NOTES ON SHEAVES

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1. Basic Definitions

Consider a set X with a topology $\mathcal{U}$. We can give $\mathcal{U}$ a natural poset structure, described via inclusions of open sets into open sets. This gives us a category, which we will call $\mathrm{Top}(X,\mathcal{U})$.

Definition 1.1. A *presheaf* on X of objects in a category $\mathscr{C}$ or a *$\mathscr{C}$-valued presheaf* on X is a functor $F : \mathrm{Top}(X,\mathcal{U})^{op} \to \mathscr{C}$.

Note on Notation 1. Usually, the topology $\mathcal{U}$ is understood to be the natural one on the space X, and will be omitted. Sometimes, even X might be omitted from the notation. The omission will be fairly heuristic, though.

Definition 1.2. $\mathrm{Pre}(X,\mathscr{C})$ is the category of $\mathscr{C}$-valued presheaves on X, with the morphisms being natural transformations. By convention, for any presheaf $\mathscr{F} \in \mathrm{Pre}(X,\mathscr{C})$, $\mathscr{F}(\phi)$ will be the final object in the category $\mathscr{C}$, if such an object exists.

Definition 1.3. Given $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$, where $\mathscr{C}$ is a concrete category, then a *section* of $\mathscr{F}$ over an open set $U \subset X$ is simply an element $s \in \mathscr{F}(U)$.

Note on Notation 2. Given an inclusion $i_{V,U} : V \hookrightarrow U$, and any presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$, we will, abusing notation, use $\mathrm{res}_{U,V}$ to refer to the morphism $\mathscr{F}(i_{V,U}) : \mathscr{F}(U) \to \mathscr{F}(V)$, independent of which presheaf or category we're talking about, except in cases where confusion is likely.

Here's an important class of categories, the one we'll be most concerned with.

Definition 1.4. A category $\mathscr{C}$ is *abelian* if the following axioms hold:

- Ab-1 $\mathscr{C}$ has a 0-object, that is, an object that's both initial and final.
- Ab-2 For every pair of objects $X, Y \in \mathrm{Ob}\,\mathscr{C}$, the set of morphisms $\mathscr{C}(X, Y)$ is an abelian group, and if Z is another object, then composition $\mathscr{C}(X, Y) \times \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$ is a bilinear map.
- Ab-3 $\mathscr{C}$ has all finite products.
- Ab-4 Every morphism in $\mathscr{C}$ has a kernel and a cokernel.
- Ab-5 For every monic f, $\ker(\operatorname{coker} f) = f$ and for every epi g, $\operatorname{coker}(\ker g) = g$.

presheaf-abelian

Proposition 1.5. *If $\mathscr{C}$ is an abelian category, then so is* $\mathrm{Pre}(X, \mathscr{C})$.

Proof. This is pretty clear. It's just a matter of checking that the obvious choices for kernels and cokernels work.

Or one can just use the general fact that any functor category consisting of functors into an abelian category is also abelian. $\square$

So much for presheaves. We will now define the objects we really care about. From now on we'll assume that all our categories are complete.

Definition 1.6. A $\mathscr{C}$-valued *sheaf* on X is a presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$ that satisfies the following axiom:

Suppose $U \in \mathcal{U}$ is a union of open sets $\{U_i : i \in I\}$. Then, the following diagram is an equalizer:

$$\mathscr{F}(U) \xrightarrow{\prod_{i \in I} \mathrm{res}_{U,U_i}} \prod_{i \in I} \mathscr{F}(U_i) \underset{\prod_j \mathrm{res}_{U_j, U_j \cap U_i}}{\overset{\prod_i \mathrm{res}_{U_i, U_i \cap U_j}}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_i \cap U_j).$$

Remark 1.7. If $\mathscr{C}$ is a concrete category, then all this is saying is that if I have a bunch of elements $t_i \in \mathscr{F}(U_i)$ that agree on the intersections $U_i \cap U_j$, then I can patch them together to get a unique $t \in \mathscr{F}(U)$ that restricts in turn to each of the t_i.

Remark 1.8. Instead of $\mathrm{Top}(X)$ we could have looked at more general categories (with a suitable notion of covering (read étale)) to define our sheaves over. In this case, a natural replacement for the intersections $U_i \cap U_j$ would be the fiber product $U_i \times_U U_j$.

Definition 1.9. $\mathrm{Shf}(X, \mathscr{C})$ is the full subcategory of $\mathrm{Pre}(X, \mathscr{C})$ whose objects are sheaves.

2. Sheafification

In the cases that we're concerned with, namely when $\mathscr{C}$ is a concrete and complete category, we will be able to construct what's called the sheafification functor, that's a left adjoint to the forgetful functor from $\mathrm{Shf}(X, \mathscr{C})$ to $\mathrm{Pre}(X, \mathscr{C})$. There are a few ways of doing this. We'll first use a construction that generalizes well to other contexts (like the étale topology), but later we'll see a more quick and dirty way of doing it that's useful sometimes. Before we build our sheafification, we'll need a class of objects that are intermediate between presheaves and sheaves.

Definition 2.1. A presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$ is *separated* if, for every open set U and every open cover $\{U_i\}$ of U, the following morphism is monic:

$$\mathscr{F}(U) \xrightarrow{\prod_i \mathrm{res}_{U,U_i}} \prod_i \mathscr{F}(U_i).$$

We name the subcategory of separated sheaves $\mathrm{Sp}(X, \mathscr{C})$.

What this is saying for concrete categories is that two sections that restrict to the same section over every open subset in an open cover of U must be equal in $\mathscr{F}(U)$. It's also called the Identity Axiom.

For the sequel, we'll assume that $\mathscr{C}$ is a concrete and complete category, and that bijective morphisms are invertible in $\mathscr{C}$.

We take a tiny detour now.

Definition 2.2. Given any open set $U \in \mathcal{U}$ the *directed system associated to* U is the directed set $D(U) = \{\mathcal{V} : \mathcal{V} \text{ an open cover of } U\}$, with the ordering, $\mathcal{V} \preccurlyeq \mathcal{W}$ iff $\mathcal{W}$ is a refinement of $\mathcal{V}$. By convention, an open cover of an open set U can only contain subsets of U.

cofinal-system

Remark 2.3. Observe that for every $\mathcal{V} \in D(U)$, the subset of $D(U)$ formed by refinements of $\mathcal{V}$ is a cofinal system. This will be useful very soon.

Definition 2.4. For $U \in \mathcal{U}$ a *weak covering sieve* on U is an open cover $\mathcal{V} \in D(U)$ such that whenever $V, W \in \mathcal{V}$, we have $V \cap W \in \mathcal{V}$. In other words, $\mathcal{V}$ is closed under intersections.

