RINGED SPACES

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Contents

1. Ringed spaces and Sheaves of Modules

RS

A primary concept leading to the definition of schemes is the notion of a ringed space.

Definition 1.1. A ringed space is a pair $(X, \mathscr{O}_X)$, where X is a topological space, and $\mathscr{O}_X \in \mathrm{Shf}(X, \mathrm{Ring})$.

More interesting is the notion of a morphism between ringed spaces.

Definition 1.2. If $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are ringed spaces, then a morphism between them is a triple $(f, f^\sharp, f^\flat)$, where $f : X \to Y$ is a continuous map, $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a morphism in $\mathrm{Shf}(Y, \mathrm{Ring})$, and $f^\flat : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ is the morphism corresponding to $f^\sharp$ under the adjunction betwen f^{-1} and f_*. This gives us the category of *ringed spaces*

Put more concretely, $f^\sharp$ is a collection of maps $f^\sharp_U : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ for open subsets $U \subset Y$.

Remark 1.3. Of course, $f^\flat$ is determined by $f^\sharp$ (and vice versa); so most of the time we will represent a morphism of ringed spaces by the pair $(f, f^\sharp)$. Also, suppose we had morphisms $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\sharp) : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$, then we see that $f^\sharp \circ g^\sharp$ defines a map $\mathscr{O}_Z \to g_* f_* \mathscr{O}_X = (g \circ f)^* \mathscr{O}_X$. Thus, composition of maps of ringed spaces is well-defined.

Notice that for any topological space X, we have a natural ringed structure given by the pair $(X, \underline{\mathbb{Z}})$.

Remark 1.4. Sometimes this map $f^\sharp$ can be naturally obtained from f. For example, if we take $\mathscr{O}_X$ and $\mathscr{O}_Y$ to be the sheaves of continuous functions over X and Y, then $f^\sharp$ can be obtained by postcomposition by f. As we'll see in the case of schemes, the map $f^\sharp$ must be specified separately.

We also have an analogue for modules over ringed spaces.

defn-sheaf-of-modules **Definition 1.5.** A *sheaf of modules* over a ringed space $(X, \mathscr{O}_X)$ is a pair $(\mathscr{M}, \phi)$, where $\mathscr{M} \in \mathrm{Shf}(X, \mathrm{Ab})$, and $\phi : \mathscr{O}_X \to \underline{\mathrm{End}}(\mathscr{M})$ is a morphism of rings of sheaves. To conserve effort, we may also say that $\mathscr{M}$ is a $\mathscr{O}_X$-module.

Note that this equivalent to giving a 'ring action' map of sheaves $\phi : \mathscr{O}_X \times \mathscr{M} \to \mathscr{M}$ that's a morphism of sheaves of abelian groups in each co-ordinate, satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathscr{O}_X \times \mathscr{O}_X \times \mathscr{N} & \xrightarrow{m \times 1_N} & \mathscr{O}_X \times \mathscr{N} \\ \Big\downarrow {\scriptstyle 1_{\mathscr{O}_X} \times \phi} & & \Big\downarrow {\scriptstyle \phi} \\ \mathscr{O}_X \times \mathscr{N} & \xrightarrow[\phi]{} & \mathscr{F} \end{array}$$

where $m : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$ is the multiplication morphism.

In either case, we say that ϕ is the *defining morphism* of $\mathscr{M}$.

Remark 1.6. There are advantages to both definitions. With the first, it is very easy, as we'll see later, to show that certain sheaves are $\mathscr{O}_X$-modules. With the second, other things are easier to show; for example, the fact that if an additive functor F takes a ring of sheaves $\mathscr{O}$ to another ring of sheaves $\mathscr{S}$, then it will take a module over $\mathscr{O}$ to a module over $\mathscr{S}$, because it will preserve all the diagrams that define a module of sheaves.

The first definition is the same as saying that $\underline{\mathrm{End}}(\mathscr{M})$, in a suitable sense, is an algebra over $\mathscr{O}_X$. Since this sense is easy to clarify, let's do it now.

Definition 1.7. An *algebra* over a ringed space $(X, \mathscr{O}_X)$ is a pair $(\mathscr{A}, \phi)$, where $\mathscr{A}$ is a sheaf of rings over X, and $\phi : \mathscr{O}_X \to \mathscr{A}$ is a morphism of sheaves of rings. In this case, we'll say that $\mathscr{A}$ is an $\mathscr{O}_X$-algebra.

Remark 1.8. Observe that if $\mathscr{A}$ is an $\mathscr{O}_X$-algebra, then any $\mathscr{A}$-module is automatically an $\mathscr{O}_X$-module.

abgrp-module-lcshf *Remark* 1.9. Note that any sheaf of abelian groups over any space X is a module over the ringed space $(X, \underline{\mathbb{Z}})$.

Now that we have modules, we want morphisms between them. This is easy: we just sheafify the usual definitions for module homomorphisms.

Definition 1.10. If we have two $\mathscr{O}_X$-modules, $\mathscr{M}$ and $\mathscr{N}$, then a *a morphism of $\mathscr{O}_X$-modules* from $\mathscr{M}$ to $\mathscr{N}$ is a map $\phi : \mathscr{M} \to \mathscr{N}$ such that the following diagram commutes:

$$\begin{CD} \mathscr{O}_X \times \mathscr{M} @>>> \mathscr{M} \\ @V{1_{\mathscr{O}_X} \times \phi}VV @VV{\phi}V \\ \mathscr{O}_X \times \mathscr{N} @>>> \mathscr{N} \end{CD}$$

Definition 1.11. The category $\mathscr{O}_X$-mod, or the *category of $\mathscr{O}_X$-modules*, is the category whose objects are $\mathscr{O}_X$-modules with the morphisms between modules as defined above.

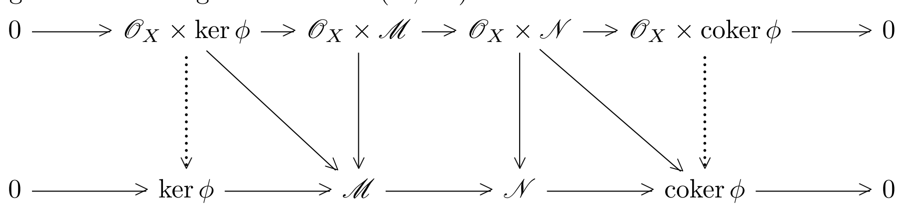
We will use $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ to refer to the group of morphisms between $\mathscr{M}$ and $\mathscr{N}$ in this category.

The next Proposition should be predictable.

Proposition 1.12. *The category $\mathscr{O}_X$-mod is abelian.*

Proof. Follows for pretty much the same reason that regular R-mod is abelian: namely, the direct sums and products have a natural $\mathscr{O}_X$-module structure, and the kernels and cokernels of $\mathscr{O}_X$-module morphisms are also naturally $\mathscr{O}_X$-modules.

The first statement is easy to check, but ugly. To see why the second is true, suppose $\phi : \mathscr{M} \to \mathscr{N}$ is a morphism of $\mathscr{O}_X$-modules; then we have the following diagram for the ring actions in $\mathrm{Shf}(X, \mathrm{Ab})$:



where we get the dotted maps from the universal properties of kernels and cokernels. It is these dotted maps that give us the natural $\mathscr{O}_X$-module structures on $\ker \phi$ and $\mathrm{coker}\, \phi$. □

We'll now make a few standard definitions analogous to those in the study of modules over rings.

Definition 1.13. A *$\mathscr{O}_X$-submodule* of an $\mathscr{O}_X$-module $\mathscr{M}$ is simply a subsheaf $\mathscr{N}$ of $\mathscr{M}$ such that the composition $\mathscr{O}_X \to \mathrm{End}(\mathscr{M}) \to \underline{\mathrm{Hom}}(\mathscr{N}, \mathscr{M})$ factors through $\underline{\mathrm{End}}(\mathscr{N})$.

In other words, the ring action map $\mathscr{O}_X \times \mathscr{N} \to \mathscr{M}$ factors through $\mathscr{N}$.

Definition 1.14. An *ideal* or a *sheaf of ideals* of $\mathscr{O}_X$ is just a $\mathscr{O}_X$-submodule $\mathscr{I}$ of $\mathscr{O}_X$.

It's clear with these definitions that for every open set U, $\mathscr{N}(U)$ is an $\mathscr{O}_X(U)$-submodule of $\mathscr{M}(U)$ in the usual sense, and that $\mathscr{I}(U)$ is an ideal of $\mathscr{O}_X(U)$, again, in the usual sense.

Now, for any $\mathscr{O}_X$-module $\mathscr{M}$, the kernel of the defining morphism $\phi : \mathscr{O}_X \to \underline{\mathrm{End}}(\mathscr{M})$ is a sheaf of ideals in $\mathscr{O}_X$.

Definition 1.15. For any $\mathscr{O}_X$-module $\mathscr{M}$, the kernel of the defining morphism $\phi : \mathscr{O}_X \to \underline{\mathrm{End}}(\mathscr{M})$ is called the *annihilator* of $\mathscr{M}$. We denote it by $\underline{\mathrm{Ann}}(\mathscr{M})$.

1.1. **Gluing Ringed Spaces.** Just as we can glue sheaves together, we can also glue ringed spaces along morphisms.

Proposition 1.16. *Suppose $\mathcal{V} = \{V_i\}$ is an open cover for X, and suppose that, for each i, V_i has a ringed space structure $(V_i, \mathscr{O}_{V_i})$, and, for every pair i, j, we have isomorphisms of ringed spaces $(\phi_{ij}, \phi_{ij}^\sharp) : (V_i \cap V_j, \mathscr{O}_{V_i}|_{V_i \cap V_j}) \to (V_i \cap V_j, \mathscr{O}_{V_j}|_{V_i \cap V_j})$ such that two conditions hold:*

1. $(\phi_{ii}, \phi_{ii}^\sharp) = (1_{V_i}, 1_{\mathscr{O}_{V_i}})$.
2. *For each triple i, j, k, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $V_i \cap V_j \cap V_k$.*

Then, there is, upto isomorphism, a unique ringed space structure $(X, \mathscr{O}_X)$ on X, and isomorphisms $(\psi_i, \psi_i^\sharp) : (V_i, \mathscr{O}_X|_{V_i}) \to (V_i, \mathscr{O}_{V_i})$, such that for each pair i, j, we have $\psi_j = \phi_{ij} \circ \psi_i$.

The next proposition is an easy, technical argument, but very essential.

gluing-morphisms

Proposition 1.17 (Gluing Morphisms)**.** *Suppose $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are two ringed spaces. Then, to give a morphism $X \to Y$ is equivalent to choosing an open cover $\mathcal{V} = \{V_i\}$ of X and giving morphisms $f_i : V_i \to Y$ such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$.*

Proof. It's clear that a morphism between X and Y will give rise to such data. Conversely, suppose we're given such data. Then, we have continuous maps $f_i : V_i \to Y$, and maps of sheaves $f_i^\sharp : \mathscr{O}_Y \to f_{i*}\mathscr{O}_{V_i}$. If we can manage to glue a global map from all this data, then it will have to be unique, because, as a map of topological spaces, it's definitely determined by the f_i, and as a map of sheaves it's determined by the $f_i^\sharp$, since they already determine how the glued together map is going to act on stalks. So it's enough to show the existence of such a gluing. For the map of topological spaces, we can just glue together the f_i in the usual fashion to get a global continuous map $f : X \to Y$, such that $f|_{V_i} = f_i$.

It remains to glue together a map of sheaves. Given an open set $U \subset Y$, $f_i^\sharp$ gives a map $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U) \cap V_i)$. Let $\phi : \mathscr{O}_X(U) \to (\mathcal{W} \cap U)(\mathscr{O}_X)$ be the natural isomorphism, where $\mathcal{W}$ is the weak covering sieve generated by $\mathcal{V}$. By the condition on the maps $f_i^\sharp$, we see that we get a map $\widetilde{f} : \mathscr{O}_Y \to f_*\mathscr{F}$, where $\mathscr{F}$ is the sheaf $U \mapsto (\mathcal{W} \cap U)(\mathscr{O}_X)$ (see [NOS, 2.7]), given by taking the direct limits over the $f_i^\sharp$. Coupled with the natural isomorphisms $\mathscr{O}_X \to \mathscr{F}$, we get the following picture for open sets $U \subset V \subset Y$.

$$\begin{array}{ccccc}
\mathscr{O}_Y(V) & \xrightarrow{\widetilde{f}_V} & \mathscr{F}(f^{-1}(V)) & \xrightarrow{\cong} & \mathscr{O}_X(f^{-1}(V)) \\
\downarrow{\scriptstyle \mathrm{res}_{V,U}} & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\
\mathscr{O}_Y(U) & \xrightarrow{\widetilde{f}_U} & \mathscr{F}(f^{-1}(U)) & \xrightarrow{\cong} & \mathscr{O}_X(f^{-1}(U))
\end{array}$$

We set the horizontal composition of maps to be $f^\sharp$. The diagram tells us that it's a morphism of sheaves. Moreover, if $V \subset V_i$, then it's easy to see that $f_V^\sharp = (f_i^\sharp)_V$. This finishes our gluing process. □

2. Operations on Sheaves of Modules

In this longish section, we'll define a few main operations on sheaves of modules and explore the relationships between them.

2.1. Sheaf Hom.

Proposition 2.1. *For any sheaf of abelian groups $\mathscr{M}$ and any $\mathscr{O}_X$-module $\mathscr{N}$, the sheaf hom $\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})$ has a natural $\mathscr{O}_X$-module structure.*

Proof. This follows from the fact that we have a natural morphism of rings of sheaves $\underline{\mathrm{End}}(\mathscr{N}) \to \underline{\mathrm{End}}(\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N}))$ given on open sets $V \subset U \subset X$ by

$$\begin{aligned} \mathrm{End}(\mathscr{N}|_U) &\to \mathrm{End}(\mathrm{Hom}(\mathscr{M}|_V,\mathscr{N}|_V)) \\ \tau &\mapsto (\phi \mapsto \mathrm{res}_{U,V}(\tau)\circ\phi) \end{aligned}$$

In other words, $\underline{\mathrm{End}}(\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N}))$ is an algebra over $\underline{\mathrm{End}}(\mathscr{N})$, which in turn is an algebra over $\mathscr{O}_X$. □

Just as in the category modules over a ring, we have a specialization of the sheaf hom to $\mathscr{O}_X$-mod.

