 1, 2$.

Thm: $\exists$ flat family

$$\begin{array}{c} \mathcal{Y} \\ \downarrow F \\ H^2(Y, \mathbb{C}) \end{array}$$

and a relative closed 2-form Ω on $\mathcal{Y}$.

s.t.

- $Y_0 \simeq Y$ $\quad (Y_c = F^{-1}(c)$, for $c \in H^2(Y, \mathbb{C}))$

  $\Omega|_{Y_0} \simeq \omega$

- The period map

$$c \longmapsto [\Omega|_{Y_c}] \in H^2_{dR}(Y_c)$$

  gives a horizontal section of $H^2_{dR}(\mathcal{Y})$ w.r.t. to the Gauss-Manin connection.

- For each c, $(Y_c, \Omega|_{Y_c})$ is a symplectic manifold.

- $\exists$ a sheaf $\mathcal{A}$ of assoc. algebras over $(F^{-1}\mathcal{O}_{H^2(Y,\mathbb{C})})[t]$ s.t. $\mathcal{A}/t\mathcal{A} \simeq \mathcal{O}_{\mathcal{Y}}$.

  and the Poisson bracket on $\mathcal{O}_{\mathcal{Y}}$ induced by $\mathcal{A}$ is just the canonical Poisson bracket on $\mathcal{O}_{\mathcal{Y}}$ induced by the symplectic structure.

Let X be a not necessarily smooth variety ~~simple~~ with Poisson structure.

Def^n A resolution of singularities $f: \tilde{X} \to X$ is called a symplectic resolution if:

(1) $\tilde{X}$ has a symplectic structure

(2) $f^*: f^{-1}\mathcal{O}_X \to \mathcal{O}_{\tilde{X}}$ respects the Poisson structures on both sides.

(3) X is affine, normal.

Suppose $\tilde{X} \xrightarrow{f} X$ is a symplectic resolution. Then, we have:

(a) Grauert-Riemenschneider $\Rightarrow H^i(\mathcal{O}_{\tilde{X}}) = 0, \forall i > 0$ (since X is affine and $\tilde{X}$ is Calabi-Yau)

(b) normality of $X \Rightarrow f^*: \mathbb{C}[X] \xrightarrow{\sim} \mathbb{C}[\tilde{X}]$.

(c) X has finitely many symplectic leaves.

Let $\mathfrak{g}$ be a Lie algebra; recall:

$U_t(\mathfrak{g})$: flat deformation of $U(\mathfrak{g})$ over $\mathbb{C}[t]$ with $U_0(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})$.

$\Rightarrow$ we get a Poisson bracket on $\mathrm{Sym}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ called the Kirillov-Kostant structure

So: $\mathfrak{g}^*$ is a Poisson variety and $Z^{Poisson}(\mathbb{C}[\mathfrak{g}^*]) = \mathbb{C}[\mathfrak{g}^*]^G$.

- Moreover, the symplectic leaves of $\mathfrak{g}^*$ are the G-orbits.

  In particular, each G-orbit in $\mathfrak{g}^*$ is an even-dimensional symplectic manifold.

- $\pi : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*//G = \operatorname{Spec} Z^{Poisson}(\mathbb{C}[\mathfrak{g}^*])$

  Each fiber of π inherits a Poisson structure.

Now, suppose $\mathfrak{g}$ is semi-simple and identify $\mathfrak{g} \simeq \mathfrak{g}^*$.

Then: $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}//G$ and $\pi^{-1}(0) = \mathcal{N}$ inherits a Poisson structure.

Claim: $\mu : \tilde{\mathcal{N}} \longrightarrow \mathcal{N}$ is a symplectic resolution.

Indeed: $\tilde{\mathcal{N}} \xrightarrow{\sim} \mathbb{V}(\Omega^1_{\mathcal{B}})$ has a canonical symplectic structure and μ is a moment map, and thus has to respect Poisson structures, $\mathcal{N}$ is normal, by Kostant.

Deduce: $\mathcal{N}$ is a finite union of G-orbits and each orbit is even-dimensional.