Back to presheaves:

Definition 2.5. If we have a presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$, then for every open set $U \in \mathcal{U}$ and every $\mathcal{V} \in D(U)$, $\mathcal{V}(\mathscr{F})$ is the inverse limit

$$\mathcal{V}(\mathscr{F}) = \varprojlim_{U_i \in \mathcal{V}} \mathscr{F}(U_i).$$

We have a natural morphism $\mathscr{F}(U) \to \mathcal{V}(\mathscr{F})$ given by the inverse limit of the restriction maps res_{U,U_i}. Moreover, $\mathscr{F} \mapsto \mathcal{V}(\mathscr{F})$ is functorial in $\mathscr{F}$, since the inverse limit is functorial.

sheaf-reformulation

Remark 2.6. With these definitions in hand we can reformulate the sheaf axiom in a way that's suitable both for generalization and for the construction of sheafification. These may be treated as the standard definitions this point onward:

A presheaf $\mathscr{F}$ is *separated* iff for every open set U and every $\mathcal{V} \in D(U)$, the natural morphism $\mathscr{F}(U) \to \mathcal{V}(\mathscr{F})$ is an injection.

A presheaf $\mathscr{F}$ is a *sheaf* iff for every open set U and every weak covering sieve $\mathcal{V} \in D(U)$, the natural morphism $\mathscr{F}(U) \to \mathcal{V}(\mathscr{F})$ is an isomorphism.

Now, suppose $\mathcal{V} \in D(U)$ and $\mathcal{W}$ is a refinement of $\mathcal{V}$. Let $\mathcal{V} = \{V_i : i \in I\}$ and $\mathcal{W} = \{W_j : j \in J\}$. Then, we have a map $\sigma : J \to I$ such that, for every $j \in J$, $W_j \subset V_{\sigma(j)}$. This gives a natural morphism $\mathcal{V}(\mathscr{F}) \to \mathcal{W}(\mathscr{F})$ that just restricts each section $s \in \mathscr{F}(V_i)$ to $\mathscr{F}(W_j)$, for $j \in \sigma^{-1}(i)$. Thus $\{\mathcal{V}(\mathscr{F}) : \mathcal{V} \in D(U)\}$ is a directed system of objects in $\mathscr{C}$.

If we have an open set $V \subset U$, then we define for $\mathcal{W} \in D(U)$, $\mathcal{W} \cap V = \{W_i \cap V : W_i \in \mathcal{W}\} \in D(W)$. Accompanying this, we have a natural morphism $\mathcal{W}(\mathscr{F}) \to (\mathcal{W} \cap V)(\mathscr{F})$ taking every section $s \in \mathscr{F}(W_i)$ to its restriction in $\mathscr{F}(W_i \cap V)$.

sieve-sheaf *Remark* 2.7. Observe that this gives us a presheaf on U defined by $V \mapsto (\mathcal{W} \cap V)(\mathscr{F})$. Then it's easy to check that the natural map $\mathscr{F}(V) \to (\mathcal{W} \cap V)(\mathscr{F})$ then gives us a morphism of a presheaves.

We are now ready to take the first step towards defining the sheafification functor.

Definition 2.8. For a presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$, we define the *separation* $\mathrm{Sp}\,\mathscr{F}$ to be the presheaf defined on an open set $U \in \mathcal{U}$ by the direct limit

$$\mathrm{Sp}\,\mathscr{F}(U) = \varinjlim_{\mathcal{V} \in D(U)} \mathcal{V}(\mathscr{F}).$$

The restriction morphism $\mathrm{Sp}\,\mathrm{res}_{U,V}$ is just the direct limit of the natural morphisms $\mathcal{W}(\mathscr{F}) \to (\mathcal{W} \cap V)(\mathscr{F})$ defined above.

It's clear that this in fact defines a functor $\mathrm{Sp} : \mathrm{Pre}(X, \mathscr{C}) \to \mathrm{Sp}(X, \mathscr{C})$. We call this the separation functor.

Remark 2.9. We have a natural morphism $\mathrm{Sp}_{\mathscr{F}} : \mathscr{F} \to \mathrm{Sp}\,\mathscr{F}$: Just take the direct limit of the natural morphisms $\mathscr{F}(U) \to \mathcal{V}(\mathscr{F})$. By the definition of a direct limit, and the fact that for any $\mathcal{V} \in D(U)$, the set $\{\mathcal{W} \in D(U) : \mathcal{W} \succcurlyeq \mathcal{V}\}$ is a cofinal system, two sections with the same restrictions in some open cover $\mathcal{V}$ go to the same element in $\mathrm{Sp}\,\mathscr{F}(U)$. Essentially, that's precisely what this construction does: it identifies all elements that have the same restriction to some covering of U.

separation **Proposition 2.10.** *The separation functor* Sp *has the following properties:*

1. *$\mathscr{F}$ is separated iff $\mathrm{Sp}_{\mathscr{F}}$ is monic.*
2. *$\mathscr{F}$ is a sheaf iff $\mathrm{Sp}_{\mathscr{F}}$ is an isomorphism.*
3. *For any presheaf $\mathscr{F}$, $\mathrm{Sp}\,\mathscr{F}$ is separated.*
4. *If $\mathscr{F}$ is already separated, then $\mathrm{Sp}\,\mathscr{F}$ is a sheaf.*

In particular, for any presheaf $\mathscr{F}$, $\mathrm{Sp}(\mathrm{Sp}(\mathscr{F}))$ is a sheaf.

Proof. We will repeatedly use standard properties of direct limits without explicit mention. Going through the minute details of this proof is actually a great way to get comfortable with direct and inverse limits!

1. As observed in Remark 2.6, a presheaf is separated iff for every open set U, and every $\mathcal{V} \in D(U)$, the natural morphism $\mathscr{F}(U) \to \mathcal{V}(\mathscr{F})$ is an injection. But this can happen iff $\mathrm{Sp}_{\mathscr{F}U} : \mathscr{F}(U) \to \mathrm{Sp}\,\mathscr{F}(U)$ is an injection, for all open sets U.
2. Again, by Remark 2.6, a presheaf $\mathscr{F}$ is a sheaf iff the natural morphism $\mathscr{F}(U) \to \mathcal{V}(\mathscr{F})$ is an isomorphism for every weak covering sieve $\mathcal{V} \in D(U)$. Since the weak covering sieves form a cofinal system in $D(U)$, this implies that a presheaf $\mathscr{F}$ is a sheaf iff $\mathrm{Sp}_{\mathscr{F}U} : \mathscr{F}(U) \to \mathrm{Sp}\,\mathscr{F}(U)$ is an isomorphism, for all open sets U.