Proposition 2.2. *If $\mathscr{M},\mathscr{N} \in \mathscr{O}_X$-mod, then the subpresheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$ of $\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})$ defined on an open set U by $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})(U) = \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{M}|_U,\mathscr{N}|_U)$ is also a sheaf.*

Proof. Suppose we have an open set U with a weak covering sieve $\mathcal{V} = \{V_i\}$, and suppose $(\phi_i) \in \mathcal{V}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N}))$. Then, since $\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})$ is a sheaf, we have a unique $\phi \in \underline{\mathrm{Hom}}(\mathscr{M}|_U,\mathscr{N}|_U)$ such that the following diagram commutes for every $W \subset U$.

$$\begin{array}{ccccccc}
\mathscr{O}_X(W)\times\mathscr{M}(W) & \xrightarrow{\cong} & (\mathcal{V}\cap W)(\mathscr{O}_X\times\mathscr{M}) & \longrightarrow & (\mathcal{V}\cap W)(\mathscr{M}) & \xrightarrow{\cong} & \mathscr{M}(W) \\
{\scriptstyle 1_{\mathscr{O}_X(W)}\times\phi_W}\downarrow & & \downarrow{\scriptstyle (1_{\mathscr{O}_X(W\cap V_i)}\times\phi_{W\cap V_i})} & & \downarrow{\scriptstyle (\phi_{W\cap V_i})} & & \downarrow{\scriptstyle \phi_W} \\
\mathscr{O}_X(W)\times\mathscr{N}(W) & \xrightarrow[\cong]{} & (\mathcal{V}\cap W)(\mathscr{O}_X\times\mathscr{N}) & \longrightarrow & (\mathcal{V}\cap W)(\mathscr{N}) & \xrightarrow[\cong]{} & \mathscr{N}(W)
\end{array}$$

Hence ϕ is a morphism of $\mathscr{O}_X$-modules, and we're done. □

Till now, we've made no commutativity assumptions on $\mathscr{O}_X$. If we do, then we can say more about $\underline{\mathrm{Hom}}_{\mathscr{O}_X}$.

sheaf-hom-module **Proposition 2.3.** *If $\mathscr{O}_X$ is a sheaf of commutative rings, then $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$ is an $\mathscr{O}_X$-submodule of $\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})$.*

Before we show this, we need a lemma.

commring-defining-map **Lemma 2.4.** *With the hypotheses of the proposition, the defining morphism of an $\mathscr{O}_X$-module $\mathscr{M}$ is actually a morphism from $\mathscr{O}_X$ into $\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{M})$.*

Proof. We want to show that, for every open subset $U \subset X$, and every $r \in \mathscr{O}_X(U)$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{O}_X(U) \times \mathscr{M}(U) & \longrightarrow & \mathscr{M}(U) \\ \Big\downarrow{\scriptstyle 1_{\mathscr{O}_X(U)} \times r} & & \Big\downarrow{\scriptstyle r} \\ \mathscr{O}_X(U) \times \mathscr{M}(U) & \longrightarrow & \mathscr{M}(U) \end{array}$$

But this follows immediately from commutativity of $\mathscr{O}_X(U)$ □

Proof of Proposition 2.3. This follows immediately from the lemma, because the composition

$$\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{N}) \to \underline{\mathrm{End}}(\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})) \to \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N}), \underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N}))$$

factors through $\underline{\mathrm{End}}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N}))$. This follows since the composition of two $\mathscr{O}_X$-module morphisms is again an $\mathscr{O}_X$-module morphism. □

Definition 2.5. The *sheaf hom* between two $\mathscr{O}_X$-modules $\mathscr{M}$ and $\mathscr{N}$ is the sheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$.

$\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\_\_,\_\_) : \mathscr{O}_X\text{-mod}^{op} \times \mathscr{O}_X\text{-mod} \to \mathrm{Shf}(X,\mathrm{Ab})$ gives us a functor, which is left exact in both variables. In case $\mathscr{O}_X$ is a sheaf of commutative rings, then it is actually a functor into $\mathscr{O}_X$-mod.

Note on Notation 1. From now on, when we refer to the sheaf hom between two $\mathscr{O}_X$-modules, this is the object we will be referring to.

We also have the notion of a bimodule. For this, we assume that the space X has two different ringed structures, $(X,\mathscr{O}_X)$ and $(X,\mathscr{S}_X)$.

Definition 2.6. An $(\mathscr{S}_X,\mathscr{O}_X)$*-bimodule* is a sheaf $\mathscr{M}$ over X that is both a $\mathscr{S}_X$-module and a right $\mathscr{O}_X$-module, such that the defining morphism $\phi : \mathscr{S}_X \to \underline{\mathrm{End}}(\mathscr{M})$ actually factors through $\underline{\mathrm{End}}_{\mathscr{O}_X^{op}}(\mathscr{M})$.

shfhom-module-struct

Proposition 2.7. *Suppose now that $\mathscr{M}$ is an $(\mathscr{O}_X,\mathscr{S}_X)$-bimodule and $\mathscr{N}$ is an $\mathscr{O}_X$-module. Then the sheaf hom $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$ has a natural $\mathscr{S}_X$-module structure.*

Proof. We have a natural map $\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{M})^{op} \to \underline{\mathrm{End}}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N})$, given on open sets $V \subset U \subset X$ by:

$$\mathrm{End}_{\mathscr{O}_X|_U}(\mathscr{M}|_U)^{op} \to \mathrm{End}(\mathrm{Hom}_{\mathscr{O}_X|_V}(\mathscr{M}|_V,\mathscr{N}|_V))$$
$$\tau \mapsto (\phi \mapsto \phi \circ (\mathrm{res}_{U,V}(\tau))$$

As noted above, the defining map for the right $\mathscr{S}_X$-module $\mathscr{M}$ factors through $\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{M})$, thus making it an $\mathscr{S}_X^{op}$-algebra. Thus, since the map above gives $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{N}))$ a right $\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{M})$-module structure, it also makes it into a right $\mathscr{S}_X^{op}$-module and hence into an $\mathscr{S}_X$-module. □

Here's another property of sheaf hom that's completely analogous to the situation in the case of R-mod.

hfhom-from-ring-isomorph

Proposition 2.8. *For any $\mathscr{O}_X$-module $\mathscr{N}$, $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{N})$ is canonically isomorphic to $\mathscr{N}$.*

Proof. Consider, for any open $U \subset X$, the map

$$\begin{aligned} \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{O}_X|_U, \mathscr{N}|_U) &\to \mathscr{N}(U) \\ \phi &\mapsto \phi_U(1_{\mathscr{O}_X(U)}) \end{aligned}$$

We have a map going in the other direction also. Given $n \in \mathscr{N}(U)$, let $\psi(n) \in \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{O}_X|_U, \mathscr{N}|_U)$ be the map that acts on open sets $V \subset U$ in the following fashion:

$$\begin{aligned} \mathscr{O}_X(V) &\to \mathscr{N}(V) \\ 1_{\mathscr{O}_X(V)} &\to \mathrm{res}_{U,V}(n) \end{aligned}$$

It's easy to check that these maps are inverses of each other. □

2.2. **Tensor Product.** Any reasonable category of modules should have a notion of a tensor product, and we have such a notion for $\mathscr{O}_X$-modules. Before we do that, let's define the equivalent of a right R-module.

Definition 2.9. Given a ringed space $(X, \mathscr{O}_X)$, we define the *opposite ringed space* $(X, \mathscr{O}_X^{op})$ to be the space X equipped with the sheaf of rings $\mathscr{O}_X^{op}$ that assigns to every open set U, the opposite ring $\mathscr{O}_X(U)^{op}$.

Remark 2.10. Note that if $\mathscr{O}_X$ is commutative, then the ringed space $(X, \mathscr{O}_X)$ is equal to its opposite.

Definition 2.11. A *right* $\mathscr{O}_X$-module is a pair $(\mathscr{N}, \psi)$, such that $\mathscr{N}$ is a sheaf of abelian groups on X and $\psi : \mathscr{O}_X^{op} \to \underline{\mathrm{End}}(\mathscr{N})$ is a morphism of rings of sheaves.

Observe that this just turns every $\mathscr{N}(U)$ into a right $\mathscr{O}_X(U)$-module. We'll usually think of ψ as defining a 'ring action map' $\psi : \mathscr{N} \times \mathscr{O}_X \to \mathscr{N}$. In effect, a right $\mathscr{O}_X$-module is just a $\mathscr{O}_X^{op}$-module.

Note on Notation 2. In general, when we use the term $\mathscr{O}_X$-module without any qualification, we will still be referring to our original definition.

tensor-product

Definition 2.12. Given an $\mathscr{O}_X$-module $\mathscr{M}$ and a right $\mathscr{O}_X$-module $\mathscr{N}$, we define their *tensor product* $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}$ to be the *sheafification* of the presheaf $\mathscr{G}$ that assigns to every open set $U \subset X$, the abelian group $\mathscr{N}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{M}(U)$.

Just like the regular tensor product in R-mod, this one also satisfies a universal property. But before that, a definition.

Definition 2.13. Let $(\mathscr{N}, \phi)$ be an $\mathscr{O}_X$-module, let $(\mathscr{M}, \psi)$ be a right $\mathscr{O}_X$-module, and let $\mathscr{F}$ be a presheaf of abelian groups. Then an *$\mathscr{O}_X$-balanced map* $\alpha : \mathscr{M} \times \mathscr{N} \to \mathscr{F}$ is a morphism of presheaves α, which is a morphism of sheaves of abelian groups in each co-ordinate, and is also such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{M} \times \mathscr{O}_X \times \mathscr{N} & \xrightarrow{1_M \times \phi} & \mathscr{M} \times \mathscr{N} \\ \Big\downarrow{\scriptstyle \psi \times 1_N} & & \Big\downarrow{\scriptstyle \alpha} \\ \mathscr{M} \times \mathscr{N} & \xrightarrow[\alpha]{} & \mathscr{F} \end{array}$$

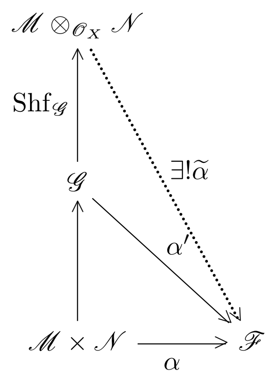
Observe that if $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} = \operatorname{Shf} \mathscr{G}$ (see Definition 2.12), then the natural map $\iota : \mathscr{M} \times \mathscr{N} \to \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ given by the composition $\mathscr{M} \times \mathscr{N} \to \mathscr{G} \to \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ is $\mathscr{O}_X$-balanced. This follows from the fact that the corresponding property holds for the tensor product of modules over a ring.

tensor-product-univ-prp

Proposition 2.14. *The natural map $\iota : \mathscr{M} \times \mathscr{N} \to \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ satisfies the following universal property:*

For any other $\mathscr{O}_X$-balanced map $\alpha : \mathscr{M} \times \mathscr{N} \to \mathscr{F}$, there is a unique morphism $\widetilde{\alpha} : \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ of sheaves of abelian groups such that $\widetilde{\alpha} \circ \iota = \alpha$.

Proof. It's enough to show that with $\mathscr{G}$ as in Definition 2.12, there is a unique presheaf morphism $\alpha' : \mathscr{G} \to \mathscr{F}$ such that the bottom half of this diagram commutes:



Then, by the universal property of sheafification (see [NOS, 2.13]), we get the unique map $\widetilde{\alpha}$ that makes everything work. But we get the map α' simply from the corresponding universal property of tensor products for modules over rings! $\square$

Suppose now that we have a $(\mathscr{S}_X, \mathscr{O}_X)$-bimodule $\mathscr{M}$ and a $\mathscr{O}_X$-module $\mathscr{N}$. Observe that if we set $\operatorname{Bal}_{\mathscr{O}_X}^{\mathscr{M} \times \mathscr{N}}(U)$ to be the group of $\mathscr{O}_X|_U$-balanced maps $\mathscr{M}|_U \times \mathscr{N}|_U \to \mathscr{M}|_U \times \mathscr{N}|_U$, then we get a presheaf of abelian groups $\operatorname{Bal}_{\mathscr{O}_X}^{\mathscr{M} \times \mathscr{N}}$, which has a natural inclusion into the sheaf $\underline{\operatorname{End}}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N})$ given to us by the universal property of the tensor product (Proposition 2.14). Moreover, we have a map $\underline{\operatorname{End}}_{\mathscr{O}_X^{op}}(M) \to \operatorname{Bal}_{\mathscr{O}_X}^{\mathscr{M} \times \mathscr{N}}$ given by $\tau \mapsto \tau \times 1_{\mathscr{N}}$. Putting all this together, we find that we have a natural map

$$\mathscr{S}_X \to \underline{\operatorname{End}}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}).$$

The last paragraph can be condensed into the following proposition.

or-product-module-struct

Proposition 2.15. *Given an $(\mathscr{S}_X, \mathscr{O}_X)$-bimodule $\mathscr{M}$ and an $\mathscr{O}_X$-module $\mathscr{N}$, the tensor product $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ has a natural $\mathscr{S}_X$-module structure. Moreover, let $\alpha : \mathscr{M} \times \mathscr{N} \to \mathscr{F}$ be an $\mathscr{O}_X$-balanced map, and let $\mathscr{F}$ be an $\mathscr{S}_X$-module, such that the* *following diagram commutes.*

$$\begin{array}{ccc} \mathscr{S}_X \times \mathscr{M} \times \mathscr{N} & \xrightarrow{1_{\mathscr{S}_X} \times \alpha} & \mathscr{S}_X \times \mathscr{F} \\ {\scriptstyle \phi \times 1_{\mathscr{N}}} \downarrow & & \downarrow {\scriptstyle \psi} \\ \mathscr{M} \times \mathscr{N} & \xrightarrow[\alpha]{} & \mathscr{F} \end{array}$$

where ϕ and ψ are the defining morphisms of $\mathscr{M}$ and $\mathscr{F}$ respectively.