$\Rightarrow \mathcal{N} \setminus \mathcal{N}^{reg}$ has codim ≥ 2.

We will apply the Bezrukavnikov Kaledin theorem to $\tilde{N}$.

First: $H^2(\tilde{N}, \mathbb{C}) = H^2(\mathbb{V}(\Omega^1_{\mathcal{B}}), \mathbb{C})$
$= H^2(\mathcal{B}, \mathbb{C}) \underset{\text{Borel}}{\cong} \mathfrak{h}^*$

1st Chern class of $\mathcal{O}(\lambda)$ $\leftrightarrow$ λ

So we set:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, phantom, "\simeq"] \arrow[d, "F"] & \tilde{\mathfrak{g}} \arrow[d, "\nu"] \\ \mathfrak{h}^* \arrow[r, phantom, "\simeq"] & \mathfrak{h} \end{tikzcd}$$

and $Y_h = \nu^{-1}(h)$
$\cong G \times^B (h+\mathfrak{u})$
has a natural symplectic structure.

By the theorem, we can quantise $\mathcal{Y}$ to a flat $\nu^{-1}\mathcal{O}_{\mathfrak{h}}[t]$-algebra $\mathcal{A} \simeq \tilde{\mathcal{D}}_t$.

Beilinson-Bernstein theorem

$\chi : Z(\mathfrak{g}) \to \mathbb{C}$ a central character.

$\mathcal{U}^{\chi}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathbb{C}_{\chi}$.

$\mathcal{U}^{\chi}$-mod : Abelian category of f.g. left $\mathcal{U}^{\chi}(\mathfrak{g})$-modules.

$\lambda \in \mathfrak{h}^{*}$; $\mathcal{D}_{\lambda} = \tilde{\mathcal{D}} \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}_{\lambda}$: sheaf of λ-twisted differential operators over $\mathcal{B}$.

$\mathcal{D}^{\lambda}$-mod : Abelian category of sheaves of coherent $\mathcal{D}^{\lambda}$-modules (left modules).

Warning: Coherent over $\mathcal{D}^{\lambda}$!

By Harish-Chandra, (modified) $\lambda \in \mathfrak{h}^{*} \mapsto \chi_{\lambda} : Z(\mathfrak{g}) \to \mathbb{C}$.

and we saw:

Thm $\mathcal{U}^{\chi_{\lambda}}(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathcal{B}, \mathcal{D}_{\lambda})$.

We have a pair of adjoint functors:

$$\mathcal{D}^{\lambda}\text{-mod} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{U}^{\chi_{\lambda}}\text{-mod}$$

Δ is left adjoint to Γ.

$$\mathcal{M} \overset{\Gamma}{\longmapsto} \Gamma(\mathcal{B}, \mathcal{M})$$

$$\mathcal{D}^{\lambda} \otimes_{\mathcal{U}^{\chi_{\lambda}}} N \overset{\Delta}{\longleftarrow\!\!\!\shortmid} N$$

Thm (B-B): (1) λ s.t. $\lambda-\rho$ is <u>anti-dominant</u>, then:

For any $\mathcal{F}\in\mathcal{D}^\lambda$-mod, $H^i(\mathcal{B},\mathcal{F})=0$, for $i>0$.

(2) λ is <u>anti-dominant</u>, then:

Γ is an <u>equivalence of categories</u> with inverse Δ.

Identify $\mathcal{B}\simeq G/B$ for a <u>fixed</u> Borel $B\subset G$.

Given a finite dimensional B-rep V

$\rightsquigarrow$ G-equivariant vector bundle $G\times^B V$ over G/B; $\underline{V}$.

e.g. if $\mu: B\to\mathbb{C}^\times$, then $\underline{\mathbb{C}_\mu}\simeq\mathcal{O}(\mu)$.

<u>Lie Thm for B</u> $\Rightarrow$ V admits a complete B-stable flag with factors isomorphic to $\mathbb{C}_{\nu_i}$, for $\nu_i: B\to\mathbb{C}^\times$.

$\Rightarrow$ $\underline{V}$ admits a complete G-stable flag with factors isomorphic to $\mathcal{O}(\nu_i)$, $\nu_i: B\to\mathbb{C}^\times$.

