TOPOLOGY FOR SCHEME THEORY

KEERTHI MADAPUSI

Contents

1. Irreducibility

secn:irreducible

Definition 1.1. A topological space X is *reducible* if there exist proper closed subsets $Y_1, Y_2 \subsetneq X$ such that $X = Y_1 \cup Y_2$. By convention, the empty set is reducible.

A topological space X is *irreducible* if it is not reducible.

equiv-prps-irred

Proposition 1.2. *The following statements are equivalent:*

(1) *X is irreducible.*
(2) *Every open set $U \subset X$ is dense in X.*

Moreover, in this situation, every open subset U of X is also irreducible.

Proof. An open set $\emptyset \neq U \subset X$ is not dense in X iff $\overline{U} \neq X$ iff $X = \overline{U} \cup Z$, where $Z \not\subseteq \overline{U}$ is a closed proper subset of X. The last implication needs some checking in one direction. For this we just take $Z = \overline{X - \overline{U}}$; this can't be the whole of X, since it doesn't contain U.

For the second statement, suppose $U = (U \cap Z_1) \cup (U \cap Z_2)$, where the Z_i are closed proper subsets of X. Then, we see that

$$Z_1 \cup Z_2 \supset \overline{U \cap Z_1} \cup \overline{U \cap Z_2} = \overline{U} = X,$$

which is a contradiction. □

irred-closure-img-irred

Proposition 1.3. *Let X be a topological space.*

(1) *A subspace $Z \subset X$ is irreducible if and only if the closure $\overline{Z}$ is irreducible.*
(2) *The irreducible closed subsets of an open subset $U \subset X$ are in bijective correspondence with the irreducible closed subsets of X intersecting non-trivially with U.*

(3) *If $f : X \to Y$ is a continuous map of topological spaces, then $f(Z)$ is irreducible for every irreducible subspace $Z \subset X$.*

Proof. (1) Suppose $\overline{Z} = Z_1 \cup Z_2$, where $Z_1, Z_2 \subset \overline{Z}$ are closed proper subsets. Then, since Z is irreducible, we must have, without loss of generality, $Z_1 \cap Z = Z$. But in that case, $\overline{Z} \subset Z_1$, which is a contradiction. Now, suppose $\overline{Z}$ is irreducible, but Z is not. Then $Z = Z_1 \cup Z_2$, with, say, $\overline{Z_1} = \overline{Z}$, but $Z_i \neq Z$, for $i = 1, 2$. But observe that $\overline{Z_i} \cap Z$ is the closure of Z_i in Z. Since Z_1 is closed, it follows that $Z_1 = Z$.

(2) For any irreducible closed subset $Z \subset X$ such that $Z \cap U \neq \emptyset$, we see that $Z \cap U$ is dense in Z, and so it's clear that if $Z \subsetneq W$, then $Z \cap U \subsetneq W \cap U$.

(3) Now, suppose Z is irreducible, and $f(Z) = (W_1 \cap f(Z)) \cup (W_2 \cap f(Z))$, where $W_i \subset Y$ are closed. Then, we have

$$Z = f^{-1}(f(Z)) \cap Z = (f^{-1}(W_1) \cap Z) \cup (f^{-1}(W_2) \cap Z).$$

So without loss of generality, we can assume that $Z \subset f^{-1}(W_1)$, and so $f(Z) \subset W_1$, showing that $f(Z)$ is also irreducible.

□

maximal-irreducible

Corollary 1.4. *Every irreducible subset of a space X is contained in a maximal irreducible subset. Moreover, every such maximal irreducible subset is closed.*

Proof. The first part is an application of Zorn's lemma, observing that the union of an increasing chain of irreducible subsets is still irreducible. For the second, if $Z \subset X$ is an irreducible subset, then so is $\overline{Z}$. Therefore, if Z is maximal irreducible then $Z = \overline{Z}$. □

Definition 1.5. A maximal irreducible subset of X is called an *irreducible component* of X.

secn:generic-points

2. Generic Points and Quasi-Zariski spaces

Definition 2.1. Let X be a topological space. For every element $x \in X$, we set $V(x) = \overline{\{x\}}$.

A *generic point* of a topological space X is a point $x \in X$ such that $V(x)$ is an irreducible component of X.

specialization

Definition 2.2. Let X be a topological space. Then, given points $x, y \in X$, we say that x *specializes* to y, or, equivalently, that y *generizes* to x, if $y \in V(x)$. We denote this by $x \rightsquigarrow y$.

open-closed-generization

Lemma 2.3. *Let X be a topological space. Then an open subset $U \subset X$ contains all generizations of its elements, and a closed subset $Z \subset X$ contains all specializations of its elements.*

Proof. Pick an element $x \in U$, and suppose y is a generization of x. Then $x \in V(y)$, and hence every open neighborhood of x contains y; in particular, U contains y. The case for the closed subset Z is similar. □

Definition 2.4. A topological space X is *quasi-Zariski* if every non-empty closed, irreducible subspace has a unique generic point. In particular, for any quasi-Zariski space, there is a bijection between its set of generic points and the collection of its irreducible components.

Remark 2.5. We will see in [AG, **??**] that every scheme is quasi-Zariski.

quasi-zariski-prps

Proposition 2.6. *Let X be a quasi-Zariski space.*

(1) *X satisfies the T_0 axiom.*
(2) *The relation $\leq$, defined on X by $x \leq y$ if and only if $x \in V(y)$, defines a partial ordering on X.*
(3) *If X is irreducible, then its generic point is contained in every non-empty open subset of X.*

Proof. (1) Let $x, y \in X$ be two distinct elements. Suppose $x \in V(y)$; we must show that $y \notin V(x)$. But if this were true, then $V(y) = V(x)$ would have two distinct generic points, which is impossible.
(2) By the previous part, the relation is reflexive. We must show that it is transitive. But if $x \in V(y)$ and $y \in V(z)$, then it follows that $x \in V(z)$.
(3) This follows from (2.3).