Then the lifting $\widetilde{\alpha} : \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \to \mathscr{F}$ is in fact a map of $\mathscr{S}_X$-modules.

Proof. The proof of the first part is contained in the above discussion. For the second part, observe that we can insert $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ into the commutative diagram above quite simply:

$$\begin{array}{ccccc} \mathscr{S}_X \times \mathscr{M} \times \mathscr{N} & \xrightarrow{1_{\mathscr{S}_X} \times \iota} & \mathscr{S}_X \times \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} & \xrightarrow{1_{\mathscr{S}_X} \times \widetilde{\alpha}} & \mathscr{S}_X \times \mathscr{F} \\ {\scriptstyle \phi \times 1_{\mathscr{N}}} \downarrow & & {\scriptstyle \widetilde{\phi}} \downarrow & & \downarrow {\scriptstyle \psi} \\ \mathscr{M} \times \mathscr{N} & \xrightarrow[\iota]{} & \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} & \xrightarrow[\widetilde{\alpha}]{} & \mathscr{F} \end{array}$$

where $\widetilde{\phi}$ is the defining morphism for $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$. The square on the left commutes; so we find that

$$\widetilde{\alpha} \circ \widetilde{\phi} \circ (1_{\mathscr{S}_X} \times \iota) = \psi \circ (1_{\mathscr{S}_X} \times \widetilde{\alpha}) \circ (1_{\mathscr{S}_X} \times \iota).$$

Then, by the uniqueness of liftings, we see that we must have

$$\widetilde{\alpha} \circ \widetilde{\phi} = \psi \circ (1_{\mathscr{S}_X} \times \widetilde{\alpha}),$$

which shows that $\widetilde{\alpha}$ is a morphism of $\mathscr{S}_X$-modules. □

Here's another analogue from the world of R-mod.

tensor-ring-isomorph **Proposition 2.16.** *For any $\mathscr{O}_X$-module $\mathscr{M}$, the sheaf $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is also an $\mathscr{O}_X$-module, and it's naturally isomorphic to $\mathscr{M}$.*

Proof. The first part of the proposition follows from Proposition 2.15 and the fact that $\mathscr{O}_X$ is naturally an $(\mathscr{O}_X, \mathscr{O}_X)$-bimodule. For the second part, observe that the defining map $\phi : \mathscr{O}_X \times \mathscr{M} \to \mathscr{M}$ is $\mathscr{O}_X$-balanced. This lifts uniquely, by Proposition 2.15, to a map of $\mathscr{O}_X$-modules, $\widetilde{\phi} : \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \mathscr{M}$ (Observe that the diagram that ϕ has to satisfy is simply expressing the associativity of the ring action). Now, we also have a natural map in the other direction given by the composition

$$\psi : \mathscr{M} \xrightarrow{i_2} \mathscr{O}_X \times \mathscr{M} \xrightarrow{\iota} \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}$$

At the level of open sets it's easy to check that these two maps are inverses to each other. □

Now we are ready to explore the relationship between tensor product and sheaf hom. The next result should be familiar from the study of modules over a ring. But before we do that, let's fix some notation. Suppose $\mathscr{M}$ is an $(\mathscr{S}_X, \mathscr{O}_X)$-bimodule. Then $\mathscr{M}$ induces a functor

$$\mathscr{M} \otimes_{\mathscr{O}_X} \text{--} : \mathscr{O}_X\text{-mod} \to \mathscr{S}_X\text{-mod}$$

by Proposition 2.15, and a functor

$$\underline{\mathrm{Hom}}_{\mathscr{S}_X}(\mathscr{M}, \text{--}) : \mathscr{S}_X\text{-mod} \to \mathscr{O}_X\text{-mod}$$

by Proposition 2.7. This leads us to the following proposition.

tensor-shfhom-adj

Proposition 2.17. *The functor $\mathscr{M} \otimes_{\mathscr{O}_X} \text{--}$ is left adjoint to the functor $\underline{\mathrm{Hom}}_{\mathscr{S}_X}(\mathscr{M}, \text{--})$.*

Proof. Given an $\mathscr{O}_X$-module $\mathscr{N}$ and an $\mathscr{S}_X$ module $\mathscr{P}$, we need to construct a natural isomorphism

$$\mathrm{Hom}_{\mathscr{S}_X}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}, \mathscr{P}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{N}, \underline{\mathrm{Hom}}_{\mathscr{S}_X}(\mathscr{M}, \mathscr{P})).$$

To do this, it'll be easier if we introduce the category of *presheaves* of modules over $\mathscr{S}_X$. All the definitions are the same, with the word 'sheaf' replaced everywhere by 'presheaf'. What also remains true is that the sheafification functor is still a left adjoint to the forgetful functor from the category of presheaves of modules to $\mathscr{O}_X$-mod. So to find our natural isomorphism, it will suffice to find a natural isomorphism of groups

$$\mathrm{Hom}_{\mathscr{S}_X}(\mathscr{G}, \mathscr{P}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{N}, \underline{\mathrm{Hom}}_{\mathscr{S}_X}(\mathscr{M}, \mathscr{P})),$$

where $\mathscr{G}$ is as in Definition 2.12. Note that we implicitly used this principle in the proof of Proposition 2.14.

Suppose we're given a ϕ on the left hand side. This is a collection of $\mathscr{S}_X(U)$-module homomorphisms $\phi_U : \mathscr{M}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{N}(U) \to \mathscr{P}(U)$, for each open set $U \subset X$. Given such a natural transformation, we send it to the morphism $\Psi(\phi) : \mathscr{N} \to \underline{\mathrm{Hom}}_{\mathscr{S}_X}(\mathscr{M}, \mathscr{P})$ that, for an open set U, sends each $n \in \mathscr{N}(U)$ to the following morphism $\Psi(\phi)_U(n) : \mathscr{M}|_U \to \mathscr{P}|_U$

$$\begin{aligned}(\Psi(\phi)_U(n))_V : \mathscr{M}(V) &\to \mathscr{P}(V)\\ m &\mapsto \phi_V(m \otimes \mathrm{res}_{U,V}(n))\end{aligned}$$

Now, suppose we're given ψ on the right hand side. This is a collection of $\mathscr{O}_X(U)$-module homomorphisms $\psi_U : \mathscr{N}(U) \to \mathrm{Hom}_{\mathscr{S}_X|_U}(\mathscr{M}|_U, \mathscr{P}|_U)$, for each open set $U \subset X$. We send this ψ to the morphism $\Phi(\psi) : \mathscr{G} \to \mathscr{P}$ that, for an open set U, sends each $m \otimes n \in \mathscr{G}(U)$ to $(\psi_U(n))_U(m)$.

In one direction, we have

$$\begin{aligned}(\Phi\Psi\phi)_U(m \otimes n) &= ((\Psi\phi)_U(n))_U(m)\\ &= \phi_U(m \otimes n)\end{aligned}$$

In the other direction, we have

$$\begin{aligned}((\Psi\Phi\psi)_U(n))_V(m) &= (\Phi\psi)_V(m \otimes \mathrm{res}_{U,V}(n))\\ &= (\psi_V(\mathrm{res}_{U,V}(n)))_V(m)\\ &= (\mathrm{res}_{U,V}(\psi_U(n)))_V(m)\\ &= (\psi_U(n))_V(m)\end{aligned}$$

Hence, Ψ and Φ are inverses of each other, and we have the isomorphism that we sought. □

Corollary 2.18. *With the notation as in the Proposition, we have*

$$\underline{\mathrm{Hom}}_{\mathscr{S}_X}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}, \mathscr{P}) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{N}, \underline{\mathrm{Hom}}_{\mathscr{S}_X}(\mathscr{M}, \mathscr{P})).$$

Proof. Just observe that, by the Proposition, there are natural isomorphisms between each side specialized to an open set U. □

Corollary 2.19. *For any right $\mathscr{O}_X$-module $\mathscr{M}$, the functor*

$$\mathscr{M} \otimes_{\mathscr{O}_X} \text{--} : \mathscr{O}_X\text{-}mod \to \mathrm{Shf}(X, \mathrm{Ab})$$

is right exact.

Proof. Recall from Remark 1.9 that $\mathrm{Shf}(X, \mathrm{Ab})$ is the same as the category $\underline{\mathbb{Z}}$-mod. Now, from the Proposition, we have a right adjoint to $\mathscr{M} \otimes_{\mathscr{O}_X} \text{--}$ given by:

$$\underline{\mathrm{Hom}}(\mathscr{M}, \text{--}) : \mathrm{Shf}(X, \mathrm{Ab}) \to \mathscr{O}_X\text{-mod}$$

From this, the Corollary follows. □

Now, for the associativity of the tensor product.

tensor-product-assoc

Proposition 2.20. *Given an $\mathscr{O}_X$-module $\mathscr{M}$, a right $\mathscr{S}_X$-module $\mathscr{N}$, and an $(\mathscr{S}_X, \mathscr{O}_X)$-bimodule $\mathscr{P}$, there is a canonical isomorphism of sheaves of abelian groups*

$$(\mathscr{N} \otimes_{\mathscr{S}_X} \mathscr{P}) \otimes_{\mathscr{O}_X} \mathscr{M} \xrightarrow{\cong} \mathscr{N} \otimes_{\mathscr{S}_X} (\mathscr{P} \otimes_{\mathscr{O}_X} \mathscr{M}).$$

Proof. This is proved most easily by specializing on an open set and noting that we have a canonical isomorphism there that we get from module theory. □

Note on Notation 3. Given this canonical isomorphism, we'll usually write the tensor product as $\mathscr{N} \otimes_{\mathscr{S}_X} \mathscr{P} \otimes_{\mathscr{O}_X} \mathscr{M}$, without any parantheses.

Just as in the case for sheaf hom, if we impose some commutativity constraints, then we can get some stronger results.

twist-tensor-commutative

Proposition 2.21. *If $\mathscr{O}_X$ is a sheaf of commutative rings, then, for any $\mathscr{O}_X$-modules $\mathscr{M}$ and $\mathscr{N}$, the tensor product $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ has a natural $\mathscr{O}_X$-module structure. Moreover, we have a canonical isomorphism*

$$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \underset{\cong}{\xrightarrow{\tau}} \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}$$

Proof. The first part follows from Proposition 2.15 and the fact that over any commutative sheaf of rings $\mathscr{O}_X$, an $\mathscr{O}_X$-module is automatically an $(\mathscr{O}_X, \mathscr{O}_X)$-bimodule.

For the second, simply note that we have a canonical isomorphism

$$\mathscr{M}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{N}(U) \xrightarrow{\cong} \mathscr{N}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{M}(U).$$

on any open set $U \subset X$. □

2.3. **The Direct and Inverse Image Functors.** Of course, there's no reason why we need to stick with one topological space. If $\mathscr{M}$ is an $\mathscr{O}_X$-module, and we have a morphism of ringed spaces $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, then we can treat $f_*\mathscr{M}$ as an $\mathscr{O}_Y$-module via the composition

$$\mathscr{O}_Y \xrightarrow{f^\sharp} f_*\mathscr{O}_X \to f_*(\underline{\mathrm{End}}(M)) = \underline{\mathrm{End}}(f_*M).$$

See [NOS, 9.6] for the last equality.

Definition 2.22. The functor $f_* : \mathscr{O}_X\text{-mod} \to \mathscr{O}_Y\text{-mod}$ given by $\mathscr{M} \mapsto f_*\mathscr{M}$ will from now on be called the *direct image functor*.

We can now specialize [NOS, 9.6].

Proposition 2.23. *If $\mathscr{M}, \mathscr{N} \in \mathscr{O}_X$-mod and $f : X \to Y$ is a continuous map, then we have a natural isomorphism:*

$$f_* \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{f_*\mathscr{O}_X}(f_*\mathscr{M}, f_*\mathscr{N}).$$

Proof. If we go back to the proof of [NOS, 9.6], then we'll see that the only thing that needs proving is, given an open set $U \subset X$, the isomorphism

$$\mathrm{Hom}_{\mathscr{O}_X|_{f^{-1}(U)}}(\mathscr{M}|_U, \mathscr{N}|_U) \xrightarrow{\cong} \mathrm{Hom}_{f_*\mathscr{O}_X|_U}(f_*\mathscr{M}|_U, f_*\mathscr{N}|_U).$$

Just as in that proof, we have a natural map easily defined in one direction. The map in the other direction is defined as in the earlier proof. We just have to check that the map we get is a map of $\mathscr{O}_X$-modules. Given a ϕ on the right hand side, we get a ψ on the left, so that this diagram commutes (the notation is from the proof of the earlier proposition.).

$$\begin{array}{ccccccc}
\mathscr{O}_X(W) \times \mathscr{M}(W) & \xrightarrow{\cong} & (\mathcal{V} \cap W)(\mathscr{O}_X \times \mathscr{M}) & \longrightarrow & (\mathcal{V} \cap W)(\mathscr{M}) & \xrightarrow{\cong} & \mathscr{M}(W) \\
\downarrow {\scriptstyle 1_{\mathscr{O}_X(W)} \times \psi_W} & & \downarrow {\scriptstyle (1_{\mathscr{O}_X(f^{-1}(V))} \times \phi_V)} & & \downarrow {\scriptstyle (\phi_V)} & & \downarrow {\scriptstyle \psi_W} \\
\mathscr{O}_X(W) \times \mathscr{N}(W) & \xrightarrow{\cong} & (\mathcal{V} \cap W)(\mathscr{O}_X \times \mathscr{N}) & \longrightarrow & (\mathcal{V} \cap W)(\mathscr{N}) & \xrightarrow{\cong} & \mathscr{N}(W)
\end{array}$$