(3) Let $s, t \in \operatorname{Sp}\mathscr{F}(U)$ be sections. Then, we can consider them to be lying in $\mathcal{V}(\mathscr{F})$ for some $\mathcal{V} \in D(U)$. Suppose we have $W_i \subset U$ such that $\operatorname{Sp}\operatorname{res}_{U,W_i}(s) = \operatorname{Sp}\operatorname{res}_{U,W_i}(t)$. Then, this means that we can find a refinement $\mathcal{W}_i$ of $\mathcal{V}$ such that the image of s equals that of t in $(\mathcal{W}_i \cap W_i)(\mathscr{F})$. If we now have an open cover $\{W_i\}$ of U such that $\operatorname{Sp}\operatorname{res}_{U,W_i}(s) = \operatorname{Sp}\operatorname{res}_{U,W_i}(t)$, for all i, then we have an open cover $\mathcal{W} = \bigcup_i \{V \cap W_i : V \in \mathcal{W}_i\} \in D(U)$ that's a refinement of $\mathcal{V}$, but is such that the natural morphism $\mathcal{V}(\mathscr{F}) \to \mathcal{W}(\mathscr{F})$ now sends both s and t to the same element. Hence $s = t$ in $\operatorname{Sp}\mathscr{F}(U)$, and $\operatorname{Sp}\mathscr{F}$ is indeed separated.

(4) Suppose that U is open and that $\mathcal{V} \in D(U)$ is a covering sieve. We want to show that the natural morphism $\operatorname{Sp}\mathscr{F}(U) \to \mathcal{V}(\operatorname{Sp}\mathscr{F})$ is an isomorphism. By statement (3) we know that $\operatorname{Sp}\mathscr{F}$ is separated, and so we see from Remark 2.6 that this natural morphism is injective. So, assuming that $\mathscr{F}$ is separated, we need to show that it is surjective.

Let $\mathcal{V} = \{V_i\}$: then an element s of $\mathcal{V}(\operatorname{Sp}\mathscr{F})$ is given by coherent sequence of coherent sequences(!) of the form $((s_{i,\alpha}))$, where $\mathcal{V}_i = \{V_{i,\alpha}\}$ is an open cover of V_i and $s_{i,\alpha} \in \mathscr{F}(V_{i,\alpha})$. The coherence condition they satisfy on one level is that for every i, the sequence $(s_{i,\alpha})$ is an element of $\mathcal{V}_i(\mathscr{F})$. On the next level, if $V_i \cap V_j = V_k$, then there is a refinement $\mathcal{W}_{ij}$ of $\mathcal{V}_k$ such that both $(s_{i,\alpha})$ and $(s_{j,\beta})$ restrict to the same (coherent) sequence in $\mathcal{W}_{ij}(\mathscr{F})$.

Now, if it happens that $V_{i,\alpha} = V_{j,\beta}$ for some i, j, α, β, then we will show that $s_{i,\alpha} = s_{j,\beta}$. The only information we have is that $\mathscr{F}$ is separated. So suppose $V_i \cap V_j = V_k$. Then, we find that the restrictions of both sections to each $W_{k,\gamma} \in \mathcal{W}_{ij}$, for the open refinement $\mathcal{W}_{ij}$ of $\mathcal{V}_k$ guaranteed by second level coherence, are equal. Since $\mathcal{W}_{ij}$ is a cover of V_k, separatedness tells us that $s_{i,\alpha} = s_{j,\beta} \in V_{i,\alpha} \subset V_k$.

Consider now the open cover $\mathcal{W} = \{V_{i,\alpha}, \forall i, \alpha\} \in D(U)$. What we've shown above is that $t = (s_{i,\alpha}) \in \mathcal{W}(\mathscr{F})$, since we always had coherence of t, but we didn't know that it was well-defined, i.e. if we didn't have a conflict between $s_{i,\alpha}$ and $s_{j,\beta}$ if $V_{i,\alpha} = V_{j,\beta}$. What we did in the last paragraph is precisely to show that there is no such conflict, and so we're good. Notice, moreover, that the image of t under the natural morphism $\mathcal{W}(\mathscr{F}) \to \operatorname{Sp}\mathscr{F}(U) \to \mathcal{V}(\operatorname{Sp}(\mathscr{F}))$ is just s, as is easily checked. So we have proved surjectivity, and are thus thankfully done.

□

Remark 2.11. The same argument, with simple translations using an appropriate dictionary, goes through nicely for the general cases we've vaguely hinted at earlier. Just think étale.

Definition 2.12. The *sheafification* functor Shf is the functor assigning to every presheaf $\mathscr{F}$, the sheaf $\operatorname{Shf}\mathscr{F} = \operatorname{Sp}(\operatorname{Sp}(\mathscr{F}))$, with the obvious accompanying action on morphisms.

sheafification-adjoint

Theorem 2.13. *The sheafification* Shf *is a left adjoint to the forgetful functor* $U : \operatorname{Shf}(X, \mathscr{C}) \to \operatorname{Pre}(X, \mathscr{C})$.

Proof. It suffices to show that for every presheaf $\mathscr{F}$, we have a universal arrow to U, given by $\operatorname{Shf}_{\mathscr{F}} = \operatorname{Sp}_{\operatorname{Sp}_{\mathscr{F}}} : \mathscr{F} \to \operatorname{Shf}\mathscr{F}$, where of course we have omitted U, implicitly treating sheaves as objects in $\operatorname{Pre}(X, \mathscr{C})$.

Suppose $f : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, with $\mathscr{G}$ a sheaf. Then, by Proposition 2.10, $\mathscr{G}$ is isomorphic to $\mathrm{Shf}\,\mathscr{G}$. So we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\mathrm{Shf}_{\mathscr{F}}} & \mathrm{Shf}\,\mathscr{F} \\ {\scriptstyle f}\downarrow & \exists \widetilde{f} \swarrow & \downarrow{\scriptstyle \mathrm{Shf}\, f} \\ \mathscr{G} & \underset{\mathrm{Shf}_{\mathscr{G}}}{\overset{\cong}{\rightleftarrows}} & \mathrm{Shf}\,\mathscr{G} \end{array}$$

To show that $\mathrm{Shf}_{\mathscr{F}}$ is a universal arrow, we only need to show that $\widetilde{f}$ is unique. We will postpone the proof of uniqueness to the next section. □

Remark 2.14. In what follows, we will use the forgetful functor U rather erratically. Just keep in mind that the functor basically does nothing. In fact, since sheaves are a full subcategory, it doesn't even increase the number of possible morphisms like the forgetful functor, say, from Grp to Set. All it does is mess with surjectivity of morphisms. This will be clear very soon.