For $\nu: B\to\mathbb{C}^\times$, $\mathcal{F}$ an $\mathcal{O}_{\mathcal{B}}$-module; set $\mathcal{F}(\nu)=\mathcal{F}\otimes\mathcal{O}(\nu)$.

We have
$$\mathcal{O}(\nu_1)\hookrightarrow\underline{V}\xrightarrow{p}\mathcal{O}(\nu_n)\ (\text{not exact}).$$
and so we also have:
$$\mathcal{O}_{\mathcal{B}}\xhookrightarrow{j}\underline{V}(-\nu_1)\twoheadrightarrow\mathcal{O}(\nu_n-\nu_1)$$
[illegible]

Now, suppose V is in fact a G-rep^n; then we have

$$\underline{V} \underset{G\text{-equivariant}}{\xrightarrow{\sim}} \mathcal{O}_{\mathcal{B}} \otimes_{\mathbb{C}} V \quad \underline{\text{canonically}}.$$

Now, let $\mu \in P^{++}$, $V = V_\mu$ an irred. rep^n w. highest wt μ.

We have:
- $\mathcal{O}_{\mathcal{B}} \xhookrightarrow{j} V_\mu \otimes_{\mathbb{C}} \mathcal{O}(-\mu)$
- $V_\mu \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{B}} \xrightarrow{p} \mathcal{O}(w_0\mu)$.

For any $\mathcal{O}_{\mathcal{B}}$-module $\mathcal{F}$, we have:

- $\mathcal{F} \xhookrightarrow{j_{\mathcal{F}}} V_\mu \otimes_{\mathbb{C}} \mathcal{F}(-\mu)$
- $V_\mu \otimes_{\mathbb{C}} \mathcal{F} \xrightarrow{p_{\mathcal{F}}} \mathcal{F}(w_0\mu)$

<u>Key lemma:</u> $\mathcal{F} \in \mathcal{D}^\lambda$-mod.

(1) $\lambda - \rho \in -P^{++}$; then $j_{\mathcal{F}}$ admits a splitting $j'_{\mathcal{F}}$ as sheaves (<u>not</u> $\mathcal{O}$-linear)

(2) $\lambda \in -P^{++}$; then $p_{\mathcal{F}}$ also admits a splitting $p'_{\mathcal{F}}$. as sheaves (<u>not</u> $\mathcal{O}$-linear).

These arise as elements of $Z(\mathfrak{g})$ acting as differential operators.

~~Deduci~~
Deducing theorem from Key Lemma:

Key lemma (1) $\Rightarrow H^i(\mathcal{B}, \mathcal{F}) = 0, \ \forall i > 0$.

Suffices to show: For any coherent $\mathcal{O}$-module $\mathcal{L}_0$ and any morphism $\mathcal{L}_0 \xrightarrow{\varphi} \mathcal{F}$ of $\mathcal{O}$-modules, $H^i(\varphi) = 0$, for $i > 0$.

We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{L}_0 & \xhookrightarrow{j_{\mathcal{L}_0}} & V_\mu \otimes_{\mathbb{C}} \mathcal{L}_0(-\mu) \\ \varphi \downarrow & & \downarrow 1 \otimes \varphi(-\mu) \\ \mathcal{F} & \xhookrightarrow[j_{\mathcal{F}}]{} & V_\mu \otimes_{\mathbb{C}} \mathcal{F}(-\mu) \end{array}$$

Since $j_{\mathcal{F}}$ admits a splitting, it suffices to show that

$$H^i(\mathcal{L}_0) \xrightarrow{H^i(j_{\mathcal{L}_0})} V_\mu \otimes H^i(\mathcal{L}_0(-\mu)) \xrightarrow{H^i(1 \otimes \varphi(-\mu))} V_\mu \otimes H^i(\mathcal{F}(-\mu))$$

is 0.

so, it suffices to show that $H^i(\varphi(-\mu)) = 0$ for $i > 0$.

Just choose μ so ~~~~ dominant that $H^i(\mathcal{L}_0(-\mu)) = 0$.

Done!