The square in the middle commutes, precisely because ϕ is a morphism of $f_*\mathscr{O}_X|_U$-modules. This tells us that the whole diagram commutes, and so ψ is indeed a morphism of $\mathscr{O}_X$-modules. □

Now we turn to the inverse image functor. Fix for now a map of ringed spaces $(f, f^\sharp, f^\flat) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. Observe that f^{-1} gives a functor $\mathscr{O}_Y\text{-mod} \to f^{-1}\mathscr{O}_Y\text{-mod}$. And f_* gives a functor $f^{-1}\mathscr{O}_Y\text{-mod} \to \mathscr{O}_Y\text{-mod}$. The adjunction between them is still maintained. So let's record that in the following proposition.

module-dirinv-adjunction

Proposition 2.24. *Given an $\mathscr{O}_Y$-module $\mathscr{M}$ and an $f^{-1}\mathscr{O}_Y$-module $\mathscr{N}$, there is a natural isomorphism*

$$\mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{M}, \mathscr{N}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{M}, f_*\mathscr{N})$$

Proof. □

le-dirinv-adjunction-cor

Corollary 2.25. *With the hypotheses as in the Proposition, we have the following natural isomorphism:*

$$f_*(\underline{\mathrm{Hom}}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{M}, \mathscr{N})) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathscr{O}_Y}(\mathscr{M}, f_*\mathscr{N})$$

Proof. Specializing on an open set $U \subset Y$ on either side, we find that we have to show:

$$\mathrm{Hom}_{(f^{-1}\mathscr{O}_Y)|_{f^{-1}(U)}}((f^{-1}\mathscr{M})|_{f^{-1}(U)}, \mathscr{N}|_{f^{-1}(U)}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{O}_Y|_U}(\mathscr{M}|_U, (f_*\mathscr{N})|_U).$$

But this is just the situation of Proposition 2.24 with f being replaced by the map $f|_{f^{-1}(U)} : f^{-1}(U) \to U$. □

Just as the direct image functor commutes with $\underline{\mathrm{Hom}}$, the inverse image functor commutes with tensor products.

invimage-commute-tensor

Proposition 2.26. *Given an $\mathscr{O}_Y$-module $\mathscr{N}$ and a right $\mathscr{O}_Y$-module $\mathscr{M}$, we have a natural isomorphism*

$$f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N} \xrightarrow{\cong} f^{-1}(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}).$$

Proof. Note that since f^{-1} is additive, we have a natural map $f^{-1}\mathscr{M} \times f^{-1}\mathscr{N} \to f^{-1}(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N})$, given by the composition

$$f^{-1}\mathscr{M} \times f^{-1}\mathscr{N} = f^{-1}(\mathscr{M} \times \mathscr{N}) \to f^{-1}(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}).$$

Now, this lifts to a unique map

$$f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N} \to f^{-1}(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}).$$

And the easiest way to see that this map is an isomorphism is to descend to stalks. For a point $x \in X$, this will just be the identity map

$$\mathscr{M}_{f(x)} \otimes_{\mathscr{O}_{Y\,f(x)}} \mathscr{N}_{f(x)} \to (f^{-1}(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}))_x = \mathscr{M}_{f(x)} \otimes_{\mathscr{O}_{Y\,f(x)}} \mathscr{N}_{f(x)}.$$

So the isomorphism is proved. □

Notice that the map $f^\flat : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ gives $\mathscr{O}_X$ an $(\mathscr{O}_X, f^{-1}\mathscr{O}_Y)$-bimodule structure. So given an $\mathscr{O}_Y$-module $\mathscr{M}$, we can consider the $\mathscr{O}_X$-module $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}$. Given an $\mathscr{O}_X$-module $\mathscr{N}$, we have the following sequence of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{M}, \mathscr{N}) &\cong \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{O}_Y, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{N})) \\ &\cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y, f_*\mathscr{N})\end{aligned}$$

where the first isomorphism follows from Proposition 2.17 and the second follows from Propositions 2.8 and 2.24.

This suggests a definition.

Definition 2.27. The *inverse image functor* $f^* : \mathscr{O}_Y\text{-mod} \to \mathscr{O}_X\text{-mod}$ is the functor that assigns to every $\mathscr{O}_Y$-module $\mathscr{M}$, the $\mathscr{O}_X$-module $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M}$.

Remark 2.28. From now on, whenever we talk about direct and inverse image functors, these are the ones we'll be referring to.

We've proved an important fact about f^* above.

Proposition 2.29. *The functor $f_* : \mathscr{O}_X\text{-mod} \to \mathscr{O}_Y\text{-mod}$ is right adjoint to the functor $f^* : \mathscr{O}_Y\text{-mod} \to \mathscr{O}_X\text{-mod}$. In particular, f_* is left exact and f^* is right exact.*

Proof. Found above. □

Example 2.1. Although f^{-1} was exact, f^* need not necessarily be exact, for the simple reason that tensor product is not exact. For example take any topological space X and put two ringed structures on it, corresponding to $\underline{\mathbb{Z}}_2$ and $\underline{\mathbb{Z}}$. Then, the identity map I from $(X, \underline{\mathbb{Z}}_2)$ to $(X, \underline{\mathbb{Z}})$ is a morphism of ringed spaces, and in the latter, we have an exact sequence:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\times 2} \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_2 \to 0.$$

When we act upon this with I^*, we get

$$0 \to \underline{\mathbb{Z}}_2 \xrightarrow{0} \underline{\mathbb{Z}}_2 \to \underline{\mathbb{Z}}_2 \to 0,$$

which is definitely not exact on the left.

With this adjointness property, we can use the same proof as in [NOS, 7.10] to get the following proposition.

Proposition 2.30. *If $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\sharp) : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ are morphisms of ringed spaces, then $(g \circ f)^* = f^* g^*$.*

restrict-invimage **Corollary 2.31.** *If $V \subset Y$ is an open set, and $\mathscr{M}$ is an $\mathscr{O}_Y$-module, then*

$$(f^*\mathscr{M})|_{f^{-1}(V)} = (f|_{f^{-1}(V)})^*(\mathscr{M}|_V)$$

where $f|_{f^{-1}(V)} : f^{-1}(V) \to V$.

Proof. Note that we have the following commutative diagram:

$$\begin{array}{ccc} f^{-1}(V) & \hookrightarrow & X \\ {\scriptstyle f|_{f^{-1}(V)}} \downarrow & & \downarrow {\scriptstyle f} \\ V & \hookrightarrow & Y \end{array}$$

Now, just use the Proposition, noting that for an inclusion $j : V \to Y$, $j^*\mathscr{M} = \mathscr{M}|_V$. □

We can also strengthen Proposition 2.26 in the case where $\mathscr{O}_Y$ is commutative.

-invimage-commute-tensor **Proposition 2.32.** *If $\mathscr{O}_Y$ is a sheaf of commutative rings, then, for any $\mathscr{O}_Y$-modules $\mathscr{M}$ and $\mathscr{N}$, we have a canonical isomorphism*

$$f^*(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}) \cong f^*\mathscr{M} \otimes_{\mathscr{O}_X} f^*\mathscr{N}$$

Proof. We have the following sequence of isomorphisms:

$$\begin{aligned} f^*(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}) &= \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}) \\ &\cong \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N} \\ &\cong f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N} \\ &\cong f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{N} \\ &\cong f^*\mathscr{M} \otimes_{\mathscr{O}_X} f^*\mathscr{N} \end{aligned}$$

where we used Proposition 2.21 for the third and last steps, Proposition 2.16 in the second to last step, Proposition 2.26 in the second step, and Proposition 2.20 just about in every step. □

2.4. **Extensions by Zero, Restrictions and Sections with Local Support.** Here we'll investigate the relationships between the functors i_*, j_*, $j_!$ and $\mathscr{H}^0_Z(\_\_)$ defined earlier, and the tensor product and sheaf hom over $\mathscr{O}_X$ that we've defined in this section.

The functor $j_!$ will be easiest to handle, mainly because of the following proposition that follows immediately from [NOS, 8.8].

extzero-restrict-adj **Proposition 2.33.** *For any open set $U \subset X$, any $\mathscr{O}_X|_U$-module $\mathscr{M}$ and any $\mathscr{O}_X$-module $\mathscr{N}$, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathscr{O}_X}(j_!\mathscr{M}, \mathscr{N}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{M}, \mathscr{N}|_U).$$

where $j : U \hookrightarrow X$ is the inclusion map.

Proof. Just as in the proof of [NOS, 8.8]. □

extzero-restrict-adj-cor **Corollary 2.34.** *With the hypotheses as above, we have an isomorphism of sheaves:*

$$\underline{\mathrm{Hom}}_{\mathscr{O}_X}(j_!\mathscr{M},\mathscr{N}) \xrightarrow{\cong} j_*(\underline{\mathrm{Hom}}_{\mathscr{O}_X|_U}(\mathscr{M},\mathscr{N}|_U)).$$

Proof. Follows from the Proposition and the same kind of argument that was used in Corollary 2.25. □

Now, we're ready to say everything we want in the following proposition.

tzero-localsup-relations **Proposition 2.35.** *Suppose $U \subset X$ is open, with inclusion map $j : U \hookrightarrow X$, and suppose $i : Z \hookrightarrow X$ is the inclusion of $Z = X \setminus U$ into X. Suppose also that $\mathscr{M} \in \mathscr{O}_X$-mod. Then, we have the following natural isomorphisms:*

(1) $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(j_!(\mathscr{O}_X|_U),\mathscr{M}) \cong j_*(\mathscr{M}|_U)$

(2) $j_!(\mathscr{O}_X|_U) \otimes_{\mathscr{O}_X} \mathscr{M} \cong j_!(\mathscr{M}|_U)$

(3) $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(i_*(\mathscr{O}_X|_Z),\mathscr{M}) \cong \mathscr{H}^0_Z(\mathscr{M})$

(4) $i_*(\mathscr{O}_X|_Z) \otimes_{\mathscr{O}_X} \mathscr{M} \cong i_*(\mathscr{M}|_U)$

Proof.

(1) We have the following sequence of isomorphisms

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathscr{O}_X}(j_!(\mathscr{O}_X|_U),\mathscr{M}) &\cong j_*(\underline{\mathrm{Hom}}_{\mathscr{O}_X|_U}(\mathscr{O}_X|_U,\mathscr{M}|_U))\\ &\cong j_*(\mathscr{M}|_U)\end{aligned}$$

where the first isomorphism follows from Corollary 2.34, and the second from Proposition 2.8.

(2) We will use Yoneda's Lemma ([CT, 1.3]). Let $\mathscr{N}$ be any $\mathscr{O}_X$-module. Then we have the following sequence of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{O}_X}(j_!(\mathscr{O}_X|_U)\otimes_{\mathscr{O}_X}\mathscr{M},\mathscr{N}) &\cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{M},\mathrm{Hom}_{\mathscr{O}_X}(j_!(\mathscr{O}_X|_U),\mathscr{N}))\\ &\cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{M},j_*(\mathscr{N}|_U))\\ &\cong \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{M}|_U,\mathscr{N}|_U)\\ &\cong \mathrm{Hom}_{\mathscr{O}_X}(j_!(\mathscr{M}|_U),\mathscr{N})\end{aligned}$$

where the first isomorphism follows from Proposition 2.17, the second from part (1), the third from Proposition 2.24 and the last from Proposition 2.33. Yoneda's Lemma does the rest of the work for us.

(3) Observe that if we apply the functor $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\_\_,\mathscr{M})$ to the exact sequence

$$0 \to j_!(\mathscr{O}_X|_U) \to \mathscr{O}_X \to i_*(\mathscr{O}_X|_Z) \to 0,$$

and if we use part (1) and Proposition 2.8, then we get an exact sequence:

$$0 \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(i_*(\mathscr{O}_X|_Z) \to \mathscr{O}_X \to j_*(\mathscr{O}_X|_U).$$

Comparing this sequence with the one in [NOS, 8.13], we find the isomorphism that was claimed.

(4) Start with the same exact sequence as in the last part, but this time apply the functor $\_\_\otimes_{\mathscr{O}_X}\mathscr{M}$ to get the exact sequence

$$0 \to j_!(\mathscr{M}|_U) \to \mathscr{M} \to i_*(\mathscr{O}_X|_Z)\otimes_{\mathscr{O}_X}\mathscr{M} \to 0$$

where exactness on the left is checked by descending to stalks.

Comparing this with the exact sequence from [NOS, 8.5], we find the isomorphism we need.

□

3. Locally Free Sheaves of Modules

We'll define an important class of $\mathscr{O}_X$-modules in this section, and explore its properties as a first step towards coherence.

Definition 3.1. A *free* $\mathscr{O}_X$-module is an $\mathscr{O}_X$-module that is isomorphic to $\mathscr{O}_X^I$, for some indexing set I, where $\mathscr{O}_X^I$ is the direct sum $\oplus_{i\in I}\mathscr{O}_X$.

In this case, if I is finite, we say that the *rank* of $\mathscr{M}$ is $\#I$. Otherwise, we say that it has *infinite rank*.

But for sheaves, as always, it's more useful to define things on a local level.

Definition 3.2. A *locally free* $\mathscr{O}_X$-module is an $\mathscr{O}_X$-module $\mathscr{M}$ such that for every point $x \in X$, there is a neighborhood $U \ni x$ such that $\mathscr{M}|_U$ is isomorphic to a free $\mathscr{O}_X|_U$-module.

The *rank* of $\mathscr{M}$ in such a neighborhood U is just the rank $\mathscr{M}|_U$ as a free $\mathscr{O}_X|_U$-module. We say that a locally free sheaf $\mathscr{M}$ is of *finite rank* if the rank in every such neighborhood is finite.

Suppose now that $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module, and suppose U and V are two open sets such that $\mathscr{M}$ is free when restricted to both. If $U \cap V \neq \phi$, then the rank of $\mathscr{M}$ in U is the same as the rank of $\mathscr{M}$ in V, since the ranks must agree on $U \cap V$. So, in particular, the rank of $\mathscr{M}$ is well defined on any connected open set.