3. Stalks

Definition 3.1. If $\mathscr{F}$ is a presheaf, $x \in X$, then the *stalk* of $\mathscr{F}$ at x, denoted $\mathscr{F}_x$, is the direct limit $\lim_{x \ni U} \mathscr{F}(U)$ over open neighborhoods of x.

Remark 3.2. Here's a fun (but probably not very useful) way of looking at the definition of $\mathrm{Sp}\,\mathscr{F}$: We can take the set of all collections of open sets of X, and order it by refinement. Then $\mathcal{V} \mapsto \mathcal{V}(\mathscr{F})$ defines a presheaf on this poset, and what we're doing is simply looking at the stalk of this presheaf at U.

Definition 3.3. For $t \in \mathscr{F}(U)$, $x \in U$, we define the *germ* of t at x to be its image $t_x \in \mathscr{F}_x$ in the direct limit $\mathscr{F}_x$.

Here is an important property of sheaves that's illustrative of their local nature:

zerostalk-zeromap

Proposition 3.4. *Suppose $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{Shf}(X, \mathscr{C})$. Then, for every $x \in X$, we have an induced map of stalks, $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$, such that $\phi = 0$ iff $\phi_x = 0$, for all $x \in X$. In particular, a morphism of sheaves is determined completely by its induced maps on stalks.*

Proof. The ϕ_x are defined in the obvious fashion, as the direct limit of maps ϕ_U, for open sets $U \ni x$. One direction of the claimed equivalence is clear. So let's look at the non-trivial part: suppose $\phi_x = 0$, for all $x \in X$. Let $s \in \mathscr{F}(U)$ be a section; then, by the properties of direct limits, we can find, for every $x \in U$ a neighborhood $U_x \subset U$ of x such that $0 = \phi_{U_x}(\mathrm{res}_{U,U_x}(s)) = \mathrm{res}_{U,U_x}(\phi_U(s))$. But $\{U_x : x \in U\}$ is a covering of U, and so $\phi_U(s) = 0$, by the sheaf property. Since s and U were arbitrary, this implies that $\phi = 0$. □

Remark 3.5. The proof of the proposition goes through without any changes with the weaker assumption that $\mathscr{F}$ is any presheaf and that $\mathscr{G}$ is separated.

fification-iso-on-stalks

Lemma 3.6. *The map $\mathrm{Shf}_{\mathscr{F}} : \mathscr{F} \to \mathrm{Shf}\,\mathscr{F}$ induces isomorphisms on stalks.*

Proof. It's enough to show that $\mathrm{Sp}_{\mathscr{F}}$ induces isomorphisms on stalks. Suppose $x \in X$. Then, by our definitions, an element of $\mathrm{Sp}\,\mathscr{F}_x$ is represented by an element of $\mathcal{V}(\mathscr{F})$ for some open cover $\mathcal{V}$ of some neighborhood U of x. But an element of $\mathcal{V}(\mathscr{F})$ is just a coherent sequence of sections that restrict to the same germ at x. So we have a natural moprhism from $\mathrm{Sp}\,\mathscr{F}_x$ to $\mathscr{F}_x$ taking an element s to the germ corresponding to a coherent sequence that represents it. This is an inverse for the natural map induced by $\mathrm{Sp}_{\mathscr{F}}$ from $\mathscr{F}_x$ to $\mathrm{Sp}\,\mathscr{F}_x$. □

We can now complete the proof of Theorem 2.13.

Proof of Theorem 2.13 *(concluding part).* With all the notation is as in the first part of the proof, suppose $f' : \mathrm{Shf}\,\mathscr{F} \to \mathscr{G}$ is a morphism such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\mathrm{Shf}_{\mathscr{F}}} & \mathrm{Shf}\,\mathscr{F} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & \\ \mathscr{G} & & \end{array}$$

Then for every $x \in X$, we have the following picture for stalks:

$$\begin{array}{ccc} \mathscr{F}_x & \xrightarrow{\mathrm{Shf}_{\mathscr{F}_x}} & \mathrm{Shf}\,\mathscr{F}_x \\ {\scriptstyle f_x}\downarrow & \swarrow {\scriptstyle f'_x} & \\ \mathscr{G}_x & & \end{array}$$

But the top arrow is an isomorphism by Lemma 3.6, and so the map f'_x is uniquely determined by f. But Proposition 3.4 then tells us that f' is uniquely determined by f, and we're done. □

4. Local properties of morphisms of sheaves

From now on we will assume that all our categories $\mathscr{C}$ are concrete, complete and abelian. In other words, they are complete subcategories of Ab.

Definition 4.1. The *stalk* functor at x, $\mathscr{F} \mapsto \mathscr{F}_x$ on $\mathrm{Pre}(X, \mathscr{C})$ is the functor that sends a presheaf to its stalk at a point $x \in X$.

Remark 4.2. Note that the stalk functor is exact on $\mathrm{Pre}(X, \mathscr{C})$, since the direct limit functor is exact on abelian groups. More is true: it is in fact exact as a functor on $\mathrm{Shf}(X, \mathscr{C})$. Of course, we haven't shown that $\mathrm{Shf}(X, \mathscr{C})$ is abelian yet, so we can't really talk about exactness.

Before we proceed, we need to formalize some of our definitions first.

Definition 4.3. The *image* of a morphism of sheaves ϕ, $\mathrm{im}\,\phi$, is by definition $\mathrm{Shf}(\mathrm{im}\,U\phi)$, where $U : \mathrm{Shf}(X, \mathscr{C}) \to \mathrm{Pre}(X, \mathscr{C})$ is the forgetful functor. We will use $\mathrm{im}\,\phi$ interchangeably for both the map itself and for its codomain.

The *cokernel* of a morphism of sheaves ϕ, $\mathrm{coker}\,\phi$, is by definition $\mathrm{Shf}(\mathrm{coker}\,U\phi)$. The same caveat about abuse of notation applies to the cokernel.

Remark 4.4. [CT, 2.4] tells us that these are indeed the right definitions, in the sense that $\operatorname{coker}\phi$ really is $\operatorname{coker}\phi$!