Key lemma (2) $\Rightarrow$ ~~~~ for $\lambda \in -P^{++}$,

$\mathcal{F} \in \mathcal{D}_\lambda\text{-mod}$, $\mathcal{F} \neq 0 \Rightarrow \Gamma(\mathcal{F}) \neq 0$.

Consider $\gamma_{\mathcal{F}} : V_\mu \otimes_{\mathbb{C}} \mathcal{F} \twoheadrightarrow \mathcal{F}(w_0\mu)$

This admits a splitting; so it suffices to show that $\Gamma(\mathcal{F}(w_0\mu)) \neq 0$.

But this can be arranged by making μ dominant enough.

Done!

So, if $\lambda \in -\rho^{++}$, Γ is an exact functor s.t. $\Gamma(\mathcal{F}) = 0 \Rightarrow \mathcal{F} = 0$.

LECTURE 13 05/10/10

Thm (B-B) $\mathcal{D}^{\lambda}_{\mathcal{B}}\text{-mod} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{U}^{\chi_\lambda}(\mathfrak{g})\text{-mod}$ adjoint functors.

① If $\lambda - \rho \in -\mathfrak{h}^{++}$ (anti-dominant), then

$$H^i(\mathcal{B}, M) = 0, \quad \forall M \in \mathcal{D}^{\lambda}\text{-mod}.$$

② If $\lambda \in -\mathfrak{h}^{++}$, then Γ is an equivalence of categories.

Pf: We showed last time:

① is true

②: if $\lambda \in -\mathfrak{h}^{++}$ then Γ is exact

& $\Gamma(M) = 0 \Rightarrow M = 0$.

We have the unit

$$a : \mathbb{1}_{\mathcal{U}^{\chi_\lambda}(\mathfrak{g})\text{-mod}} \longrightarrow \Gamma \circ \Delta.$$

if $M = \mathcal{U}^{\chi_\lambda}(\mathfrak{g})$, then:

$$a : M \longrightarrow (\Gamma\Delta)(M) \text{ is an } \underline{\text{isomorphism}}.$$

For general M, take a free presentation and use right exactness of $\Gamma \circ \Delta$.

We also have the co-unit.

$$b : \Delta \circ \Gamma \longrightarrow \mathbb{1}_{\mathcal{D}^{\lambda}\text{-mod}}.$$

We want to show:

$$b_M : \Delta(\Gamma(M)) \longrightarrow M \text{ is an isom.}$$

for every $M \in \mathcal{D}^{\lambda}\text{-mod}$.

i.e. $\mathcal{D}^\lambda \otimes_{U^{\chi_\lambda}(\mathfrak{g})} \Gamma(\mathcal{M}) \xrightarrow{b_{\mathcal{M}}} \mathcal{M}$ is an isomorphism

for all $\mathcal{M} \in \mathcal{D}^\lambda$-mod.

(a) $b_{\mathcal{M}}$ is surjective : i.e. $\mathcal{M}$ is generated by global sections.

Suppose $\mathcal{C} = \mathrm{Coker}(b_{\mathcal{M}})$; we get, on applying Γ, an exact sequence:

$$\Gamma(\mathcal{M}) \xrightarrow[\text{identity}]{\Gamma(b_{\mathcal{M}})} \Gamma(\mathcal{M}) \to \Gamma(\mathcal{C}) \to 0$$

$\Rightarrow \Gamma(\mathcal{C}) = 0 \Rightarrow \mathcal{C} = 0$

Applying the same argument to $\mathcal{K} = \ker(b_{\mathcal{M}})$, we get $\Gamma(\mathcal{K}) = 0 \Rightarrow \mathcal{K} = 0$.

So, the main content of B-B is the following

Key Lemma: (1) If $\lambda - \rho \in -\mathfrak{h}^{++}$, $\mu \in P^{++}$, then $\mathcal{F} \hookrightarrow V_\mu \otimes_{\mathbb{C}} \mathcal{F}(-\mu)$ admits a $\mathbb{C}$-linear splitting for $\mathcal{F} \in \mathcal{D}^\lambda$-mod.