□

Now, we'll investigate a little more the relation that we defined in the Proposition above.

specialization-ordering

Proposition 2.7. *Let X be a quasi-Zariski space given a partial ordering as in the Proposition above.*

(1) *Any closed point of X is minimal in this ordering. Conversely, any minimal element is a closed point.*
(2) *Any generic point of X is maximal in this ordering. Conversely, any maximal element is a generic point.*
(3) *If $f : X \to Y$ is a continuous map of quasi-Zariski spaces, then it is order preserving.*

Proof. (1) One implication is clear. Suppose y is a minimal element, and suppose $x \in V(y)$. Then, by minimality of y, $x = y$, which shows that $V(y) = \{y\}$, and that y is therefore a closed point. Note, however, that minimal elements need not exist.
(2) Again, one implication is clear. Now, let y be a maximal element; then it follows that $V(y)$ is a maximal irreducible subset of X and is thus an irreducible component.
(3) Indeed if $x \in V(y)$, then $f(x) \in V(f(y))$.

□

secn:noetherian

3. Noetherian Spaces

Definition 3.1. A topological space X is *Noetherian* if every descending chain of closed subspaces stabilizes.

A topological space X is *locally Noetherian* if every point $x \in X$ has a Noetherian neighborhood U.

A topological space X is *quasicompact* if every open cover of X has a finite subcover.

Remark 3.2. With this definition, it's clear that every subspace of a Noetherian space is also Noetherian.

equiv-noetherian

Proposition 3.3. *The following statements are equivalent for a topological space X:*

(1) *X is Noetherian.*
(2) *Every non-empty collection of closed subsets of X contains a minimal element.*
(3) *Every open subset of X is quasicompact.*

Proof. (1) $\Rightarrow$ (2): Standard application of Zorn's Lemma.

(2) $\Rightarrow$ (3): Let U be an open subset of X, and let $\mathcal{V} = \{V_i : i \in I\}$ be an open cover of U. Define

$$\mathscr{M} = \{U - \bigcup_{i \in F} V_i : F \subset I \text{ finite}\}.$$

Then, $\mathscr{M}$ has a minimal element Z. If $Z \neq \emptyset$, then there is some open set V_j that it intersects non-trivially, in which case $Z \setminus V_j$ would be a smaller element of $\mathscr{M}$. This contradicts the minimality of Z. So $Z = \emptyset$, and U can be covered by finitely many of the V_i.

(3) $\Rightarrow$ (1): Let $Z_1 \supset Z_2 \supset \dots$ be a descending chain of closed sets in X. Let $U_i = X \setminus Z_i$; then we see that $U = X \setminus (\bigcap_i Z_i) = \bigcup_i U_i$. Let $\{U_{i_1}, \dots, U_{i_n}\}$ be a finite subcover of this open cover with $U_{i_k} \subset U_{i_{k+1}}$; then we see that

$$\bigcap_i Z_i = X \setminus (\bigcup_{k=1}^{n} U_{i_k}) = Z_{i_n},$$

showing that X is Noetherian. ☐

irred-decomp-noetherian

Proposition 3.4. *Every non-empty closed subset Y of a Noetherian space X can be expressed as a finite union $Y = Y_1 \cup \ldots \cup Y_n$, with Y_i closed and irreducible. If we require that $Y_i \not\supseteq Y_j$, for $i \neq j$, then the Y_i are uniquely determined: they will be the irreducible components of Y. In particular, any Noetherian space has only finitely many irreducible components.*

Proof. Standard descent argument. ☐

noeth-iff-fin-irred-comp

Corollary 3.5. *A space X is Noetherian iff it has finitely many irreducible components, each of which is Noetherian.*

Proof. One direction follows from the Proposition; for the other, just observe that a finite union of Noetherian spaces is again Noetherian (using characterization (3) from (3.3) would seem to be the quickest way of doing it). ☐

noetherian-induction

Proposition 3.6 (Noetherian Induction)**.** *Let P be a property of closed subsets of a Noetherian space X. Suppose P satisfies the condition that whenever P is true for all proper closed subsets of a closed subset $Y \subset X$, it's true for Y. Then P is true for every closed subset of X.*

Proof. Let $\mathcal{W}$ be the collection of closed subsets of X for which P is not true, and suppose this is non-empty. Let Z be the minimal element of $\mathcal{W}$. Then P is true for all proper closed subsets of Z, and hence it's true for Z. Contradiction! Hence $\mathcal{W}$ is empty, and the statement is proved. ☐

Finally, here's a useful fact we'll need later.

losed-points-qc-locnoeth

Proposition 3.7. *Let X be a quasi-Zariski space, and suppose X is either quasicompact or locally Noetherian. Then every closed subset of X contains a closed point.*

Proof. Since either of these conditions descends to all non-empty closed subsets of X, it will suffice to show that any quasicompact (or locally Noetherian) quasi-Zariski space contains a closed point. □

4. Zariski Spaces

Definition 4.1. A topological space X is *Zariski* if it's Noetherian and if every non-empty closed irreducible subset of X has a unique generic point.

prps-zariski-space

Proposition 4.2. *Let X be a Zariski space. Then, the following statements are true:*

(1) *Any minimal nonempty closed subset of X is a singleton.*
(2) *X satisfies the T_0 axiom.*
(3) *If X is irreducible, then its generic point is contained in every non-empty open subset of X.*

Proof.
(1) It suffices to show that any non-singleton irreducible closed subset has a proper nonempty closed subset. So let Z be an irreducible closed set with generic point ξ such that $Z \neq \{\xi\}$. Let z be any other point in Z. Then, since the generic point is unique, we see that $V(z) \subsetneq Z$ is a proper closed subset, and we're done.
(2) Let $x, y \in X$ be two distinct points. Now, suppose $y \in V(x)$; then we claim that $x \notin V(y)$. Indeed, if this were the case, then we'd have
$$V(x) = V(y).$$
But, observe that y is a generic point of $\{y\}$, which is irreducible, by 1.3. Hence, by uniqueness of generic points, we see that $x = y$, which is contrary to our assumption. Hence $x \notin V(y)$, and so we can find an open set around x that doesn't contain y.
(3) Follows from the fact that the generic point is dense in X.