Locally free $\mathscr{O}_X$-modules are projective in a sense that's made precise by the following Proposition.

free-modules-projective

Proposition 3.3. *Given an* $\mathscr{M} \xrightarrow{g} \mathscr{M}'' \to 0$*, and a morphism* $\phi : \mathscr{O}_X^n \to \mathscr{M}''$*, we can find an open neighborhood* $U \subset X$ *of any point* $x \in X$*, and a morphism* $\widetilde{\phi} : \mathscr{O}_X|_U^n \to \mathscr{M}|_U$*, such that* $g|_U \circ \widetilde{\phi} = \phi|_U$*.*

Proof. Observe that $\mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{O}_X|_U^n, \mathscr{M}|_U) \cong \mathscr{M}(U)^n$ (see Proposition 2.8). So to prove the assertion, it's enough to show that for every set of n elements $\{s_i : 1 \leq i \leq n\}$ in $\Gamma(X, \mathscr{M}'')$, we can find an open set U, such that $\{\mathrm{res}_{X,U}(s_i) : 1 \leq i \leq n\}$ is contained in the image of g_U. We'll do this by induction on n; for $n = 1$, this is [NOS, 4.8]. Suppose we can do this for $n-1$ elements; then we're essentially back to the case of one element, and so we're done again. □

invimg-free

Proposition 3.4. *If* $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *is a morphism of ringed spaces, then the following statements hold:*

(1) *If* $\mathscr{M}$ *is a free* $\mathscr{O}_Y$*-module, then* $f^*\mathscr{M}$ *is a free* $\mathscr{O}_X$*-module.*
(2) *If* $\mathscr{M}$ *is a locally free* $\mathscr{O}_Y$*-module, then* $f^*\mathscr{M}$ *is a locally free* $\mathscr{O}_X$*-module.*

Proof. (1) This follows from two facts: $f^*\mathscr{O}_Y \cong \mathscr{O}_X$ and f^* is additive.
(2) This follows from (1) and Proposition 2.31.

□

Definition 3.5. The *dual* $\check{\mathscr{M}}$ of an $\mathscr{O}_X$-module $\mathscr{M}$ is the right $\mathscr{O}_X$-module $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$. If $\mathscr{M}$ is actually a right $\mathscr{O}_X$-module, then the dual becomes an $\mathscr{O}_X$-module structure.

The dual is very useful in exploring the properties of locally free modules of finite rank, mainly because of the two propositions that follow.

Proposition 3.6. *If $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module of finite rank, then $\mathscr{M}$ is isomorphic ; that is, $\mathscr{M} \cong \check{\check{\mathscr{M}}}$.*

Proof. We have a natural map

$$\mathscr{M} \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X), \mathscr{O}_X)$$

given by the usual double dual construction. Since locally $\mathscr{M}$ is free of finite rank, it's easy to see that this natural map is actually a local isomorphism, which is sufficient for it to be an isomorphism. In more detail, since this is a local question, we can assume $\mathscr{M} \cong \mathscr{O}_X^n$, for some $n \in \mathbb{N}$, and so we get the following commutative diagram:

$$\begin{array}{ccc} \mathscr{M} & \longrightarrow & \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X), \mathscr{O}_X) \\ \cong \big\downarrow & & \big\downarrow \cong \\ \mathscr{O}_X^n & \xrightarrow[\cong]{} & \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{O}_X), \mathscr{O}_X) \end{array}$$

The bottom row is a canonical isomorphism, since

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{O}_X), \mathscr{O}_X) &\cong \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{O}_X) \\ &\cong \mathscr{O}_X^n \end{aligned}$$

where we used the isomorphism from Proposition 2.8 twice, along with the additivity of sheaf hom.

Thus, the map on the top is also an isomorphism, which is what we had claimed. □

Proposition 3.7. *If $\mathscr{M}$ is locally free of finite rank, then, for any $\mathscr{O}_X$-module $\mathscr{N}$, we have an isomorphism*

$$\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) \cong \check{\mathscr{M}} \otimes_{\mathscr{O}_X} \mathscr{N}$$

Proof. We have a natural map

$$\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{N} \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}).$$

that's induced by the natural map on the level of $\mathscr{O}_X(U)$-modules. We prove just as above, using Propositions 2.8 and 2.16, that this map is a local isomorphism, and is thus an isomorphism. □

This proposition has an important corollary. For this, we specialize now to the case where $\mathscr{O}_X$ is *commutative*, and make the following definition.

Definition 3.8. An *invertible* $\mathscr{O}_X$-module $\mathscr{M}$ is one for which we can find an $\mathscr{O}_X$-module $\mathscr{N}$ such that $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \cong \mathscr{O}_X$.

$\mathscr{N}$ is then known as an *inverse* for $\mathscr{M}$.

locally-free-invertible **Corollary 3.9.** *Any locally free $\mathscr{O}_X$-module $\mathscr{M}$ of constant rank 1 is invertible, with inverse its dual $\check{\mathscr{M}}$.*

Proof. From the Proposition, it suffices to show that

$$\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M},\mathscr{M}) \cong \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{M}) \cong \mathscr{O}_X.$$

Now, since $\mathscr{O}_X$ is commutative, the defining morphism for $\mathscr{M}$ factors through $\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{M})$ (See Lemma 2.4). Just as before, we just need to check that this gives a local isomorphism. But this is clear, since, locally, $\mathscr{M} \cong \mathscr{O}_X$, and $\underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{O}_X) \cong \mathscr{O}_X$ canonically. □

While locally free $\mathscr{O}_X$-modules are very nice, unfortunately they impose too strict a constraint. As a subcategory of $\mathscr{O}_X$-mod, they're not closed under taking of kernels, cokernels and extensions. To remedy this we need the unbelievably important notion of a coherent $\mathscr{O}_X$-module. This corresponds roughly to the notion of a finitely presented module over a ring.

quasicoherent-sheaves

4. Quasicoherent and Coherent Sheaves of Modules

This section is based largely on Serre's exposition in his famous Faisceaux Algébriques Cohérents.

4.1. Modules of Finite Type.

Definition 4.1. An $\mathscr{O}_X$-module $\mathscr{M}$ is *of finite type* if, for ever $x \in X$, there is a neighborhood U of x such that $\mathscr{M}|_U$ is generated locally by global sections over U, in the sense that there is an exact sequence of the following form

$$\mathscr{O}_X|_U^n \to \mathscr{M}|_U \to 0$$

for some $n \in \mathbb{N}$.

Note that n is dependent on U.

One sees with this definition that if $x \in U$, and $\{(s_i)_x : 1 \leq i \leq n\}$ are the images in $\mathscr{M}_x$ of a basis for $\mathscr{O}_X|_U^n$, and thus form a generating set for $\mathscr{M}_x$ over $\mathscr{O}_x$. This in fact characterizes modules of finite type, as one can see by applying Proposition 4.2. In this case, we say that the sections $\{s_i : 1 \leq i \leq n\}$ *generate* $\mathscr{M}|_U$.

Example 4.1. Here's Grothendieck's example of an $\mathscr{O}_X$-module that's not of finite type: Take $X = \mathbb{R}$, and $\mathscr{O}_X = \underline{\mathbb{Z}}_X$. Now, let $U = X - \{0\}$, and consider $\mathscr{M} = j_!(\mathscr{O}_X|_U)$. For every open set V containing 0, we have $\mathscr{M}(V) = 0$, but clearly $\mathscr{M}|_V \neq 0$. So $\mathscr{M}$ cannot be generated locally by global sections, and is thus not of finite type.

Finite type modules are *very* local, as the next proposition will show.

e-type-stalkgen-localgen

Proposition 4.2. *Suppose $\mathscr{M}$ is an $\mathscr{O}_X$-module of finite type, and suppose for some $x \in X$ the stalk $\mathscr{M}_x$ is generated by germs $\{s_x^{(i)} : 1 \leq i \leq n\}$, for some section of $\mathscr{M}|_U^n$ over some neighborhood U of x. Then, we can find a neighborhood V of x such that $\mathscr{M}|_V$ is generated by $\{\mathrm{res}_{U,V}(s^{(i)})\}$.*

Proof. Since $\mathscr{M}$ is of finite type, we can find a neighborhood $\mathcal{W}$ of x and a section $(t^{(j)})$ of $\mathscr{M}|_W^m$ that generates $\mathscr{M}|_W$. But then the germs $\{t_x^{(j)}\}$ will be generated by $\{s_x^{(i)}\}$. So, for every j, we can find a neighborhood V_j of x, and sections $\{a^{(i,j)} : 1 \leq i \leq n\}$ of $\mathscr{O}_X|_{V_j}$ such that $\mathrm{res}_{W,V_j}(t_j) = \sum_i a^{(i,j)} \mathrm{res}_{U,V_j}(s^{(i)})$. If we take V to be the intersection of the V_j, then on V we can express the restrictions of the $t^{(j)}$ as linear sums of restrictions of the $s^{(i)}$, which proves our statement. □

ero-fintype-locally-zero **Corollary 4.3.** *Suppose $\mathscr{M}$ is an $\mathscr{O}_X$-module of finite type, and for some $x \in X$, suppose $\mathscr{M}_x = 0$. Then we can find a neighborhood V of x such that $\mathscr{M}|_V = 0$.*

Proof. Clear. □

defn-support-module **Definition 4.4.** The *support* of an $\mathscr{O}_X$-module $\mathscr{M}$ is the set

$$\operatorname{Supp} \mathscr{M} = \{x \in X : \mathscr{M}_x \neq 0\} \subset X$$

ntype-module-supp-closed **Corollary 4.5.** *If $\mathscr{M}$ is an $\mathscr{O}_X$-module of finite type, then* $\operatorname{Supp} \mathscr{M}$ *is closed in X.*

Proof. Follows from the Corollary above. □

4.2. **Quasicoherent and Finitely Presented Modules.** A quasicoherent module is actually a very simple notion. It's just something that locally looks like an honest module. The definition will make this vague characterization clearer.

Definition 4.6. An $\mathscr{O}_X$-module $\mathscr{M}$ is *quasicoherent* if, for every $x \in X$, there is a neighborhood U of x such that $\mathscr{M}|_U$ is the cokernel of a morphism of free $\mathscr{O}_X|_U$-modules. In other words, if there exists an exact sequence of the following form

$$\mathscr{O}_X|_U^I \to \mathscr{O}_X|_U^J \to \mathscr{M}|_U \to 0.$$

We say that $\mathscr{M}$ is *finitely presented* if, for every such U, we can find *finite* indexing sets I and J for which the above sequence is exact.

It follows easily from this definition that any locally free $\mathscr{O}_X$-module is quasicoherent, and that any locally free $\mathscr{O}_X$-module of finite rank is finitely presented.

invimage-qc-finitepres **Proposition 4.7.** *If $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces, then the following statements hold:*

1. *If $\mathscr{M}$ is a quasicoherent $\mathscr{O}_Y$-module, then $f^*\mathscr{M}$ is a quasicoherent $\mathscr{O}_X$-module.*
2. *If $\mathscr{M}$ is a finitely presented $\mathscr{O}_Y$-module, then $f^*\mathscr{M}$ is a finitely presented $\mathscr{O}_X$-module.*

In particular, if $U \subset X$ is an open set, and $\mathscr{M}$ is quasicoherent (resp. finitely presented), then $\mathscr{M}|_U$ is also quasicoherent (resp. finitely presented).

Proof. (1) Suppose $x \in X$; then there is a neighborhood $U \subset Y$ of $f(x)$ and an exact sequence of the following form:

$$\mathscr{O}_Y|_U^I \to \mathscr{O}_Y|_U^J \to \mathscr{M}|_U \to 0.$$

Now, use Corollary 2.31 and the fact that f^* is a right exact functor (since it's a left adjoint) to get the following exact sequence:

$$(f|_{f^{-1}(U)})^*(\mathscr{O}_Y|_U^I) \to (f|_{f^{-1}(U)})^*(\mathscr{O}_Y|_U^J) \to (f^*\mathscr{M})|_{f^{-1}(U)} \to 0.$$

Now, apply Proposition 3.4 to see that this is indeed expressing f^*M locally as the cokernel of a morphism between free modules.

(2) Same as above, with I and J finite everywhere.

□

Quasicoherent sheaves, as a subcategory of $\mathscr{O}_X$-mod, are not necessarily closed under extensions. This will not be a problem with coherent sheaves, as we'll see below. But in the case that we'll be most concerned with-modules over the structure sheaf of a scheme-we'll see that quasicoherent modules are indeed stable under extensions. For the rest of these notes, we'll assume that we're in a situation where this is true.

quasicoherent-assumption

Note on Notation 4 (Standing Assumption). From now on, we'll assume that quasicoherent $\mathscr{O}_X$-modules are stable under extensions, and also that they form an abelian subcategory of $\mathscr{O}_X$-mod: that is, if $\phi : \mathscr{M} \to \mathscr{N}$ is a morphism of quasicoherent $\mathscr{O}_X$-modules, then $\ker \phi$, $\operatorname{coker} \phi$ and $\operatorname{im} \phi$ are all quasicoherent.

4.3. **Coherent Modules.** We finally arrive at our destination.

Definition 4.8. An $\mathscr{O}_X$-module $\mathscr{M}$ is *coherent* if the following conditions hold:

(1) $\mathscr{M}$ is of finite type.
(2) For every open set U, and every homomorphism $\phi : \mathscr{O}_X|_U^n \to \mathscr{M}$, with $n \in \mathbb{N}$, $\ker \phi$ is also of finite type.

An immediate corollary to this definition is that a coherent module $\mathscr{M}$ is quasicoherent and is in fact finitely presented.

If we consider Grothendieck's example above, then we see that $\underline{\mathbb{Z}}_X/\mathscr{M}$ is not coherent, since the kernel of the map from $\underline{\mathbb{Z}}_X$ is not of finite type.