(2) If $\lambda \in -\mathfrak{h}^{++}$, $\mu \in P^{++}$, then

$$V_\mu \otimes_{\mathbb{C}} \mathcal{F} \twoheadrightarrow \mathcal{F}(w_0\mu)$$ admits a $\mathbb{C}$-linear splitting for $\mathcal{F} \in \mathcal{D}^\lambda$-mod.

Pf: (1) We have

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\Theta'"] \arrow[rr, bend right, "\chi^\lambda"'] & U(\mathfrak{h}) \arrow[r, "\lambda"] & \mathbb{C} \end{tikzcd}$$

By assumption, $Z(\mathfrak{g})$ acts on $\mathcal{F}$ via χ^λ.

Claim: (1) $V_\mu \otimes \mathcal{F}(-\mu)$ has finite support as a $Z(\mathfrak{g})$-module.

i.e. $V(\mathrm{ann}_{Z(\mathfrak{g})}(V_\mu \otimes \mathcal{F}(-\mu)))$ is finite.

(2) If $\lambda - \rho \in \mathfrak{h}^{++}$, then

$\mathfrak{m}_{\chi^\lambda} \notin V(\mathrm{ann}_{Z(\mathfrak{g})}(V_\mu \otimes \mathcal{F}(\mu)/\mathcal{F}))$.

Pf of claim: We have:

$\underline{V}_1 \otimes \mathcal{F} \subseteq \underline{V}_2 \otimes \mathcal{F} \subseteq \cdots \subseteq \underline{V}_n \otimes \mathcal{F} = V_\mu \otimes \mathcal{F}$.

s.t. $\underline{V}_i / \underline{V}_{i-1} \simeq \mathcal{O}(\nu_i)$

and so:

$(\underline{V}_i \otimes \mathcal{F}(-\mu)) / (\underline{V}_{i-1} \otimes \mathcal{F}(-\mu)) \cong \mathcal{F}(\nu_i - \mu)$.

(H): $Z(\mathfrak{g}) \longrightarrow \mathbb{C}[\mathfrak{h}^*]^W$ H-C isom

$z \longmapsto P$

then: z acts on $\mathcal{F}$ via $P(w_0 \lambda)$

and on $\mathcal{F}(\nu_i - \mu)$ via $P(w_0(\lambda + \nu_i - \mu))$

(1) is clear from this.

So, the claim amounts to:

(②)

$w_0(\lambda + \nu_i - \mu) \neq w(w_0\lambda + \rho) - \rho$, for any $w \in W$.

and $i > 1$

(assuming $\lambda - \rho \in \mathfrak{h}^{++}$)

Now, $\mu - \nu_i \in \Phi^+$ and so $w_0(\nu_i - \mu) \in \Phi^+$.

(positive chamber)

$w_0(\lambda + \nu_i - \mu) + \rho$

$= w_0(\lambda - \rho + \nu_i - \mu)$

is dominant

So is $w_0(\lambda - \rho)$.

And so, they can't be in the same Weyl orbit.

$$|w_0(\lambda + \nu_i - \mu) + \rho| = |w_0\lambda + \rho + w_0(\nu_i - \mu)|$$

$$= |w_0(\lambda - \rho) + w_0(\nu_i - \mu)|$$

Since $\lambda - \rho \in -\mathfrak{h}^{++}$, $w_0(\lambda - \rho) \in \mathfrak{h}^{++}$

$$\Rightarrow |w_0(\lambda + \nu_i - \mu) + \rho| > |w_0(\lambda - \rho)|$$

The splitting is given by an idempotent projection in the f.d. quotient of $Z(\mathfrak{g})$ that acts on $V_\mu \otimes \mathcal{F}(\mu)$.

Consequences:

① If $\lambda = 0 \in -\mathfrak{h}^{++}$, then $\mathcal{U}^0(\mathfrak{g}) = \dfrac{\mathcal{U}(\mathfrak{g})}{\langle Z_+ \rangle}$.

$$\mathcal{D}_{\mathcal{B}}\text{-mod} \xrightarrow[\sim]{\Gamma} \mathcal{U}^0_{\mathfrak{g}}\text{-mod} \leftarrow \text{repns w. trivial central character}$$

② $\lambda \in P$, $\mathcal{O}(\lambda) \in \mathcal{D}^\lambda\text{-mod}$.