□

prps-specialization

Proposition 4.3. *The relation generated on the points of a Zariski space X by $x \leq y$ if and only if $x \rightsquigarrow y$ gives rise to a partial ordering. In this ordering, the minimal elements are the closed points, and the maximal elements are the generic points of X.*

Proof. We showed in the proof of (2) of the proposition above that if $x \leq y$ and $y \leq x$, then $y = x$. So this is indeed is a partial ordering. An element x is minimal in this ordering iff it is the only element satisfying $x \in V(x)$. This happens iff x is itself closed. An element y is maximal in this ordering iff there are no elements x such that $y \rightsquigarrow x$. This can only happen iff $V(y)$ is an irreducible component of X iff y is a generic point of X. □

There's a natural functor from the category of Noetherian spaces to the category of Zariski spaces that is a generalization of the process of going from an affine variety to the Spec of its co-ordinate ring. We'll describe it now.

In fact, given any topological space X, let $t(X)$ be the set of nonempty irreducible closed subsets of X. For example, if X is an affine variety, then this set is in natural bijection with the primes in $A(X)$. We see easily, that if $Y \subset X$ is a closed subspace, then $t(Y) \subset t(X)$ (this corresponds to the fact that $\operatorname{Spec} R/I$ embeds into $\operatorname{Spec} R$), and that if Y_1 and Y_2 are two different closed subsets, then $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$

$(V(I_1 \cap I_2) = V(I_1) \cup V(I_2))$. Moreover, in the same vein $t(\cap_i Y_i) = \cap_i t(Y_i)$. So we can define a topology on $t(X)$ by setting the closed subsets to be of the form $t(Y)$, for $Y \subset X$ closed. Now, suppose we have a continuous map $f : X \to Y$. Then, we can define a map $t(f) : t(X) \to t(Y)$ that sends an irreducible subset to the closure of its image in Y. Note that this is still an irreducible subset of Y, by 1.3. So t is indeed a functor.

Moreover, observe that we have a natural continuous map $\alpha_X : X \to t(X)$ that takes a point x to its closure $V(x)$. The preimage of a closed set $t(Y)$ is exactly Y. So the map is indeed continuous. In fact, α gives us a natural transformation from the identity functor to the t-functor. Let X and Y be two Noetherian spaces, and consider the following commutative diagram for a map $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "f"'] & t(X) \arrow[d, "t(f)"] \\ Y \arrow[r, "\alpha_Y"] & t(Y). \end{tikzcd}$$

We see that $x \in X$ maps to $V(f(x))$ no matter what route we take.

t-functor

Proposition 4.4. *The t-functor gives a right adjoint to the forgetful functor from the category* Noeth *of Noetherian spaces to the full subcategory* Zar *of Zariski spaces. In particular, every continuous map $\varphi : X \to Y$ from a Noetherian space to a Zariski space factors uniquely through the natural map $\alpha : X \to t(X)$.*

Proof. We will first show that, whenever X is a Noetherian space, $t(X)$ is Zariski. The descending chains of closed sets in $t(X)$ are in one-to-one correspondence with the descending chains of closed sets of X. So we see that $t(X)$ must be Noetherian. So now we must show that every irreducible closed subset of $t(X)$ has a generic point. But this is easy: $t(Y) \subset t(X)$ is irreducible iff $Y \subset X$ is irreducible. Consider the point $Y \in t(Y)$; let $V = t(X) \setminus t(Z)$ be an open subset containing a point $Y' \in t(Y)$. Then, $Y' \nsubseteq Z$, and so $Y \nsubseteq Z$, which means that $Y \in V$. So we see that $V(Y) = t(Y)$, and is thus a generic point for $t(Y)$. If $Y' \in t(Y)$ is any other point, then $t(Y') \subsetneq t(Y)$ does not contain Y, and so Y is a unique generic point.

Suppose $\varphi : X \to Y$ is a continuous map, with Y a Zariski space. Then, we have the following diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "\varphi"'] & t(X) \arrow[d, "t(\varphi)"] \\ Y \arrow[r, "\alpha_Y"] & t(Y). \end{tikzcd}$$

We claim that α_Y is a homeomorphism. Since every irreducible closed subset of Y has a generic point, it's certainly surjective, and since that generic point is unique, it's also in fact injective. It's also clear that a closed subset of Y maps to a closed subset in $t(Y)$. This proves our claim. So φ factors through α via $\alpha_Y^{-1} \circ t(\varphi)$. Uniqueness follows immediately. $\square$

5. Constructible Sets

Definition 5.1. The family of *constructible sets* on a topological space X is the smallest family $\mathfrak{C}$ of subsets of X which:

(1) contains all open sets of X,
(2) is closed under finite intersections,
(3) and contains the complement of each of its elements.

A subset of X is *constructible* if it belongs to $\mathfrak{C}$.