Here's an easy characterization of coherent sheaves that will be used a lot.

rence-finite-type-kernel

Proposition 4.9. *An $\mathscr{O}_X$-module $\mathscr{M}$ is coherent iff it is of finite type and every morphism $\phi : \mathscr{N} \to \mathscr{M}|_U$, where $\mathscr{N}$ is an $\mathscr{O}_X|_U$-module of finite type, has a kernel of finite type.*

Proof. One direction is trivial. So assume $\mathscr{M}$ is coherent, and we have a morphism ϕ as in the statement. Then, since $\mathscr{N}$ is of finite type, we can find a neighborhood $V \subset U$, and a surjection $\psi : \mathscr{O}_X|_V^n \to \mathscr{N}|_V$, giving us the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(\phi \circ \psi) & \longrightarrow & \mathscr{O}_X|_V^n & \xrightarrow{\phi \circ \psi} & \mathscr{M}|_V \\
 & & \vdots & & \downarrow{\scriptstyle \psi} & & \| \\
0 & \longrightarrow & \ker \phi|_V & \longrightarrow & \mathscr{N}|_V & \xrightarrow[\phi]{} & \mathscr{M}|_V
\end{array}$$

where we get the dotted map from the universal property of $\ker \phi$ (note that restriction is an exact functor). Since the map in the middle is surjective, and the map on the right is injective, the 4-lemma tells us that the dotted map is surjective. But, since $\mathscr{M}$ is coherent, $\ker(\phi \circ \psi)$ is of finite type. So $\ker \phi|_V$ is the image of a module of finite type, and is thus itself of finite type. □

Here's the reason coherent sheaves are worth it.

coherent-closed-ext

Proposition 4.10. *Suppose we have an exact sequence*

$$0 \to \mathscr{M}' \xrightarrow{f} \mathscr{M} \xrightarrow{g} \mathscr{M}'' \to 0$$

of $\mathscr{O}_X$-modules. Then, if any two of the modules in the sequence are coherent, so is the third. In other words, the subcategory of coherent $\mathscr{O}_X$-modules is closed under extensions.

Proof of Proposition 4.10. We'll do the three cases separately

(1) Suppose first that $\mathscr{M}$ and $\mathscr{M}''$ are coherent, and suppose we have a morphism $\phi : \mathscr{O}_X|_U^n \to \mathscr{M}'|_U$. Then, since f is injective, we see that $\ker(f \circ \phi) = \ker \phi$. But $f \circ \phi$ is a morphism to $\mathscr{M}$ from a free $\mathscr{O}_X|_U$-module, and thus has kernel of finite type. This implies that $\ker \phi$ is of finite type. This shows that $\mathscr{M}'$ satisfies the second condition for coherence.

Now, we want to show that $\mathscr{M}'$ is of finite type. So suppose $x \in X$; then we want to find a neighborhood U of x and a surjection from a free $\mathscr{O}_X|_U$-module onto $\mathscr{M}'|_U$. Since, $\mathscr{M}$ is of finite type, we can always find a neighborhood U, for which we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(g \circ \psi) & \longrightarrow & \mathscr{O}_X|_U^n & \longrightarrow & \mathscr{M}''|_U & \longrightarrow & 0 \\ & & \vdots & & \downarrow \psi & & \| & & \\ 0 & \longrightarrow & \mathscr{M}'|_U & \xrightarrow[f]{} & \mathscr{M}|_U & \xrightarrow[g]{} & \mathscr{M}''|_U & \longrightarrow & 0 \end{array}$$

where $\psi : \mathscr{O}_X|_U^n \to \mathscr{M}|_U$ is an epimorphism, and the dotted map is obtained from the universal property of kernels (in this case $\mathscr{M}' = \ker g$). Since the map in the middle is a surjection, and the map on the right is an injection, we see by the 4-lemma that the dotted map is a surjection.

But then, since $\mathscr{M}''$ is coherent, $\mathscr{K} = \ker(g \circ \phi)$ is of finite type, and so we can find a smaller neighborhood V on which there is a surjection of a free $\mathscr{O}_X|_V$-module onto $\mathscr{K}|_V$, which gives us a surjection of a free module onto $\mathscr{M}'|_V$, thus showing that $\mathscr{M}'$ is of finite type and so satisfies both conditions for coherence.

(2) Suppose now that $\mathscr{M}'$ and $\mathscr{M}''$ are coherent, and suppose we have a morphism $\phi : \mathscr{O}_X|_U^n \to \mathscr{M}|_U$. Then, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(g \circ \phi) & \longrightarrow & \mathscr{O}_X|_U^n & \xrightarrow{g \circ \phi} & \mathscr{M}''|_U & & \\ & & \widetilde{\phi} \vdots & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & \mathscr{M}'|_U & \xrightarrow[f]{} & \mathscr{M}|_U & \xrightarrow[g]{} & \mathscr{M}''|_U & \longrightarrow & 0 \end{array}$$

where we get $\widetilde{\phi}$ from the universal property of kernels, just as above.

From this, we find that $\ker \widetilde{\phi} \cong \ker \phi$. So it suffices to show that $\ker \widetilde{\phi}$ is of finite type. But since $\mathscr{M}''$ is coherent, we see that $\mathscr{K} = \ker(g \circ \phi)$ is of finite type. So, applying Proposition 4.9 to the map $\widetilde{\phi} : \mathscr{K} \to \mathscr{M}'|_U$, we see that $\ker \widetilde{\phi}$ is of finite type. This finishes the first part of showing coherence of $\mathscr{M}$.

Now, to show that $\mathscr{M}$ is of finite type, given an $x \in X$, we take a neighborhood U of x such that we have a surjection $\phi : \mathscr{O}_X|_U^n \to \mathscr{M}''|_U$, and a surjection $\psi : \mathscr{O}_X|_U^m \to \mathscr{M}'|_U$. Then, by Proposition 3.3, we see that we can lift ψ to a map $\widetilde{\psi} : \mathscr{O}_X|_V^m \to \mathscr{M}|_V$, for some neighborhood $V \subset U$

of x. So we get the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_X|_V^n & \longrightarrow & \mathscr{O}_X|_V^{n+m} & \longrightarrow & \mathscr{O}_X|_V^m & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle (f\circ\phi)+\widetilde{\psi}} & & \Big\downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & \mathscr{M}'|_V & \xrightarrow{f} & \mathscr{M}|_V & \xrightarrow{g} & \mathscr{M}''|_V & \longrightarrow & 0
\end{array}$$

Since the two maps on the right and left are surjective, it follows that the map in the middle is also surjective, and so $\mathscr{M}$ is also of finite type.

(3) Now, assume $\mathscr{M}'$ and $\mathscr{M}$ are coherent. It's easy to see that any quotient of a module of finite type is also of finite type, and so $\mathscr{M}''$ is of finite type. It remains to prove that it satisfies the second condition for coherence. So suppose we have a map $\phi : \mathscr{O}_X|_U^n \to \mathscr{M}''|_U$. Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\beta) & \longrightarrow & \mathscr{O}_X|_V^m & \xrightarrow{\beta} & \mathscr{O}_X|_V^n & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \mathscr{M}'|_V & \xrightarrow{f} & \mathscr{M}|_V & \xrightarrow{g} & \mathscr{M}''|_V & \longrightarrow & 0
\end{array}$$

where $\mathscr{A}$ is the pullback of the maps g and ϕ. By an easy diagram chase, we can see that the dotted map (found as always by the universal property of kernels) is an isomorphism. So by the Snake Lemma we get $\ker\alpha \cong \ker\phi$. But $\mathscr{M}$ is coherent, and $\mathscr{A}$ is of finite type by Lemma **??**. So by Proposition 4.9, we see that $\ker\alpha$ is of finite type, which means that $\ker\phi$ is of finite type.

□

We discovered a useful fact in the course of the proof of part (1). Let's record it in the form of a corollary.

finite-type-sub-coherent

Corollary 4.11. *Any submodule of finite type of a coherent module is itself coherent.*

As one would expect, the category of coherent sheaves is an abelian category.

coherent-abelian

Corollary 4.12. *If $\phi : \mathscr{M} \to \mathscr{N}$ is a map of coherent modules, then $\ker\phi$, $\operatorname{coker}\phi$ and $\operatorname{im}\phi$ are all coherent. In particular, the category of coherent $\mathscr{O}_X$-modules is abelian.*

Proof. Observe that $\operatorname{im}\phi$ is a submodule of finite type of the coherent module $\mathscr{N}$ and hence by the last Corollary it is coherent. Now, we get the coherence of the kernel and the cokernel from the Proposition and the two short exact sequences below:

$$0 \to \ker\phi \to \mathscr{M} \to \operatorname{im}\phi \to 0$$
$$0 \to \operatorname{im}\phi \to \mathscr{N} \to \operatorname{coker}\phi \to 0$$

□

Morphisms between coherent sheaf are local in a very fundamental sense.

coherent-morphism-local

Proposition 4.13. *Let $\phi : \mathscr{M} \to \mathscr{N}$ be a morphism between coherent modules, and let $x \in X$. Then the following statements are true:*

(1) *$\phi_x = 0$ iff $\phi|_V = 0$ for some neighborhood V of x.*
(2) *ϕ_x is injective iff $\phi|_V$ is injective in some neighborhood V of x.*
(3) *ϕ_x is surjective iff $\phi|_V$ is surjective in some neighborhood V of x.*
(4) *ϕ_x is an isomorphism iff $\phi|_V$ is an isomorphism in some neighborhood V of x.*

Proof. Just use Corollary 4.12 and apply Corollary 4.3 to $\operatorname{im}\phi$, $\ker\phi$ and $\operatorname{coker}\phi$ to get (1), (2) and (3) respectively. We get (4) by putting together (2) and (3). □

4.4. **Sheaf Hom and Tensor Product of Quasicoherent Modules.** Coherent modules are rather stable under the taking of sheaf hom and tensor products, but quasicoherent modules, since they're not necessarily closed under extensions, need not be so. But recall our Standing Assumption about $\mathscr{O}_X$ (4)!

coherent-shfhom-coherent

Proposition 4.14. *Suppose $\mathscr{M}$ is a finitely presented, and $\mathscr{N}$ is a (quasi)coherent $\mathscr{O}_X$-module; then $\underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is (quasi)coherent.*

Proof. Since $\mathscr{M}$ is finitely presented, for every x, we can find a neighborhood V of x and a finite presentation

$$\mathscr{O}_X|_V^n \to \mathscr{O}_X|_V^m \to \mathscr{M}|_V \to 0$$

of $\mathscr{M}|_V$. Applying the functor $\underline{\operatorname{Hom}}_{\mathscr{O}_X|_V}(\_\_, \mathscr{N}|_V)$ to this sequence, we get an exact sequence

$$0 \to \underline{\operatorname{Hom}}_{\mathscr{O}_X|_V}(\mathscr{M}|_V, \mathscr{N}|_V) \to \underline{\operatorname{Hom}}_{\mathscr{O}_X|_V}(\mathscr{O}_X|_V^m, \mathscr{N}|_V) \to \underline{\operatorname{Hom}}_{\mathscr{O}_X|_V}(\mathscr{O}_X|_V^n, \mathscr{N}|_V).$$

Using Proposition 2.8, this becomes the exact sequence

$$0 \to \underline{\operatorname{Hom}}_{\mathscr{O}_X|_V}(\mathscr{M}|_V, \mathscr{N}|_V) \to \mathscr{N}|_V^m \to \mathscr{N}|_V^n.$$

Hence, $\underline{\operatorname{Hom}}_{\mathscr{O}_X|_V}(\mathscr{M}|_V, \mathscr{N}|_V)$ is the kernel of a map between (quasi)coherent modules, and is thus (quasi)coherent, by Corollary 4.12. (Actually, quasicoherence follows from our Standing Assumption, but we will not stress this point). □

shfhom-stalks-coherent

Proposition 4.15. *Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{O}_X$-modules, with $\mathscr{M}$ finitely presented. Then, for any $x \in X$, we have*

$$\underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})_x \cong \operatorname{Hom}_{\mathscr{O}_x}(\mathscr{M}_x, \mathscr{N}_x)$$

where $\mathscr{O}_x$ stands for the stalk of $\mathscr{O}_X$ at x.

Remark 4.16. Compare with [IM, 1.2].

Proof. Note that we have a natural map in one direction, where, if $g \in \underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})_x$ is represented by $\phi : \mathscr{M}|_U \to \mathscr{N}|_U$, then we take it to $\phi_x : \mathscr{M}_x \to \mathscr{N}_x$. If $\psi : \mathscr{M}|_V \to \mathscr{N}|_V$ is another map such that $\operatorname{res}_{V,W}(\psi) = \operatorname{res}_{U,W}(\phi)$ for some open subset $W \subset U \cap V$, then ψ and ϕ agree in neighborhood of x, and so $\psi_x = \phi_x$. This tells us that the assignment is well-defined.

Moreover, if $\mathscr{M}|_V \cong \mathscr{O}_X|_V^n$ is locally free, then this map is in fact an isomorphism, since both sides are just $\mathscr{N}_x^n$.