If $\lambda - \rho \in -P^{++}$, then $H^i(\mathcal{B}, \mathcal{O}(\lambda)) = 0$ $\forall i > 0$.

(3) (Translation principle)

$\lambda_1, \lambda_2 \in -\mathfrak{h}^{*+}$, χ_i = image of λ_i in $\mathfrak{h}^*/W = \mathrm{Spec}\, Z(\mathfrak{g})$.

$\therefore$ If $\lambda_1 - \lambda_2 \in P$, then $\mathcal{U}^{\chi_1}\text{-mod} \xrightarrow{\sim} \mathcal{U}^{\chi_2}\text{-mod}$.

Indeed, we have:

$$\begin{tikzcd} \mathcal{D}^{\lambda_1}\text{-mod} \arrow[d, "\mathcal{O}(\lambda_2-\lambda_1)\otimes -"'] & \mathcal{U}^{\chi_1}\text{-mod} \arrow[l, "\Delta"', "\sim"] \arrow[d, "\sim"] \\ \mathcal{D}^{\lambda_2}\text{-mod} & \mathcal{U}^{\chi_2}\text{-mod.} \arrow[l, "\sim"', "\Delta"] \end{tikzcd}$$

(4) Derived B-B equivalence.

Thm (B-B) If $\lambda - \rho$ is regular, then:

$$D^b(\mathcal{D}^\lambda\text{-mod}) \underset{L\Delta}{\overset{R\Gamma}{\rightleftarrows}} D^b(\mathcal{U}^{\chi_\lambda}\text{-mod.})$$

Non-example: $G = SL_2$, $\lambda = \rho$ so that $\lambda - \rho = 0$.

$\mathcal{O}(\rho) \in \mathcal{D}^\rho\text{-mod.}$, $\mathcal{B} = \mathbb{P}^1$

$\mathcal{O}(\rho) \cong \mathcal{O}(-1)$

But $H^i(\mathcal{O}(-1)) = 0$, $\forall i$!

Almost commutative algebra.

A is a filtered k-alg, $k = \bar{k}$.

$0 = A_{-1} \subset A_0 \subset A_1 \subset \cdots$, $A = \bigcup_{i \geq 0} A_i$.

Def^n : A is almost commutative if:

1. $\mathrm{gr}\, A$ is commutative.
2. $\mathrm{gr}\, A$ is f.g. over k. (not strictly necessary)

We will let $S_A = \mathrm{Spec}\, \mathrm{gr}\, A$: this is an alg. variety equipped with a $\mathbb{G}_m$-action (from the grading).

Ex:
- $A = U(\mathfrak{g})$, $S_A \simeq \mathfrak{g}^* \circlearrowleft \underset{\text{scalars}}{\mathbb{G}_m}$.
- X: smooth affine variety, $A = \mathcal{D}(X)$
  $S_A \simeq \mathbb{V}(\Omega^1_{X/k}) \circlearrowleft \underset{\text{scalars.}}{\mathbb{G}_m}$

~~Filtered~~

Rem: A almost commutative $\Rightarrow$ A is fin. generated & left & right Noetherian.

Let M be a left A-module.

Def^n: A filtration $F_i M$ on M is good if:

(i) $(M, F_i M)$ is a filtered A-module.

(ii) $gr^F(M)$ is f.g. over $gr(A)$.

To any good $(M, F_i M)$ we can associate a $\mathbb{G}_m$-equivariant sheaf $\widetilde{gr^F(M)} \in Coh^{\mathbb{G}_m}(S_A)$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ Coh(S_A/\mathbb{G}_m)$.

Example: $Coh^{\mathbb{G}_m}(pt) \simeq$ f.d. $\mathbb{Z}$-graded vector spaces

for $i \in \mathbb{Z}$, $\mathbb{k}(i) \in Coh^{\mathbb{G}_m}(pt)$ is the 1-dim space in degree i.