Let $\mathfrak{L}$ be the collection of sets that are finite unions of locally closed subsets. Then, it's easy to see that it satisfies all three conditions above, and is moreover contained in $\mathfrak{C}$. Hence, $\mathfrak{L} = \mathfrak{C}$; that is, a subset is constructible iff it is the finite union of locally-closed subsets.

terization-constble-sets **Proposition 5.2.** *A subset Y of a Noetherian space X is constructible iff, for every irreducible closed set $Z \subset X$, either $Y \cap Z$ is not dense in Z, or $Y \cap Z$ contains a non-empty open subset of Z.*

Proof. First, suppose Y is constructible. Then $Y = \cup_i L_i$, where L_i is locally closed, for every i. So we see that for every irreducible closed subset $Z \subset X$ in which Y is dense, we have

$$Z = \overline{Y \cap Z} = \cup_i(\overline{L}_i \cap Z).$$

Since Z is irreducible, we must have $Z \subset \overline{L}_i$, for some i. In that case, if $L_i = U_i \cap C_i$, for U_i open and C_i closed, we see that $Z \subset \overline{U}_i \cap C_i \subset C_i$. Hence, $X \cap Z$ contains the open set $U_i \cap C_i = U_i \cap Z$ of Z.

Now, for the converse, we'll use Noetherian induction. So we can assume that if Y satisfies the hypotheses, then $Y \cap Z$ is constructible, for every proper closed set $Z \subset X$. If X is not irreducible, then it's covered by finitely many proper closed subsets Z_i, such that $Y \cap Z_i$ is constructible for each i. In that case, $Y = \cup_i(Y \cap Z_i)$ is also constructible. Assume, therefore, that X is irreducible. Then, either Y contains a non-empty open subset $U \subset X$, in which case, $Y \setminus U$ is constructible, since it's contained in a proper closed subset; or, Y is not dense in X, in which case it's contained in a proper closed subset of X and is therefore constructible. $\square$

preimage-constructible **Corollary 5.3.** *If $f : X \to Y$ is a continuous map, then the preimage under f of every constructible set in Y is constructible in X.*

Proof. Immediate from the Proposition. $\square$

eric-point-constructible **Corollary 5.4.** *If X is an irreducible Noetherian space, then $Y \subset X$ is a constructible, dense set iff Y contains an open subset of X. If X is Zariski and Y is constructible, then Y is dense iff it contains the generic point of X.*

Proof. The first statement was proved in the course of the proof of the proposition. For the second statement, use Proposition 4.2. $\square$

We also a get a similar criterion for open sets.

aracterization-open-sets **Proposition 5.5.** *A subset Y of a Noetherian space X is open iff, for every closed irreducible subset Z of X, either $Y \cap Z = \emptyset$, or Y contains a non-empty open subset of Z.*

Proof. One direction is trivial. For the other, again use Noetherian induction to reduce to the case where X is irreducible, in which case, the statement is again trivial. $\square$

ed-stable-specialization

Proposition 5.6. *Let X be a Zariski space. Then a constructible subset of X is closed iff it contains all specializations of its points. It is open iff it contains all generizations of its points.*

Proof. The second statement follows from the first. Indeed, suppose $U \subset X$ contains all generizations, and let $y \in X - U$. If $y \rightsquigarrow x$, then $x \in X - U$, for, if $x \in U$, then $y \in U$, since U contains all its generizations. So $X - U$ contains all its specializations, and so will be closed by the first part.

Now, let's prove the first statement. Suppose $Z \subset X$ is constructible and contains all specializations. First assume Z is irreducible; then we see from 5.4 above that Z contains the generic point of $\overline{Z}$. But since every point in $\overline{Z}$ is a specialization of the generic point, we see that $Z = \overline{Z}$ is closed. Now, if $Z = \bigcup_{i=1}^{n} Z_i$, where Z_i are closed and irreducible in Z. Then, in each Z_i is constructible and thus closed, by the first part of the argument. Hence Z is also closed. Conversely, if Z is closed, and $x \in Z$, then $x \rightsquigarrow y$ implies $y \in V(x) \subset Z$. $\square$

Definition 5.7. A continuous map $f : X \to Y$ is *constructible* if the image under f of every constructible subset of X is constructible in Y.

Definition 5.8. A map $f : X \to Y$ has the *going down* property if, for every element $x \in X$, and every generization $y \in Y$ of $f(x)$, there is a generization $x' \in X$ of x such that $f(x') = y$.

going-down-equiv-prps

Proposition 5.9. *The following are equivalent for a map $f : X \to Y$ of quasi-Zariski spaces:*

1. *f has the going down property.*
2. *For every irreducible closed subset $Z \subset Y$, $f^{-1}(Z)$ maps generically onto Z, in the sense that every generic point of $f^{-1}(Z)$ maps on to the generic point of Z.*

Proof. First suppose f has the going down property, and let x be a generic point of $f^{-1}(Z)$. Then $f(x) \in Z$ is a specialization of the generic point z of Z, and so by going down there is a generization y of x such that $f(y) = z$. But since x is a generic point of $f^{-1}(Z)$, we must have $y = x$, and thus $f(x) = z$. Conversely, suppose (2) is true, and let $y \in Y$ be a generization of $f(x)$ for some $x \in X$. Now, let $Z = \overline{\{y\}}$; then $x \in f^{-1}(Z)$, and, if x' is any generic point of $f^{-1}(Z)$ that's a generization of x, then $f(x') = y$, thus showing that f satisfies the going down property. $\square$

The next Proposition gives sufficient conditions for a continuous map to be open.

going-down-constble-open

Proposition 5.10. *Let $f : X \to Y$ be a constructible map between Zariski spaces. Then f has the going down property if and only if f is an open map.*

Proof. First suppose f is open. Let $x \in X$ and $y \in Y$ be such that y is a generization of x, and let $U \subset X$ be the subset consisting of all generizations of x. Then U is open, since it contains all its generizations (5.6). This means that $f(U)$ is also open, and so contains all its generizations, which of course includes the generization y of $f(x)$.