Now, suppose $\mathscr{M}$ is finitely presented around x, with a finite presentation

$$\mathscr{O}_X|_V^n \to \mathscr{O}_X|_V^m \to \mathscr{M}|_V \to 0,$$

then, applying the $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\_\_,\mathscr{N})$ functor, followed by the stalk functor at x, we get the following picture with exact rows.

$$\begin{array}{ccccccc} 0 \longrightarrow & \underline{\mathrm{Hom}}_{\mathscr{O}_X|_V}(\mathscr{M}|_V,\mathscr{N}|_V)_x & \longrightarrow & \underline{\mathrm{Hom}}_{\mathscr{O}_X|_V}(\mathscr{O}_X|_V^m,\mathscr{N}|_V)_x & \longrightarrow & \underline{\mathrm{Hom}}_{\mathscr{O}_X|_V}(\mathscr{O}_X|_V^n,\mathscr{N}|_V)_x \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_{\mathscr{O}_x}(\mathscr{M}_x,\mathscr{N}_x) & \longrightarrow & \mathrm{Hom}_{\mathscr{O}_x}(\mathscr{O}_x^m,\mathscr{N}_x) & \longrightarrow & \mathrm{Hom}_{\mathscr{O}_x}(\mathscr{O}_x^n,\mathscr{N}_x). \end{array}$$

As we saw earlier, the two vertical maps on the right and in the middle are isomorphisms, and, therefore, so it the map on the left. □

Now, we turn to the tensor product.

tensor-qc-qc

Proposition 4.17. *If $\mathscr{M}$ is a quasicoherent $(\mathscr{O}_X,\mathscr{O}_X)$-bimodule, and $\mathscr{N}$ is a (quasi)coherent $\mathscr{O}_X$-module, then $\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{N}$ is also (quasi)coherent.*

Proof. Since $\mathscr{M}$ is quasicoherent, for every $x\in X$, we have a neighborhood U of x and a free presentation

$$\mathscr{O}_X|_U^I \to \mathscr{O}_X|_U^J \to \mathscr{M}|_U \to 0$$

of $\mathscr{M}|_U$. Now, tensor this sequence with $\mathscr{N}|_U$, and use Propositions 2.32 and 2.16 to see that we get an exact sequence

$$\mathscr{N}|_U^I \to \mathscr{N}|_U^J \to (\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{N})|_U \to 0$$

Now we see from Corollary 4.12, that $\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{N}$ is (quasi)coherent. □

4.5. **Coherent Sheaves of Rings.**

Definition 4.18. We say that $\mathscr{O}_X$ is a *coherent ring of sheaves* or, just coherent, if it is coherent as an $\mathscr{O}_X$-module.

Proposition 4.19. *Suppose $\mathscr{O}_X$ is a coherent sheaf of rings. Then, every sheaf of ideals of $\mathscr{O}_X$ of finite type is coherent. Every locally free $\mathscr{O}_X$-module is coherent.*

Proof. Follows from Corollary 4.11 and Proposition 4.10. □

Coherent modules over coherent rings of sheaves are more simply described.

nt-ring-finpres-coherent

Proposition 4.20. *Suppose $\mathscr{O}_X$ is a coherent sheaf of rings. Then an $\mathscr{O}_X$-module $\mathscr{M}$ is coherent iff it is finitely presented.*

Proof. One direction is clear. For the other, suppose $\mathscr{M}$ is finitely presented. Then it's locally the cokernel of two coherent modules, by the last Proposition, and so is itself coherent. □

invimage-coherent

Corollary 4.21. *Suppose $(f,f^\sharp):(X,\mathscr{O}_X)\to(Y,\mathscr{O}_Y)$ is a map of ringed spaces, and suppose $\mathscr{O}_X$ is coherent. Then, for every coherent $\mathscr{O}_Y$-module $\mathscr{M}$, $f^*\mathscr{M}$ is a coherent $\mathscr{O}_X$-module.*

Proof. Follows from the last Proposition and Proposition 4.7, since any coherent module is finite presented, and so $f^*\mathscr{M}$ will be finitely presented, since f^* preserves that property. But then it will be coherent, since $\mathscr{O}_X$ is coherent. □

Example 4.2. The direct image functor is not quite so well-behaved. For example, take any topological space X and give it the ringed structure $(X, \underline{\mathbb{Z}}_X)$. Now, take any singleton set p and endow it with the ringed structure given by the constant sheaf $\underline{\mathbb{Z}}_2$. If we take the constant map f from X to p, and take $f^\sharp : \underline{\mathbb{Z}}_2 \to f_*\underline{\mathbb{Z}}_X$ to be the 0 map, then that makes $f_*\underline{\mathbb{Z}}_X$ a module over $\underline{\mathbb{Z}}_2$. We claim that $f_*\underline{\mathbb{Z}}_X$ is not coherent. Since we're over a one-point set, we can forget all about sheaves. We're just looking at $\mathbb{Z}$ as a trivial module over $\mathbb{Z}_2$, and it's clear that $\mathbb{Z}$ has no finite presentation as a $\mathbb{Z}_2$-module.

4.6. **Quotient Rings.** Suppose now that we have a sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$. Then the sheaf $\mathscr{O}_X/\mathscr{I}$ is also a sheaf of rings. Moreover, any $\mathscr{O}_X/\mathscr{I}$-module is naturally an $\mathscr{O}_X$-module. So we can wonder how the properties of $\mathscr{O}_X/I$-modules translate when they're looked upon as $\mathscr{O}_X$-modules. This is especially important in geometry, because the natural sheaf of rings on a subvariety (or subscheme, for that matter) is a quotient of the sheaf of regular functions on its ambient variety.

`quotient-relations`

Proposition 4.22. *Suppose $\mathscr{I} \subset \mathscr{O}_X$ is a quasicoherent sheaf of ideals, and let $\mathscr{S}_X = \mathscr{O}_X/\mathscr{I}$. Let $\mathscr{M}$ be an $\mathscr{S}_X$-module. Then $\mathscr{M}$ belongs to the following classes of $\mathscr{S}_X$-modules iff it belongs to the corresponding class of $\mathscr{O}_X$-modules.*

(1) *Quasicoherent modules*
(2) *Modules of finite type*

In addition, if $\mathscr{I}$ is of finite type, then this is also true for the class of coherent modules.

Proof. (1) Suppose $\mathscr{M}$ is a quasicoherent $\mathscr{S}_X$-module. Since the question is local, we can assume that we have a presentation of the form

$$\mathscr{S}_X^I \to \mathscr{S}_X^J \to \mathscr{M} \to 0$$

Observe that $\mathscr{S}_X$ is quasicoherent since it's the cokernel of the map $\mathscr{I} \to \mathscr{O}_X$. That $\mathscr{M}$ is a quasicoherent $\mathscr{O}_X$-module now follows from our Standing Assumption(4). Conversely, suppose $\mathscr{M}$ is quasicoherent as an $\mathscr{O}_X$-module; then, again, we can assume that we have a free presentation of $\mathscr{M}$ as an $\mathscr{O}_X$-module. Since every map from $\mathscr{O}_X$ to $\mathscr{M}$ factors through $\mathscr{S}_X$, we get the following picture

$$\begin{array}{ccccccc} \mathscr{O}_X^I & \longrightarrow & \mathscr{O}_X^J & \longrightarrow & \mathscr{M} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \\ \mathscr{S}_X^I & \longrightarrow & \mathscr{S}_X^J & \longrightarrow & \mathscr{M} & \longrightarrow & 0 \end{array}$$

with the top row exact. Now, it's just an easy diagram chase to see that the bottom row is exact, too.

(2) Similar argument as in (1); just note that $\mathscr{S}_X$ is of finite type as an $\mathscr{O}_X$-module.

Suppose now that $\mathscr{O}_X$ is coherent, and let $\mathscr{M}$ be an $\mathscr{S}_X$-module. We know that $\mathscr{M}$ is of finite type as an $\mathscr{O}_X$-module iff it's of finite type as an $tRegX$-module, by part (2) above. Let $\phi : \mathscr{O}_X^n \to \mathscr{M}$ be a morphism. As always, it has to factor through $\mathscr{S}_X$, giving us the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker\phi & \longrightarrow & \mathscr{O}_X^n & \xrightarrow{\phi} & \mathscr{M} \\ & & \vdots & & \downarrow & & \| \\ 0 & \longrightarrow & \ker\widetilde{\phi} & \longrightarrow & \mathscr{S}_X^n & \xrightarrow{\widetilde{\phi}} & \mathscr{M} \end{array}$$

Using the Snake Lemma, we get a short exact sequence

$$0 \to \mathscr{I}^n \to \ker\phi \to \ker\widetilde{\phi} \to 0.$$

Hence, $\ker\phi$ is of finite type iff $\ker\widetilde{\phi}$ is of finite type, which means that $\mathscr{M}$ is coherent as an $\mathscr{O}_X$-module iff it's coherent as an $\mathscr{S}_X$-module □

quotient-coherent

Corollary 4.23. *If $\mathscr{O}_X$ is coherent and $\mathscr{I} \subset \mathscr{O}_X$ is a sheaf of ideals of finite type, then $\mathscr{O}_X/\mathscr{I}$ is also coherent.*

Proof. $\mathscr{O}_X/I$ is coherent as an $\mathscr{O}_X$-module since it's the cokernel of a map $\mathscr{I} \hookrightarrow \mathscr{O}_X$ between coherent $\mathscr{O}_X$-modules. That $\mathscr{I}$ is coherent follows from Corollary 4.11. Now, apply the Proposition to see that $\mathscr{O}_X/\mathscr{I}$ is coherent over itself. □

5. Locally Ringed Spaces

Till now, we've been working in such generality that we haven't needed any real tools for our results. The time has now come to impose some stricter conditions on our ringed spaces, so that we can prove some deeper statements. In this section, we'll assume that all our rings are commutative.

Definition 5.1. A *locally ringed space* is a ringed space $(X, \mathscr{O}_X$ where, for every $x \in X$, the stalk $\mathscr{O}_x$ is a local ring.

Remark 5.2. Note that this is *not* the same as a sheaf of local rings, but we will indiscriminately abuse notation and sometimes call it just that.

Most of the ringed spaces we meet in real life: a manifold with its sheaf of smooth functions, a variety with its sheaf of regular functions, or a Riemann surface with its sheaf of holomorphic functions, are all locally ringed spaces. So this is a very important sub-class of the class of ringed spaces.

Note on Notation 5. If we have a locally ringed space $(X, \mathscr{O}_X)$, then we'll denote the maximal ideal of $\mathscr{O}_x$ by $\mathfrak{m}_x$, and the residue field $\mathscr{O}_x/\mathfrak{m}_x$ by $k(x)$, for every $x \in X$.

Definition 5.3. The *cotangent space* at a point $x \in X$ of a locally ringed space is the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$. The *tangent space* is the dual $\mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$.

Definition 5.4. Suppose $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are two locally ringed spaces; then a *morphism* of locally ringed spaces is a morphism of ringed spaces $(f, f^\flat) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, such that the map $f^\flat_{f(x)} : \mathscr{O}_{f(x)} \to \mathscr{O}_x$ maps $\mathfrak{m}_{f(x)}$ into $\mathfrak{m}_x$. In other words, the map induced on stalks by $f^\flat$ is a local homomorphism.

Remark 5.5. It is clear that the composition of two morphisms of locally ringed spaces is again a morphism of locally ringed spaces. So we indeed have a *category* of locally ringed spaces.

Also observe that every map of locally ringed spaces induces maps on the residue fields, the cotangent spaces and the tangent spaces at every point (in the other direction, of course).

For the rest of this section, we'll assume that $(X, \mathscr{O}_X)$ is a locally ringed space.

One main advantage of locally ringed spaces, is that, if we're given a coherent module over the space, then we can use Proposition 4.13 in conjunction with Nakayama's Lemma to transform statements on the stalk level to statements on a more global level.

Here's an example:

nak-surj-implies-iso

Proposition 5.6. *Suppose $\mathscr{M}$ and $\mathscr{N}$ are locally free $\mathscr{O}_X$-modules of rank n, and let $\phi : \mathscr{M} \to \mathscr{N}$ be a map of $\mathscr{O}_X$-modules. Let $x \in X$; if $\phi_x : \mathscr{M}_x \to \mathscr{N}_x$ is surjective, then there is a neighborhood around x on which ϕ is an isomorphism.*

Proof. Observe that, by Proposition 4.13, it suffices to show that ϕ_x is an isomorphism; or, equivalently, that $\ker \phi_x = 0$. Now, ϕ_x induces an isomorphism of vector spaces $\mathscr{M}_x \otimes k(x) \to \mathscr{N}_x \otimes k(x)$. This implies that $\ker \phi_x \subset \mathfrak{m}_x \mathscr{M}_x$. But since $\mathscr{N}_x$ is free, we have a splitting map $\psi : \mathscr{N}_x \to \mathscr{M}_x$ such that

$$\mathscr{M}_x = \operatorname{im} \psi \oplus \ker \phi_x = \operatorname{im} \psi + \mathfrak{m}_x \mathscr{M}_x,$$

which implies, by Nakayama, that $\operatorname{im} \psi = \mathscr{M}_x$, and so $\ker \phi_x = 0$. □

sec-loc-free-coherent

5.1. **Locally Free Modules over a Coherent Sheaf of Rings.** In this section, we'll characterize locally free modules over coherent sheaves of local rings. So we'll be assuming throughout that $\mathscr{O}_X$ is local and coherent.

loc-free-nbd-free

Proposition 5.7. *Suppose $\mathscr{M}$ is a coherent $\mathscr{O}_X$-module, and suppose $\mathscr{M}_x$ is a free $\mathscr{O}_x$-module of rank n, for some $x \in X$. Then, there is a neighborhood V of x such that $\mathscr{M}|_V$ is a free $\mathscr{O}_X|_V$-module of rank n.*

Proof. Suppose M_x is a free module over the basis $\{m_i : 1 \leq i \leq n\}$, and suppose W is a neighborhood of x such that each m_i is represented by some $s_i \in \Gamma(W, \mathscr{M})$. Define a map $\phi : \mathscr{O}_X|_W^n \to \mathscr{M}|_W$ that, for every open $U \subset W$, sends the generators of $\Gamma(U, \mathscr{O}_X^n)$ to the restrictions of the s_i to U. Then, $\phi_x : \mathscr{O}_x^n \to \mathscr{M}_x$ is a surjection. By 5.6, ϕ is an isomorphism on a neighborhood of x. This implies that $\mathscr{M}$ is free on that neighborhood, and has rank n. □

stalkfree-locallyfree

Corollary 5.8. *A coherent $\mathscr{O}_X$-module $\mathscr{M}$ is locally free iff $\mathscr{M}_x$ is a free $\mathscr{O}_x$-module, for every $x \in X$.*

Proof. Immediate from the Proposition. □

See [IM, 1.7] for an analogous statement for finitely presented modules over a ring.

sec-inv-modules

5.2. **Invertible Modules and the Picard Group.** The following proposition explains why locally free modules of rank 1 are usually called invertible modules right off the bat.

vertible-iff-locallyfree

Proposition 5.9. *For a coherent $\mathscr{O}_X$-module $\mathscr{M}$, the following are equivalent:*

(1) *$\mathscr{M}$ is invertible.*
(2) *$\mathscr{M}$ is locally invertible.*
(3) *$\mathscr{M}$ is locally free of rank* 1*.*

Proof. (1) $\Rightarrow$ (2) is trivial. (3) $\Rightarrow$ (1) is Corollary 3.9. So we only have to show (2) $\Rightarrow$ (3). But if $\mathscr{M}$ is locally invertible, then for every x, we can find a neighborhood U and an $\mathscr{O}_X|_U$-module $\mathscr{N}$ such that $\mathscr{M}|_U \otimes_{\mathscr{O}_X|_U} \mathscr{N} \cong \mathscr{O}_X|_U$. This implies that for every $x \in X$, we can find a $\mathscr{O}_x$-module N_x such that $\mathscr{M}_x \otimes_{\mathscr{O}_x} N_x \cong \mathscr{O}_x$. So to finish our proof, it suffices to show, by Corollary 5.8, that $\mathscr{M}_x$ is free of rank 1 for every $x \in X$.