Here's a proposition that's analogous to isomorphism being a local property in the category of commutative rings. The analogy will be made more precise when we've defined schemes.

loc-criterion-shfmap

Proposition 4.5. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. For each $x \in X$, we have the induced map of stalks, $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$. Then, the following statements are true:*

(1) *ϕ is injective iff ϕ_x is injective, for all $x \in X$.*
(2) *ϕ is an isomorphism iff ϕ_x is an isomorphism, for all $x \in X$.*
(3) *ϕ is surjective iff ϕ_x is surjective, for all $x \in X$.*

Remark 4.6. Note that a morphism of sheaves is surjective iff the *sheafification* of its image presheaf is the entire codomain(or equivalently, if the sheafification of its cokernel presheaf is 0). So a morphism of sheaves can be non-surjective as a morphism of presheaves, while still being surjective in the category of sheaves. In the examples later, we'll see that this makes a bunch of difference, and gives us examples of the unfaithfulness of sheafification.

Remark 4.7. If we knew that $\operatorname{Shf}(X, \mathscr{C})$ was an abelian category, then this proposition would tell us that the stalk functor is exact.

We also need a lemma:

surjectivity-criterion

Lemma 4.8. *A morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is surjective iff for every section $s \in \mathscr{G}(U)$, we can find an open covering $\{U_i\}$ of U and sections $t_i \in \mathscr{F}(U_i)$ such that $\phi_{U_i}(t_i) = \operatorname{res}_{U,U_i}(s)$.*

Proof. Note that ϕ is surjective, iff $\operatorname{Shf}\operatorname{im}U\phi = \mathscr{G}$. But observe that $\operatorname{im}U\phi$, being a subpresheaf, of $\mathscr{G}$ is already separated. Hence $\operatorname{Shf}\operatorname{im}U\phi = \operatorname{Sp}\operatorname{im}U\phi$, by Lemma 2.10. So we have that ϕ is surjective iff $\operatorname{Sp}\operatorname{im}U\phi = \mathscr{G}$. But this is nothing but a rewording of the lemma (Recall the definition of Sp). □

quotient-sheaf

Remark 4.9. This would be a good place to describe what a quotient sheaf over a concrete, abelian category looks like. If $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$, then the quotient presheaf $\mathscr{G}$ is separated, for, if $\{U_i\}$ is an open cover of an open set U, and $s \in \mathscr{F}(U)$ is such that $\operatorname{res}_{U,U_i}(s) \in \mathscr{F}'(U_i)$, for all i, then in fact $s \in \mathscr{F}'(U)$, since $\mathscr{F}'$ is also a sheaf. This means that any section $s \in (\mathscr{F}/\mathscr{F}')(U)$ can be represented by a coherent sequence $(s_i) \in \mathcal{V}(\mathscr{G})$, for some open cover $\mathcal{V} = \{U_i\} \in D(U)$, which basically is a sequence $(s_i) \in \prod_i \mathscr{F}(U_i)$ such that $\operatorname{res}_{U_i,U_i\cap U_j}(s_i) - \operatorname{res}_{U_j,U_j\cap U_i}(s_j) \in \mathscr{F}'(U_i \cap U_j)$, for all pairs i, j.

Proof of Proposition 4.5. Let's prove the statements one by one:

(1) Let $k : \ker\phi \to \mathscr{F}$ be the kernel of ϕ. Then we showed in Lemma 2.13 that $k = 0$ iff $k_x = 0$, for all $x \in X$. But since the kernel is just the regular old kernel presheaf, we see that $k_x : (\ker\phi)_x \to \mathscr{F}_x$ is the kernel of ϕ_x (This is important. It's not true for cokernels). Hence, we get the first statement.
(2) If we knew that $\operatorname{Shf}(X, \mathscr{C})$ was abelian, then we could get this directly from the next statement. But we don't yet, so we'll prove this independently. One direction is clear from the functoriality of direct limits. Let's consider the other one: Suppose ϕ_x is an isomorphism for every x. Then we know

already by the proof of the first statement that ϕ is injective. Now, to show that ϕ is an isomorphism, it suffices to show that ϕ is surjective as a morphism of *presheaves*, since we know for a fact that $\mathrm{Pre}(X,\mathscr{C})$ is abelian, and so injectivity and surjectivity combined imply isomorphism.

So suppose $s \in \mathscr{G}(U)$ is a section. Then, since the stalk maps are surjective, for every $x \in X$, we can find a neighborhood $U_x \subset U$ and a section $t_x \in \mathscr{F}(U_x)$ such that $\phi_{U_x}(t_x) = \mathrm{res}_{U,U_x}(s_x)$. Now, since ϕ is injective, and the sections $\phi_{U_x}(t_x)$ agree on the intersections $U_x \cap U_y$, we see that the sections t_x also agree on intersections. Since $\{U_x\}$ is an open cover of U, we have by the sheaf property an element $\widetilde{t} \in U$ such that $\mathrm{res}_{U,U_x}(\widetilde{t}) = t_x$. But then

$$\mathrm{res}_{U,U_x}(\phi_U(t)) = \phi_{U_x}(\mathrm{res}_{U,U_x}(t)) = \phi_{U_x}(t_x) = \mathrm{res}_{U,U_x}(s).$$

Hence, by the Identity Axiom, $\phi_U(t) = s$, and surjectivity is proved.

(3) We will use the criterion in Lemma 4.8. Suppose first that ϕ is surjective. Let $s_x \in \mathscr{G}_x$ be a germ at x, represented by a section $s \in \mathscr{G}(U)$. To show that ϕ_x is surjective, it will suffice to find a neighborhood $V \subset U$ of x and a section $t \in \mathscr{F}(V)$ such that $\phi_V(t) = \mathrm{res}_{U,V}(s)$. But this is easy: by the lemma, we can find an open cover $\{U_i\}$ of U and section $t_i \in U_i$ such that $\phi_{U_i}(t_i) = \mathrm{res}_{U,U_i}(s)$. Simply pick V to be a U_i that contains x.

Now suppose that ϕ_x is surjective for every x, and let $s \in \mathscr{G}(U)$ be a section. Then, for every $x \in U$, we can find a neighborhood $U_x \subset U$ of x and section $t_x \in \mathscr{F}(U_x)$ such that $\phi_{U_x}(t_x) = \mathrm{res}_{U,U_x}(s)$. Since $\{U_x\}$ is an open cover of U, we've proved precisely the criterion for surjectivity given to us by lemma 4.8.

□

5. On Abelianness and Exactness

Our goal is to prove an analogue of Proposition 1.5 for sheaves. This will not be possible to prove in all generality, but we will be able to show using the sheafification functor that $\mathrm{Shf}(X,\mathscr{C})$ is abelian for a concrete and complete abelian category $\mathscr{C}$.

Remark 5.1. Of course, by Freyd's Embedding Theorem, *every* abelian category is essentially concrete. But we still need completeness.