Conversely, suppose f is a constructible map with the going down property, and let $U \subset X$ be an open subset. We'll use the criterion from (5.6) to show that $f(U)$ is open. Indeed, let $x \in U$; we need to show that $f(U)$ contains all generizations of $f(x)$. But this follows immediately from the going down property and the fact that U contains all generizations of x. $\square$

secn:dimension

6. Dimension

6.1. First Properties.

Definition 6.1. Let X be a topological space. The *dimension* $\dim X$ of X is the supremum of all positive integers $n \in \mathbb{N}$ such that there exists a strictly ascending chain

$$Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$

of closed, irreducible subspaces of X

A space X is *finite dimensional* if $\dim X < \infty$.

A space X is *equidimensional of dimension* n if every irreducible component of X has dimension n.

We record some basic consequences of our definition in the next Proposition.

dimension-prps

Proposition 6.2. *Let X be a topological space.*

(1) *If $Z \subset X$ is any subspace, then*

$$\dim Z \leq \dim X,$$

with strict inequality holding whenever $\dim X$ is finite and Z does not contain any irreducible component of X. In particular, $\dim$ defines an order preserving function from the poset of subsets of X to $\mathbb{N} \cup \{\infty\}$.

(2) *If $\{X_i : i \in I\}$ is the collection of irreducible components of X, then*

$$\dim X = \sup_{i \in I} \dim X_i.$$

(3) *If $\{U_i : i \in I\}$ is an open cover of X, then*

$$\dim X = \sup_{i \in I} \dim U_i.$$

(4) *If $\{Y_i : 1 \leq i \leq n\}$ is a finite collection of closed subsets such that $\bigcup_{i=1}^n Y_i = X$, then*

$$\dim X = \sup_{1 \leq i \leq n} \dim Y_i.$$

Proof. (1) Suppose $Z \subset Y$ is an irreducible closed subset; then we find that $\overline{Z} \cap Y = Z$, and so given any chain of irreducible closed subsets in Y, we obtain a chain in X by taking the closures of the chain's constituents.

(2) This is clear.

(3) If $\dim U_i = \infty$, for some $i \in I$, then $\dim X = \infty$ and the equality holds. So we can assume that $\dim U_i$ is finite, for all $i \in I$. In this case, suppose

$$Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$

is a chain of irreducible closed subsets of X. Then, there is $i \in I$, such that $U_i \cap Z_0 \neq \emptyset$. In this case, we get a chain

$$Z_0 \cap U_i \subsetneq Z_1 \cap U_i \subsetneq \ldots \subsetneq Z_n \cap U_i$$

of irreducible closed subsets of U_i, and so $n \leq \dim U_i$. From this and part (1), we obtain the equality.

(4) It suffices to show that every irreducible closed subset $Z \subset X$ must be contained in Y_i, for some $1 \leq i \leq n$. Indeed, if this weren't the case, then $Z = \bigcup_{1 \leq i \leq n} (Z \cap Y_i)$ would be reducible.

□

Definition 6.3. Let X be a topological space and let $x \in X$ be a point. Then the *dimension at x* of X, denoted $\dim_x X$, is the limit $\lim_{U \ni x} \dim U$ taken over the poset of open subsets of X containing x. By the previous Proposition, this is well defined, since dim is order preserving.

dimx-prps

Proposition 6.4. *Let X be a topological space and let $x \in X$ be a point.*

(1) *There is an open neighborhood U of x such that $\dim_x X = \dim V$, for all neighborhoods V of x contained in U. Moreover, if X is locally Noetherian, we may choose this U to be such that each of its irreducible components contains x.*

(2) *Let U be a neighborhood of x and let $\{Y_i : 1 \leq i \leq n\}$ be a finite collection of closed subsets such that $U = \bigcup_i Y_i$; then*
$$\dim_x X = \sup_{1 \leq i \leq n} \dim_x Y_i,$$
where $\dim_x Y_i = -\infty$, if $x \notin Y_i$.

(3) $\dim X = \sup_{y \in X} \dim_y X$.

(4) *If X is a Zariski space and $X_0 \subset X$ is the subset of its closed points, then*
$$\dim X = \sup_{y \in X_0} \dim_y X.$$

(5) *The function $x \mapsto \dim_x X$ is upper semicontinuous on X.*

Proof. (1) Clearly, from definition, and from the discreteness of the poset $\mathbb{N} \cup \{\infty\}$,there is a neighborhood U of x such that $\dim_x X = \dim U$. If X is locally Noetherian, then U may be chosen to be Noetherian, in which case it has only finitely many irreducible components. Removing the components that don't contain x gives us a neighborhood of x, all of whose irreducible components contain x.

(2) Let $\mathscr{N}$ be the set of open neighborhoods of x in X; then we have, by part (4) of (6.2):
$$\dim_x X = \inf_{V \in \mathscr{N}} \sup_{1 \leq i \leq n} \dim (V \cap Y_i).$$
By (1), we can choose a neighborhood V_0 of x such that, for $1 \leq i \leq n$, and every neighborhood V of x contained in V_0, we have $\dim_x Y_i = \dim (V \cap Y_i)$, from which we obtain our result.

(3) Clearly $\dim X \geq \sup_{y \in X} \dim_y X$; so it suffices to prove the reverse inequality. Consider a chain of closed irreducible subsets
$$Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$
in X, and choose any point $y \in Z_0$. Then, for every open neighborhood U of y, we have $\dim U \geq n$, and so $\dim_y X \geq n$. From this the result follows.