In sum, we only have to prove the following statement from commutative algebra: If R is a local ring, and M, N are R-modules with an isomorphism $\phi : M \otimes N \to R$, then $M \cong R$. But this is easy: since R is local, there is an $m \in M$ and $n \in N$ such that $\phi(m \otimes n) = 1$. (See also [SCN, 2.3]).

This means that $M \otimes Rn \cong R$. If we now show that $\operatorname{ann}(n) = 0$, then we'll see that $Rn \cong R$ and so $M \cong M \otimes R \cong R$. But if $a \in \operatorname{ann}(n)$, then

$$0 = \phi(am \otimes n) = a\phi(m \otimes n) = a.$$

□

Compare this with the definition of an invertible module over a Noetherian ring in [IM, 2], and the subsequent discussion of the Picard group in that context.

Observe that if $\mathscr{M}$ is coherent then so is $\check{\mathscr{M}} = \underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$, by Proposition 4.14. So over a coherent ring of sheaves, every invertible coherent module has a coherent inverse, since $\mathscr{M} \otimes_{\mathscr{O}_X} \check{\mathscr{M}} \cong \mathscr{O}_X$, by Proposition 3.9. This leads to a definition.

defn-picard-grp

Definition 5.10. The *Picard group* $\operatorname{Pic}(X)$ of a coherent, locally ringed space $(X, \mathscr{O}_X)$ is the group of isomorphism classes of coherent, invertible sheaves (that is, the group of isomorphism classes of locally free $\mathscr{O}_X$-modules of rank 1), with the group operation given by tensor product.

The following Proposition gives a criterion for knowing when an invertible sheaf is in fact isomorphic to $\mathscr{O}_X$, and is thus represented by the identity in $\operatorname{Pic}(X)$.

icgrp-identity-criterion

Proposition 5.11. *An invertible $\mathscr{O}_X$-module $\mathscr{M}$ is isomorphic to $\mathscr{O}_X$ iff it has a non-vanishing global section.*

Proof. Clearly, $\mathscr{O}_X$ has a non-vanishing global section: just take the identity over each open set. Conversely, if $\mathscr{M}$ has a non-vanishing global section $s \in \Gamma(X, \mathscr{M})$, then we have a map $\phi : \mathscr{O}_X \to \mathscr{M}$ that just takes the identity to s. On stalks, this is a surjection $\phi_x : \mathscr{O}_x \to \mathscr{M}_x$. Now, 5.6, now implies that ϕ_x is an isomorphism. Since this is true for all $x \in X$, we see, by [NOS, 4.5], that ϕ is an isomorphism. □

Since the inverse image functor commutes with tensor products (Proposition 2.32), the next proposition is immediate.

picgrp-invimg-hom

Proposition 5.12. *If $(f, f^\flat) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of coherent, locally ringed spaces, then f^* induces a group homomorphism $f^* : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$.*

Proof. If $\mathscr{M}$ is an invertible $\mathscr{O}_Y$-module with inverse $\mathscr{N}$, then we see that

$$\mathscr{O}_X = f^*\mathscr{O}_Y = f^*(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}) = f^*\mathscr{M} \otimes_{\mathscr{O}_X} f^*\mathscr{N}.$$

Hence $f^*\mathscr{M}$ is also invertible, with inverse $f^*\mathscr{N}$. Since f^* commutes with tensor products, it's clear that it commutes with the group operation. □

6. Graded Modules and Multilinear Algebra on Sheaves

This will mainly be a section of definitions.

6.1. **Algebras.**

Note on Notation 6. We'll use $\underline{\mathbb{Z}}_X$ to refer to the locally constant sheaf $\mathrm{LCShf}_{\mathbb{Z}} \in \mathrm{Shf}(X, \mathrm{Ring})$.

Recall that an $\mathscr{O}_X$-algebra is a sheaf of rings $\mathscr{A}$ equipped with a morphism of rings of sheaves $\phi : \mathscr{O}_X \to \mathscr{A}$. Observe that with this definition every sheaf of rings is a $\underline{\mathbb{Z}}_X$-algebra. To see this, just take the natural map $\mathrm{CShf}_Z \to \mathscr{A}$, given by the maps $\mathbb{Z} \to \mathscr{A}(U)$, and sheafify.

Definition 6.1. A *morphism* of $\mathscr{O}_X$-algebras $\psi : \mathscr{A} \to \mathscr{A}'$ is a morphism of $\mathscr{O}_X$-modules such that $\psi \circ \phi = \phi'$.

This gives us a *category* of $\mathscr{O}_X$-algebras. We'll call this $\mathscr{O}_X$-alg. Note that $\underline{\mathbb{Z}}_X\text{-alg} = \mathrm{Shf}(X, \mathrm{Ring})$.

6.2. **Graded Algebras and Graded Modules.**

As in usual module theory, we have a notion of grading. In this section, we will assume that $\mathscr{O}_X$ is a sheaf of *commutative* rings.

Definition 6.2. A *$\mathbb{Z}$-graded $\mathscr{O}_X$-algebra* $\mathscr{A}$ is a collection $\{\mathscr{A}_n : n \in \mathbb{Z}\}$ of $\mathscr{O}_X$-modules, and a collection $\{\phi_{m,n} : (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$ of morphisms of $\mathscr{O}_X$-modules satisfying the following conditions:

(1) $\mathscr{A}_0$ is an $\mathscr{O}_X$-algebra
(2) $\phi_{m,n} : \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{A}_n \to \mathscr{A}_{m+n}$
(3) The compositions

$$\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{A}_n \to \mathscr{A}_0 \otimes_{\mathscr{O}_X} \mathscr{A}_n \xrightarrow{\phi_{0,n}} \mathscr{A}_n$$
$$\mathscr{A}_n \otimes_{\mathscr{O}_X} \mathscr{O}_X \to \mathscr{A}_n \otimes_{\mathscr{O}_X} \mathscr{A}_0 \xrightarrow{\phi_{n,0}} \mathscr{A}_n$$

are both the canonical isomorphism.

(4) For $(m,n,p) \in \mathbb{Z}^3$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{A}_n \otimes_{\mathscr{O}_X} \mathscr{A}_p & \xrightarrow{\phi_{m,n} \otimes 1_{\mathscr{A}_p}} & \mathscr{A}_{m+n} \otimes \mathscr{A}_p \\ {\scriptstyle 1_{\mathscr{A}_m} \otimes \phi_{n,p}} \downarrow & & \downarrow {\scriptstyle \phi_{m+n,p}} \\ \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{A}_{n+p} & \xrightarrow[\phi_{m,n+p}]{} & \mathscr{A}_{m+n+p} \end{array}$$

We'll usually just say that $\mathscr{A} = \oplus_{n \in \mathbb{Z}} \mathscr{A}_n$ is a graded $\mathscr{O}_X$-algebra.

Note that we used the canonical isomorphism of Proposition 2.20 in writing the triple tensor product. We will do this without comment from now on.

Definition 6.3. A *morphism* of graded $\mathscr{O}_X$-algebras $\psi : \mathscr{A} \to \mathscr{A}'$ is a collection of morphisms of $\mathscr{O}_X$-modules $\{\psi_n : n \in \mathbb{Z}\}$ satisfying the following conditions:

(1) $\psi_n : \mathscr{A}_n \to \mathscr{A}'_n$
(2) $\psi_0 : \mathscr{A}_0 \to \mathscr{A}'_0$ is a morphism of $\mathscr{O}_X$-algebras.

(3) For $(m,n) \in \mathbb{Z}^2$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{A}_n & \xrightarrow{\psi_m \otimes \psi_n} & \mathscr{A}'_m \otimes_{\mathscr{O}_X} \mathscr{A}'_n \\ \downarrow{\scriptstyle \phi_{m,n}} & & \downarrow{\scriptstyle \phi'_{m,n}} \\ \mathscr{A}_{m+n} & \xrightarrow[\psi_{m+n}]{} & \mathscr{A}'_{m+n} \end{array}$$

Remark 6.4. Note that the set of morphisms between two graded $\mathscr{O}_X$-algebras does *not* form a group. Condition (2) will not hold for the sum of two morphisms.

So now we have a *category* of graded $\mathscr{O}_X$-algebras with the morphisms defined as above. We'll call this $\mathscr{O}_X^{\mathbb{Z}}$-alg. We'll call $\underline{\mathbb{Z}}_X^{\mathbb{Z}}$-alg the category of *graded rings*.

There's also a notion of commutative graded algebra.

Definition 6.5. A *commutative* graded $\mathscr{O}_X$-algebra $\mathscr{A}$ is a graded $\mathscr{O}_X$-algebra with the following additional constraint:

For every pair $(m,n) \in \mathbb{Z}^2$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{A}_n & \xrightarrow{\tau} & \mathscr{A}_n \otimes_{\mathscr{O}_X} \mathscr{A}_m \\ & {\scriptstyle \phi_{m,n}} \searrow \quad \swarrow {\scriptstyle \phi_{n,m}} & \\ & \mathscr{A}_{m+n} & \end{array}$$

where τ is the canonical isomorphism from Proposition 2.21.

We also have graded modules over graded algebras.

Definition 6.6. A graded module $\mathscr{M}$ over a graded $\mathscr{O}_X$-algebra $\mathscr{A}$ is a collection $\{\mathscr{A}_n : n \in \mathbb{Z}\}$ of $\mathscr{O}_X$-modules, and a collection $\{\pi_{m,n} : (m,n) \in \mathbb{Z}\times\mathbb{Z}\}$ of morphisms of $\mathscr{O}_X$-modules satisfying the following conditions:

(1) $\pi_{m,n} : \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{M}_n \to \mathscr{M}_{m+n}$
(2) The composition

$$\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}_n \to \mathscr{A}_0 \otimes_{\mathscr{O}_X} \mathscr{M}_n \xrightarrow{\pi_{0,n}} \mathscr{M}_n$$

is the canonical isomorphism.
(3) For $(m,n,p) \in \mathbb{Z}^3$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{A}_n \otimes_{\mathscr{O}_X} \mathscr{M}_p & \xrightarrow{\phi_{m,n} \otimes 1_{\mathscr{M}_p}} & \mathscr{A}_{m+n} \otimes \mathscr{M}_p \\ \downarrow{\scriptstyle 1_{\mathscr{A}_m} \otimes \pi_{n,p}} & & \downarrow{\scriptstyle \pi_{m+n,p}} \\ \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{M}_{n+p} & \xrightarrow[\pi_{m,n+p}]{} & \mathscr{M}_{m+n+p} \end{array}$$

We'll simply say that $\mathscr{M} = \oplus_{n\in\mathbb{Z}} \mathscr{M}_n$ is a graded $\mathscr{A}$-module.

Definition 6.7. A *morphism* of graded $\mathscr{A}$-modules $\psi : \mathscr{M} \to \mathscr{M}'$ is a collection of morphisms of $\mathscr{O}_X$-modules $\{\psi_n : n \in \mathbb{Z}\}$ satisfying the following conditions:

(1) $\psi_n : \mathscr{M}_n \to \mathscr{M}'_n$
(2) For $(m,n) \in \mathbb{Z}^2$, the following diagram commutes:

$$\begin{CD}
\mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{M}_n @>{1_{\mathscr{A}_m} \otimes \psi_n}>> \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{M}'_n \\
@V{\pi_{m,n}}VV @VV{\pi'_{m,n}}V \\
\mathscr{M}_{m+n} @>>{\psi_{m+n}}> \mathscr{M}'_{m+n}
\end{CD}$$

So with these morphisms, we have a *category* of $\mathscr{A}$-modules. We'll call this category $\mathscr{A}$-mod.

Proposition 6.8. *The category $\mathscr{A}$-mod is abelian.*

Proof. Suppose we have a morphism $\psi : \mathscr{M} \to \mathscr{M}'$ of $\mathscr{A}$-modules. Then, we have the obvious candidates for the kernel and cokernel in $\oplus_n \ker \psi_n$ and $\oplus_n \operatorname{coker} \psi_n$. That these are indeed $\mathscr{A}$-modules follows from the following commutative diagram:

$$\begin{CD}
\mathscr{A}_m \otimes_{\mathscr{O}_X} \ker \psi_n @>>> \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{M}_n @>{1_{\mathscr{A}_m} \otimes \psi_n}>> \mathscr{A}_m \otimes_{\mathscr{O}_X} \mathscr{M}'_n @>>> \mathscr{A}_m \otimes_{\mathscr{O}_X} \operatorname{coker} \psi_n \\
@VVV @V{\pi_{m,n}}VV @VV{\pi'_{m,n}}V @VVV \\
\ker \psi_n @>>> \mathscr{M}_{m+n} @>>{\psi_{m+n}}> \mathscr{M}'_{m+n} @>>> \operatorname{coker} \psi_n
\end{CD}$$

where the dotted maps arise from the universal properties of the kernel and cokernel. □

6.3. **Multilinear Algebra.**