Theorem 5.2. *If $\mathscr{C}$ is a complete and concrete abelian category,* $\mathrm{Shf}(X,\mathscr{C})$ *is also abelian.*

Proof. By Proposition 1.5, $\mathrm{Pre}(X,\mathscr{C})$ is an abelian category. Then, by Theorem 2.13 and [CT, 2.7], we see that $\mathrm{Shf}(X,\mathscr{C})$ is pre-abelian.

So it only remains to prove Ab-5 for $\mathrm{Shf}(X,\mathscr{C})$. We want to show two things:

(1) If $\phi : \mathscr{F} \to \mathscr{G}$ is monic, then $\phi = \ker(\operatorname{coker} \phi)$.

We will use the forgetful functor U to distinguish between the cases where we think of a morphism of sheaves as a morphism of sheaves and when we think of it instead as a morphism of presheaves.

Recall from [CT, 2.4] that $\operatorname{coker} \phi = \mathrm{Shf}(\operatorname{coker} U\phi)$ (upto unique isomorphism), and from [CT, 2.5] that $0 = U(\ker \phi) = \ker U\phi$ (monic maps are injective). Hence $U\phi$ is monic, since injective maps are monic in $\mathrm{Pre}(X,\mathscr{C})$, and so $\ker(\operatorname{coker} U\phi) = U\phi$. Suppose now that $\operatorname{coker} U\phi : U\mathscr{G} \to \mathcal{C}$, coker $\phi : \mathscr{G} \to \operatorname{Shf}\mathcal{C}$, and $\ker(\operatorname{coker}\phi) : \mathcal{K} \to \mathscr{G}$. Then we have the following picture with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U\mathcal{K} & \xrightarrow{U(\ker(\operatorname{coker}\phi))} & U\mathscr{G} & \xrightarrow{U(\operatorname{coker}\phi)} & U\operatorname{Shf}\mathcal{C} & & \\
 & & \uparrow{\scriptstyle \exists\alpha} & & \uparrow{\scriptstyle 1_{U\mathscr{G}}} & & \uparrow{\scriptstyle U\operatorname{Shf}_{\mathcal{C}}} & & \\
0 & \longrightarrow & U\mathscr{F} & \xrightarrow{U\phi} & U\mathscr{G} & \xrightarrow{\operatorname{coker}U\phi} & \mathcal{C} & \longrightarrow & 0
\end{array}$$

where we get the natural map α by noting that $U(\operatorname{coker}\phi)\circ U\phi = U(\operatorname{coker}\phi\circ\phi) = 0$ and using the universal property of $U(\ker(\operatorname{coker}\phi)) = \ker(U(\operatorname{coker}\phi))$ (This equality (or unique isomorphism, rather) follows from [CT, 2.5]).

Now if we specialize to stalks, then we see that the vertical maps on the right hand side are isomorphisms of stalks, the middle one, trivially, and the sheafification, by Lemma 3.6. Hence α will also induce isomorphisms of stalks. But α is a morphism of sheaves! Hence, by Proposition 4.5, it's an isomorphism. This shows precisely that $\ker(\operatorname{coker}\phi) = \phi$.

(2) If $\psi : \mathscr{F} \to \mathscr{G}$ is epi then $\psi = \operatorname{coker}(\ker\psi)$.

Since ψ is epi, $\operatorname{coker}\psi = \operatorname{Shf}(\operatorname{coker}U\psi) = 0$. Suppose $\operatorname{coker}U\psi : U\mathscr{G} \to \mathscr{C}$, $\ker\psi : \mathcal{K} \to \mathscr{F}$ and $\operatorname{coker}(\ker\psi) : \mathscr{F} \to \mathscr{C}'$. Observe that $\operatorname{Shf}\mathscr{C} = 0$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & U\mathcal{K} & \xrightarrow{U(\ker\psi)} & U\mathscr{F} & \xrightarrow{U(\operatorname{coker}\ker\psi)} & U\mathcal{C}' & & & & \\
 & & \downarrow{\scriptstyle 1_{U\mathcal{K}}} & & \downarrow{\scriptstyle 1_{U\mathscr{F}}} & & \downarrow{\scriptstyle \exists U\beta} & & & & \\
0 & \longrightarrow & U\mathcal{K} & \xrightarrow{U(\ker\psi)} & U\mathscr{F} & \xrightarrow{U\psi} & U\mathscr{G} & \xrightarrow{\operatorname{coker}U\psi} & \mathcal{C} & \longrightarrow & 0
\end{array}$$

where $\beta : \mathscr{C}' \to \mathscr{G}$ is the natural map obtained from the universal property of $\operatorname{coker}(\ker\psi)$.

If we now specialize to stalks, then the maps on the left are identity maps and so trivially induce isomorphisms on stalks. The interesting stuff happens in the right hand side of the bottom row, where all the stalks of $\mathcal{C}$ vanish. To see this, just apply Proposition 3.4 to the identity map on $\operatorname{Shf}\mathcal{C}$ and observe that $\operatorname{Shf}_{\mathcal{C}}$ induces isomorphisms between stalks of $\mathcal{C}$ and $\operatorname{Shf}\mathcal{C}$. Hence the bottom row forms a short exact sequence, and so β induces isomorphisms of stalks, also. So β is actually an isomorphism, and we're done.

□

Now that we have an abelian category on our hands, we can talk about exactness and exact functors, bandying about images and cokernels just as we do for any old abelian category. We already have a few functors in hand that are exact, so let's show that they in fact are. Since we have all the ammunition in hand, this will be painless.

Proposition 5.3. *The stalk functor is exact on* $\operatorname{Shf}(X, \mathscr{C})$.

Proof. Follows immediately from Proposition 4.5, and the obvious fact that it is additive. □

Proposition 5.4. Shf : $\mathrm{Pre}(X, \mathscr{C}) \to \mathrm{Shf}(X, \mathscr{C})$ *is an exact functor.*

Proof. Since Shf is left adjoint, it is already right exact, by abstract nonsense. We need to show that it preserves injections. Suppose $\phi : \mathscr{F} \to \mathscr{G}$ is an injective morphism of presheaves. We want to show that $\ker \mathrm{Shf}\, \phi = 0$. Consider this commutative diagram with exact rows:

$$\begin{array}{ccccc} & & \mathrm{Shf}\,\mathscr{F} & \xrightarrow{\mathrm{Shf}(\phi)} & \mathrm{Shf}\,\mathscr{G} \\ & & {\scriptstyle \mathrm{Shf}_{\mathscr{F}}} \uparrow & & {\scriptstyle \mathrm{Shf}_{\mathscr{G}}} \uparrow \\ 0 & \longrightarrow & \mathscr{F} & \xrightarrow{\phi} & \mathscr{G} \end{array}$$

As usual, if we specialize to stalks, then the two vertical morphisms are isomorphisms, and so $\mathrm{Shf}(\phi)$ also induces injections on stalks, and therefore, by Proposition 4.5, it is itself an injection. ☐

Remark 5.5. One fact that we used repeatedly in these proofs is that *sheafification is locally an isomorphism.* This seems like a good general idea to keep at the back of one's mind, when dealing with other functors that have the same kind of property.