(4) If X is Zariski, then we can choose y to be a closed point in part (3) above (4.2).

(5) Suppose $\dim_x X \leq n < \infty$, and let U be a neighborhood of x as in (1); then, for all $y \in U$, we have $\dim_y X \leq n$. Hence, the set $\{x : \dim_x X \geq n+1\}$ is closed, for all $n \in \mathbb{N}$. Clearly, then, the set $\{x : \dim_x X < \infty\}$ is also open, which finishes the proof.

□

Definition 6.5. If $Z \subset X$ is an irreducible closed subspace, then the *codimension* $\operatorname{codim}(Z, X)$ of Z in X is the supremum of positive integers $n \in \mathbb{N}$ such that there exists a strictly ascending chain

$$Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$

of closed, irreducible subspaces of X.

If $Z \subset X$ is any closed subspace, then we define its codimension in X by the formula:

$$\operatorname{codim}(Z, X) = \inf_i \operatorname{codim}(Z_i, X),$$

where the inf is taken over the codimensions of the irreducible components of Z in X.

A closed subset $Z \subset X$ is of *pure codimension* n if, for all irreducible components $Z_i \subset Z$, $\operatorname{codim}(Z_i, X) = n$.

The space X is *equicodimensional of codimension* n if, for every minimal closed irreducible subset $Z \subset X$, $\operatorname{codim}(Z, X) = n$.

codim-prps

Proposition 6.6. *Let X be a topological space.*

(1) *If $Z_1 \subset Z_2$ is a tower of closed subsets of X, then*

$$\operatorname{codim}(Z_1, X) \geq \operatorname{codim}(Z_2, X).$$

(2) *If $Z \subset X$ is a closed subspace, then*

$$\dim X \geq \dim Z + \operatorname{codim}(Z, X).$$

(3) *We have*

$$\dim X = \sup_{x \in X} \operatorname{codim}(V(x), X).$$

(4) *If X is Zariski, and X_0 is the set of closed points of X, then*

$$\dim X = \sup_{x \in X_0} \operatorname{codim}(V(x), X).$$

(5) *If $Z \subset X$ is irreducible, and $U \subset X$ is any open subset such that $U \cap Z \neq \emptyset$, then*

$$\operatorname{codim}(Z, X) = \operatorname{codim}(U \cap Z, U).$$

Proof. (1) This follows from definition.

(2) The inequality is clear in the case where Z is irreducible. In general, if $\{Z_i : i \in I\}$ are the irreducible components of Z, then we have $\dim Z_i \leq \dim X - \operatorname{codim}(Z_i, X)$, for all $i \in I$. Now, by taking the sup on both sides over $i \in I$, and observing that $\dim Z = \sup_i \dim Z_i$ and $\operatorname{codim}(Z, X) = \inf_i \operatorname{codim}(Z_i, X)$, we get our inequality.

(3) Again, this follows from definition, and the fact that, given any non-empty closed subset $Z \subset X$, there is a closed subset of the form $V(x)$ contained in Z.

(4) In this case, we can always find a closed point in any closed subset of X.

(5) Follows from the fact that there is a bijection between the irreducible closed subsets of X intersecting non-trivially with U, and the irreducible closed subsets of U (1.3).

□

We now consider the simplest case: zero-dimensional spaces.

zero-dimension

Proposition 6.7. *Let X be a topological space.*

(1) *If X is discrete, then $\dim X = 0$.*
(2) *Conversely, if $\dim X = 0$, and X is Zariski, then X is a discrete space, and is thus finite.*

Proof. (1) Follows from the fact that the only irreducible closed subsets are

(2) Since $\dim X = 0$, for any $x \in X$, x is both a minimal and a maximal element in the ordering defined on X. Hence x is both a closed and a generic point of X. In particular, the set $\{x\}$ contains the generizations and specializations of all its elements and is thus both open and closed (5.6). The finiteness follows from the fact that X has only finitely many irreducible components.

□

Remark 6.8. There is a subtle point that we glossed over a little in the last part of the proof. The Proposition that we quoted only applies when we already know that the set under consideration is constructible. So a more legitimate line of reasoning would be to first note that $\{x\}$ is closed and then apply the Proposition to conclude that it is also open.

6.2. Going Up.

Note on Notation 1. In the sequel, we will restrict ourselves to the study of quasi-Zariski spaces. Here, we can rephrase our definition of dimension of a space X as the supremum of all positive integers $n \in \mathbb{N}$ such that there exists a strictly ascending chain of elements of the form

$$x_0 < x_1 < \ldots < x_n.$$

defn:going-up

Definition 6.9. A continuous map $f : X \to Y$ has the *going up* property if, for every point $x \in X$, f induces a surjective map from $V(x)$ to $V(f(x))$. Equivalently, f has the going up property if, for every $x \in X$, and every specialization $y \in Y$ of $f(x)$, there is a specialization $x' \in X$ of x such that $f(x') = y$.

going-up-closed

Proposition 6.10. *Let $f : X \to Y$ be a a continuous map.*

(1) *If f is closed, then f has the going up property*
(2) *Now, suppose that f has the going up property. If $Z \subset X$ is an irreducible closed subset, then $f(Z)$ is closed.*
(3) *Again, let f have the going up property. If $Z \subset X$ is a closed subset, and $\{Z_i : i \in I\}$ is its collection of irreducible components, then $\{f(Z_i) : i \in I\}$ is the collection of irreducible components of $f(Z)$.*
(4) *If f has the going up property and every subset of Y has only finitely many irreducible components (for example, if Y is Noetherian; see below), then f is a closed map.*

Proof. (1) Follows immediately from the fact that $f(V(x))$ is closed.