This gives a $\mathbb{Z}$-action on $Coh^{\mathbb{G}_m}(S_A)$ by
$i : M \mapsto M(i)$.

Def^n $K^+(Coh^{\mathbb{G}_m}(S_A))$

$\qquad\qquad \|$

$$\frac{\left\{\begin{array}{l}\text{Formal } \mathbb{Z}_{\geq 0}\text{-linear combinations} \\ \text{of elements in Isom classes of} \\ Coh^{\mathbb{G}_m}(S_A)/\mathbb{Z}\end{array}\right\}}{\left\{\begin{array}{l}[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}''] \text{ whenever} \\ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0\end{array}\right\}}$$

Given an A-module M, and a filtration $F_i M$ on M, we get, for $p \in \mathbb{Z}$, a *shifted* filtration

$$F^{[p]}_i M = F_{i+p} M \quad \text{s.t.} \quad gr^{F^{[p]}} M = gr^F M(p)$$

Prop: The class of $\widetilde{gr^F M}$ in $K^+(Coh^{G_m} S_A)$ is *independent* of choice of *good* filtration F on M.

Pf: Define:

$$R(A) = \bigoplus_{i=0}^{\infty} A_i t^i \subset A[t,t^{-1}]$$

the *Rees algebra*, a *graded flat* $k[t]$-*algebra*.

$$R(A)/R(A)t = grA.$$

$$R(A) \otimes_{k[t]} k[t,t^{-1}] = A[t,t^{-1}].$$

$\Rightarrow$ $R(A)$ is *Noetherian*.

Let $(M, F_i M)$ be a filtered A-module, and set $R(M, F_i M) = \bigoplus_{i=0}^{\infty} F_i M t^i \subset M \otimes_A A[t,t^{-1}]$.

This gives an $R(A)$-*lattice* in $M \otimes_A A[t,t^{-1}]$ if $F_i M$ is *good*; and conversely, any $R(A)$-lattice in $M \otimes_A A[t,t^{-1}]$ gives a good filtration on M.

Lemma: For any two $R(A)$-lattices $L, L' \subset M[t,t^{-1}]$, we have $[L/tL] = [L'/tL']$ in $K^+(\mathrm{Coh}^{\mathbb{G}_m} S_A)$.

Pf: For each $j \in \mathbb{Z}$, set $L_j = L + t^j L'$: this is again an $R(A)$-lattice in $M[t,t^{-1}]$.

Then: $L_j = L$, if $j >> 0$
$L_j = t^j L'$, if $j << 0$.

We have:
$t \cdot L_{j+1} \subset tL_j \subset L_{j+1} \subset L_j$
$\Rightarrow [L_j/tL_j] = [L_{j+1}/tL_{j+1}]$ in $K^+(\mathrm{Coh}^{\mathbb{G}_m} S_A)$.

Fix $d \geq 0$; $\mathrm{Coh}_d^{\mathbb{G}_m} S_A \subset \mathrm{Coh}^{\mathbb{G}_m} S_A$ is the full subcategory of sheaves whose support has $\dim \leq d$.

$K_d^{\mathbb{G}_m}(S_A) =$ Grothendieck ~~group~~ (ring) of $\mathrm{Coh}_d^{\mathbb{G}_m} S_A$.

When $S_A = pt$, then $K^{\mathbb{G}_m}(pt) \xrightarrow{\sim} \mathbb{Z}[q,q^{-1}]$
$[R(i)] \mapsto q^i$.

So, for every A, $K_d^{\mathbb{G}_m}(S_A)$ is a module over $K^{\mathbb{G}_m}(pt) = \mathbb{Z}[q,q^{-1}]$.

$A_d(S_A) =$ group of $\mathbb{G}_m$-stable alg. cycles on S of dim d (no relations).

We have a map

$$\mathrm{Coh}_d^{\mathbb{G}_m}(S_\pi) \longrightarrow A_d(S)$$
$$\mathcal{F} \longmapsto [\mathrm{Supp}(\mathcal{F})] \quad (\text{forget lower dimensions})$$

This induces a map:

$$K_d^{\mathbb{G}_m}(S) \longrightarrow A_d(S).$$