Definition 5.6. For an open set $U \in \mathcal{U}$, the *section* functor $\Gamma(U, \_\_) : \mathrm{Shf}(X, \mathscr{C}) \to \mathscr{C}$ is the functor that assigns to each sheaf $\mathscr{F}$ the object $\mathscr{F}(U)$.

`section-left-exact` **Proposition 5.7.** $\Gamma(U, \_\_)$ *is a left exact functor.*

Proof. Note simply that the functor $\mathscr{G} \mapsto \mathscr{G}(U)$ is exact from $\mathrm{Pre}(X, \mathscr{C})$ to $\mathscr{C}$, and $\Gamma(U, \_\_)$ is just the forgetful functor U, which is left exact, composed with this exact functor. Hence $\Gamma(U, \_\_)$ is left exact. ☐

Remark 5.8. In general, this functor's not exact, and the study of its right derived functors is what sheaf cohomology is about. More about this, later down the line.

6. Sheaves as bundles of stalks

There are many different ways to think about sheaves. We will now give a new (though historically older) definition of a sheaf and show that it's equivalent to our original definition. This will prove useful in some cases.

Definition 6.1. Given a topological space X, the category of *bundles over X*, $\mathrm{Bund}(X)$ is the category whose objects are pairs (h, B), where $h : B \to X$ is a continuous map. A morphism $f : (h, B) \to (h', B')$ is simply a continuous map $f : B \to B'$ such that $h' \circ f = h$.

`bundle-section-sheaf` **Definition 6.2.** Given a bundle (h, B) over X, and an open set $U \subset X$, a *section* over U is a continuous map $t : U \to B$ such that $h \circ t = i_U$, where $i_U : U \hookrightarrow X$ is the inclusion.

The set of sections of h over an open set U is denoted by $\Gamma B(U)$.

`sheaf-of-local-sections` **Proposition 6.3.** *Given any bundle (h, B) over X, the assignment $U \mapsto \Gamma B(U)$ defines a sheaf $\Gamma B \in \mathrm{Shf}(X, \mathrm{Set})$. This sheaf is called the associated sheaf of sections for (h, B).*

Proof. We need to check that for any open set U, and any weak covering sieve $\mathcal{V}$ of U, the natural map $\Gamma B(U) \to \mathcal{V}(\Gamma B)$ is an isomorphism. For this, it suffices to note, that for *any* open cover $\mathcal{V}$ of U, a coherent sequence in $\mathcal{V}(\Gamma B)$ consists of continuous maps to B that agree on the intersections of their domains of definition, and so can be patched up to define a unique continuous map on all of U, that is still of course a section. □

If $f : (h, B) \to (h', B')$ is a morphism of bundles over X, then for any open set $U \subset X$, and any section $s : U \to B$, we see that $f \circ s : U \to B'$ defines a section of (h', B') over U, since $h' \circ f \circ s = h \circ s = i_U$. This means that the assignment $(h, B) \to \Gamma B$ is functorial.

Definition 6.4. We define the *sheaf of sections* functor $\Gamma : \mathrm{Bund}(X) \to \mathrm{Shf}(X, \mathrm{Set})$ to be the functor that takes a bundle (h, B) over X to the associated sheaf of sections ΓB.

Now, observe that, given a presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$, and any section $s \in \mathscr{F}(U)$, we have a natural map $\overline{s} : U \to \coprod_{x \in U} \mathscr{F}_x$ given by $x \mapsto s_x$. Moreover, if we consider the disjoint union $\mathrm{Spc}(\mathscr{F}) := \coprod_{x \in X} F_x$, then we have a natural map $\pi : \mathrm{Spc}(\mathscr{F}) \to X$ that takes $\mathscr{F}_x$ to x. This leads to a definition:

Definition 6.5. The *espace étalé* over X associated to a presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$ is the map $\pi : \mathrm{Spc}(\mathscr{F}) \to X$, where $\mathrm{Spc}(F)$ is equipped with the finest topology such that the map $\overline{s} : U \to \mathrm{Spc}(\mathscr{F})$ is continuous for every section $s \in \mathscr{F}(U)$, over any open set $U \subset X$.

Remark 6.6. One might wonder why this makes π continuous. For this simply observe that for any section $s \in \mathscr{F}(U)$, and any open set $V \subset X$, $\overline{s}^{-1}(\pi^{-1}(V)) = U \cap V$, which is of course open. So making $\pi^{-1}(V)$ open for all open $V \subset X$ does not interfere with the continuity of $\overline{s}$. Observe also that under this definition each presheaf section $s \in \mathscr{F}(U)$ defines a *bundle* section $\overline{s} : U \to B$ as in definition 6.2. This is where the terminology comes from.

Moreover, if we have a morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$, then that induces a morphism of stalks $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$, and thus a morphism of espace étales $\mathrm{Spc}(\phi) : \mathrm{Spc}(\mathscr{F}) \to \mathrm{Spc}(\mathscr{G})$.

Definition 6.7. The *espace étalé* functor $\mathrm{Spc} : \mathrm{Pre}(X, \mathscr{C}) \to \mathrm{Bund}(X)$ is the functor that assigns to each presheaf $\mathscr{F}$, its espace étalé $(\pi, \mathrm{Spc}(\mathscr{F}))$, with the natural action on morphisms described above.

Observe that the composition of the two functors Spc and Γ gives us a functor $\Gamma \, \mathrm{Spc} : \mathrm{Pre}(X, \mathscr{C}) \to \mathrm{Shf}(X, \mathrm{Set})$. In the case where $\mathscr{C}$ is the category of modules over a ring R, we see immediately that $\Gamma \, \mathrm{Spc}$ is in fact a functor into $\mathrm{Shf}(X, \mathscr{C})$. For the rest of the section, we will assume that $\mathscr{C}$ is either Set or the category of modules over a ring R.