(2) Indeed, if $z \in Z$ is its unique generic point, then we see that

$$f(Z) = f(V(z)) \subset V(f(z)).$$

The going up condition gives us equality between the sets above on the far right and left, which of course tells us that $f(Z)$ is closed.

(3) First observe that $f(Z_i)$ is irreducible (1.3). Let W be an irreducible component of $f(Z)$, and let $w \in W$ be its unique generic point. Then $w = f(z)$, for some $z \in Z$, and there is some $j \in I$ such that $z \in Z_j$; but then $W \subset f(Z_j)$, which shows that $W = f(Z_j)$, and completes our proof.

(4) Follows immediately from part (2) and (3).

□

Definition 6.11. A map $f : X \to Y$ has the *incomparability* property if, for every point $y \in Y$, the fiber $f^{-1}(y)$ over y has dimension zero. In some sense, this is the 'relative zero dimension' case.

This next Lemma explains the terminology.

incomp-equiv

Lemma 6.12. *Let $f : X \to Y$ be a map between quasi-Zariski spaces. Then f has the incomparability property if and only if whenever $f(x) = f(x')$, for $x, x' \in X$, x and x' are incomparable under the ordering on X.*

Proof. Immediate from the definitions. □

going-up-incomp-dim

Proposition 6.13. *Let $f : X \to Y$ be a continuous map of quasi-Zariski spaces with the going up and incomparability properties.*

(1) *For every $x \in X$, we have*

$$\operatorname{codim}(V(x), X) \leq \operatorname{codim}(V(f(x)), Y).$$

In particular, $\dim X \leq \dim Y$.

(2) *If f is also closed, then* $\dim f(Z) = \dim Z$*, for all closed subsets $Z \subset X$.*

Proof. (1) Let $V(x) = V(x_0) \subsetneq \ldots \subsetneq V(x_n)$ be a chain of irreducible closed subsets of X. The going up property, via Proposition (6.10) tells us that this maps under f to a chain

$$V(f(x)) = V(f(x_0)) \subsetneq \ldots \subsetneq V(f(x_n))$$

of irreducible closed subsets of Y, where we get the strict inclusions because of the incomparability property. The second assertion follows immediately from part (3) of (6.6).

(2) For every closed subset $Z \subset X$, $f|_Z$ is also a closed map, and hence has the going up property, by (6.10). It clearly also has the incomparability property, since f does. Hence it follows that $\dim f(Z) \geq \dim Z$. Conversely, if $y_0 < \ldots < y_n$ is a chain of elements in $f(Z)$, then we can find $x_i \in Z$ such that $f(x_i) = y_i$, thus showing that $\dim f(Z) \leq \dim Z$.

□

6.3. Going Down and Dimension of Fibers.

going-down-fibers

Proposition 6.14. *Let $f : X \to Y$ be a map between quasi-Zariski spaces. If f has the going down property, and is such that $f^{-1}(y)$ is also quasi-Zariski, for all $y \in Y$, then, for every $x \in X$,*

$$\operatorname{codim}(V(x), X) \geq \operatorname{codim}(V(f(x)), Y) + \operatorname{codim}(V(x), f^{-1}(f(x))).$$

Proof. Indeed, let $x = x_0 < x_1 < \ldots < x_n$ be a chain of generizations of x in $f^{-1}(f(x))$, where $n = \operatorname{codim}(V(x), f^{-1}(f(x)))$, and let $y = y_0 < y_1 < \ldots < y_m$ be a chain of generizations of $y = f(x)$, where $m = \operatorname{codim}(V(y), Y)$. By the going down property, we can inductively lift this chain to a chain $x_n = x'_0 < x'_1 < \ldots < x'_m$ such that $f(x'_i) = y_i$. Then we get a chain

$$x_0 < x_1 < \ldots < x_n < x'_1 < \ldots < x'_m$$

of generizations of $x_0 = x$ in X, thus proving our assertion. □

6.4. **Chain Conditions.**

catenary-spaces

Definition 6.15. A topological space X is *catenary* if, for every pair Y, Z of closed irreducible subsets of X with $Y \subset Z$, we have $\operatorname{codim}(Y, Z) < \infty$, and if, for every tower $Z \subset T \subset W$ of closed irreducible subspaces of X, the following equality holds:

$$\operatorname{codim}(Z, W) = \operatorname{codim}(Z, T) + \operatorname{codim}(T, W).$$

A chain of irreducible closed subsets

$$Z_0 \subsetneq \ldots \subsetneq Z_n$$

is *saturated* if, for $1 \leq i \leq n$, $\operatorname{codim}(Z_{i-1}, Z_i) = 1$.

catenary-equiv-prps

Proposition 6.16. *Let X be a topological space. Then the following are equivalent:*

(1) *X is catenary.*
(2) *For every open subset $U \subset X$, U is catenary.*
(3) *There is an open cover $\{U_i : i \in I\}$ of X such that U_i is catenary, for each i.*
(4) *Given any two irreducible closed subsets $Z, W \subset X$ such that $Z \subsetneq W$, every saturated chain*
$$Z = Z_0 \subsetneq \ldots \subsetneq Z_n = W$$
has the same finite length.

Proof. Before we begin, we make a useful observation. Let $Z \subset T$ be two closed, irreducible subspaces of X, and let $U \subset X$ be an open subset such that $U \cap Z \neq \emptyset$. Then

$$\operatorname{codim}(Z, T) = \operatorname{codim}(Z \cap U, T \cap U).$$

This follows from part (5) of (6.6).

(1) ⇒ (2): Follows immediately from the remark above.
(2) ⇒ (3): Trivial.
(3) ⇒ (1): First let $Y, Z \subset X$ be irreducible closed subsets with $Y \subset Z$, and let $i \in I$ be such that $Y \cap U_i \neq \emptyset$. Then we have

$$\operatorname{codim}(Y, Z) = \operatorname{codim}(Y \cap U_i, Z \cap U_i) < \infty.$$

Now, let $Z \subset T \subset W$ be a tower of closed irreducible subsets of X, and let $i \in I$ be such that $U_i \cap Z \neq \emptyset$. Again, a similar argument, using the fact that U_i is catenary, gives us the equality

$$\operatorname{codim}(Z, W) = \operatorname{codim}(Z, T) + \operatorname{codim}(T, W).$$

(1) $\Rightarrow$ (4): Suppose we're given Z and W as in (4), and suppose we have two saturated chains

$$Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n = W$$
$$Z = Z'_0 \subsetneq Z'_1 \subsetneq \ldots \subsetneq Z'_m = W,$$

with $n \leq m$. We will use induction on n. If $n = 1$, then it follows that $\operatorname{codim}(Z, W) = 1$, and so $m = 1$. Suppose $n > 1$; then, by induction, $\operatorname{codim}(Z_1, W) = n - 1$, and so we have

$$\operatorname{codim}(Z, W) = \operatorname{codim}(Z, Z_1) + \operatorname{codim}(Z_1, W) = n.$$

But $\operatorname{codim}(Z, W) \geq m$, and so $n = m$.

(4) $\Rightarrow$ (1): This is easy.

□

Definition 6.17. A topological space X is *biequidimensional* if it is finite dimensional, and if every maximal chain of irreducible closed subsets of X has the same length. Evidently, this common length is $\dim X$.

quidimensional-subspaces

Lemma 6.18. *Let X be a biequidimensional space.*

(1) *X is catenary.*
(2) *For every irreducible closed subspace $Z \subset X$, we have*

$$\dim Z + \operatorname{codim}(Z, X) = \dim X.$$

(3) *Suppose every non-empty closed subset of X contains a closed point; then every irreducible closed subspace of X is also biequidimensional.*

Proof. (1) It suffices to show that any saturated chain between two fixed irreducible closed subspaces has the same length. But any such saturated chain can be extended to a maximal chain in X. Hence the result.

(2) This is equivalent to the statement that for every irreducible closed subset $Z \subset X$ there is a chain of irreducible closed subsets of length $\dim X$ containing Z. But this follows immediately from hypothesis.

(3) Let $Z \subset X$ be an irreducible closed subspace, let $z \in Z$ be a closed point, and let $X' \supset Z$ be an irreducible component of X. Then, by the catenary condition, we see that

$$\operatorname{codim}(z, Z) = \operatorname{codim}(z, X') - \operatorname{codim}(Z, X') = \dim X - \operatorname{codim}(Z, X')$$

is independent of z. Since every minimal closed subset of X is a closed point, we find that Z is equicodimensional. It is evident that Z is catenary and equidimensional.

□

uidimensional-equiv-prps

Proposition 6.19. *Let X be a finite dimensional Zariski space. Then the following are equivalent:*

(1) *X is biequidimensional.*
(2) *X is equidimensional, and for every pair of irreducible closed subsets $Y, Z \subset X$ with $Y \subset Z$, we have*

$$\dim Z = \dim Y + \operatorname{codim}(Y, Z).$$

(3) *X is equicodimensional, and for every pair of irreducible closed subsets $Y, Z \subset X$ with $Y \subset Z$, we have*

$$\operatorname{codim}(Y, X) = \operatorname{codim}(Y, Z) + \operatorname{codim}(Z, X).$$

Proof. Before we begin the rounds, note that, since X is Zariski, in every maximal chain of irreducible closed subsets of length n, Z_0 must be a closed point, and Z_n must be an irreducible component of X. Let $\{z_i : i \in I\}$ and $\{X_1, \ldots, X_r\}$ be the collections of closed points of X and irreducible components of X, respectively. Hence X is equidimensional if $\dim X_k = \dim X$, for all $1 \leq k \leq r$, X is equicodimensional if $\operatorname{codim}(\{z_i\}, X) = \dim X$, for all $i \in I$, and X is biequidimensional if $\operatorname{codim}(\{z_i\}, X_k) = \dim X$, for all pairs $i \in I$ and $k \in \{1, \ldots, r\}$, such that $z_i \in X_k$.

(1) $\Leftrightarrow$ (2): First suppose X is biequidimensional; then it's clear that X is also equidimensional. Moreover, if $Z \subset X$ is irreducible, then it follows from the lemma above (6.18) that Z is also biequidimensional, and in this case the required identity also follows from the same lemma.

Conversely, suppose (2) holds. Let $Y = \{z_i\}$, for some $i \in I$, and let $Z = X_k$, for some irreducible components X_k of X containing z_i. Then we have

$$\dim X = \dim X_k = \operatorname{codim}(\{z_i\}, X_k),$$

which shows that X is biequidimensional.

(2) $\Rightarrow$ (3): Observe that by (1) and the lemma above, we have

$$\dim Y + \operatorname{codim}(Y, X) = \dim Z + \operatorname{codim}(Z, X).$$

Now the identity in (3) follows immediately from the identity in (2).

(3) $\Rightarrow$ (1): Suppose now that (4) is true, and let $Y = \{z_i\}$, for some $i \in I$, and let $Z = X_k$, for some irreducible components X_k of X containing z_i. Then we have

$$\dim X = \operatorname{codim}(\{z_i\}, X) = \operatorname{codim}(\{z_i\}, X_k),$$

which shows that X is biequidimensional.

□