Notice then that we have a natural transformation Shf from $\mathrm{Id}_{\mathrm{Pre}(X, \mathscr{C})}$ to $\Gamma \, \mathrm{Spc}$, where we treat the latter functor as an endofunctor on $\mathrm{Pre}(X, \mathscr{C})$, postcomposing implicity by the forgetful functor U. This natural transformation is defined just by taking a section $s \in \mathscr{F}(U)$ to its corresponding bundle section $\overline{s} \in \Gamma \, \mathrm{Spc}(\mathscr{F})(U)$. The reason for naming this natural transformation in this fashion (which can, and should, be construed as abuse of notation) will be clear soon.

bundle-sheafification **Proposition 6.8.** *If $\mathscr{F} \in \operatorname{Pre}(X, \mathscr{C})$ is a presheaf, then the following statements are true:*

(1) *The morphism $\operatorname{Shf}_{\mathscr{F}} : \mathscr{F} \to \Gamma \operatorname{Spc}(\mathscr{F})$ induces isomorphisms on stalks*
(2) *The morphism $\operatorname{Shf}_{\mathscr{F}} : \mathscr{F} \to \Gamma \operatorname{Spc}(\mathscr{F})$ is monic iff $\mathscr{F}$ is separated.*
(3) *The morphism $\operatorname{Shf}_{\mathscr{F}} : \mathscr{F} \to \Gamma \operatorname{Spc}(\mathscr{F})$ is an isomorphism iff $\mathscr{F}$ is a sheaf.*

Proof. Before we begin the proof, observe that for every section $s \in \mathscr{F}(U)$, the set $\overline{s}(U) \subset \operatorname{Spc}(\mathscr{F})$ is open. If I choose any other section $t \in \mathscr{F}(V)$, then $\overline{t}^{-1}(\overline{s}(U)) = \{x \in U \cap V : t_x = s_x\}$. This is an open set, since if $t_x = s_x \in \mathscr{F}_x$, then there is a neighborhood around x on which t and s agree. Moreover, these sets form a basis for the topology on $\operatorname{Spc}(\mathscr{F})$.

(1) First we prove injectivity of the induced maps. Suppose $t, s \in \mathscr{F}(U)$ are two sections such that $\overline{t}_x = \overline{s}_x \in \Gamma \operatorname{Spc}(\mathscr{F})_x$. Then, there is a neighborhood V of x such that $\overline{t}|_V = \overline{s}|_V$, which implies that $t_y = s_y \in \mathscr{F}_y$, for all $y \in V$. But this immediately implies that $\operatorname{res}_{V,U}(t) = \operatorname{res}_{V,U}(s)$. Therefore, $t_x = s_x \in \mathscr{F}_x$, and the the induced map is indeed injectivity.

Now, for surjectivity, suppose $x \in U$ and $t_x \in \Gamma \operatorname{Spc}(\mathscr{F})_x$ is represented by a section $t \in \Gamma \operatorname{Spc}(\mathscr{F})(V)$. Then, there is a section $s \in \mathscr{F}(V)$ and a corresponding open set $\overline{s}(V) \ni t(x)$ in $\operatorname{Spc}(\mathscr{F})$. Since t is continuous, we can find a smaller open set $W \ni x$ contained in V such that $t(W) \subset \overline{s}(V)$. But then, for all $y \in W$, $t(y) = \overline{s}(y)$, which means that $t_x = \overline{s}_x$. Hence the induced map is a also surjection on stalks

(2) One direction is clear, since any subpresheaf of a sheaf is separated. Suppose then that $\mathscr{F}$ is separated. Then, we want to show that if $s, t \in \mathscr{F}(U)$ are such that $\overline{s} = \overline{t}$, then $s = t$. But our hypothesis implies that $s_x = t_x \in \mathscr{F}_x$ for all $x \in U$, which means that for all $x \in U$, we can find a neighborhood U_x such that $\operatorname{res}_{U,U_x}(s) = \operatorname{res}_{U,U_x}(t)$. Since $\{U_x\}$ is a cover of U, we see by separatedness that $s = t$.

(3) Again, one direction is clear, since a presheaf isomorphic to a sheaf is obviously a sheaf. For the other, assume that $\mathscr{F}$ is a sheaf. Then from (1) above, we know that the morphism $\operatorname{Shf}_{\mathscr{F}}$ induces isomorphisms on stalks. This finishes our proof by Proposition 4.5.

□

Proposition 6.9. *The functor $\Gamma \operatorname{Spc}$ is a left adjoint to the forgetful functor $U : \operatorname{Shf}(X, \mathscr{C}) \to \operatorname{Pre}(X, \mathscr{C})$.*

Proof. Same as the proof of Proposition 2.13. □

Remark 6.10. The last Proposition shows that $\Gamma \operatorname{Spc}$ and Shf are both left adjoints to the forgetful functor U, and are thus canonically isomorphic. We will use Shf to refer to both in the future.

7. The Direct and Inverse Image Functors

Till now, we've only been considering sheaves on a fixed topological space X. Now we will consider continuous maps between topological spaces, and two important functorial operations that such maps induce on sheaves.

Definition 7.1. Suppose $f : X \to Y$ is a continuous map of topological spaces. Then the *direct image functor* is the functor $f_* : \operatorname{Pre}(X, \mathscr{C}) \to \operatorname{Pre}(Y, \mathscr{C})$ defined on an open set $U \subset Y$ by:

$$(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U)).$$

If $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on X, then $(f_*\phi)_U = \phi_{f^{-1}(U)}$.

Proposition 7.2. *Let f be as in the definition above. Then, f_* defines a functor from* $\mathrm{Shf}(X, \mathscr{C})$ *to* $\mathrm{Shf}(Y, \mathscr{C})$. *That is, if $\mathscr{F}$ is a sheaf, then so is $f_*\mathscr{F}$.*

Proof. Recalling our reformulation of the definition of a sheaf from Remark 2.6, we have to show that if $\mathcal{V} = \{V_i\}$ is a weak covering sheaf of an open set $U \subset Y$, then the natural morphism $(f_*\mathscr{F})(U) \to \mathcal{V}(f_*(F))$ is an isomorphism. On the left hand side we have $\mathscr{F}(f^{-1}(U)$, and on the right we have $\lim_{V_i \in \mathcal{V}} \mathscr{F}(f^{-1}(V_i)$. Since $\mathscr{F}$ is a sheaf, it will suffice to show that $\mathcal{W} = \{f^{-1}(V_i)\}$ is a weak covering sieve for $f^{-1}(U)$. But this is obvious, since intersections are preserved under the taking of pre-images. □

Now, we will define the inverse image functor f^{-1}. We'll do this through the following proposition of category theory.

Proposition 7.3. *Suppose $\mathscr{D}$ and $\mathscr{D}'$ are small categories and $F : \mathscr{D}' \to \mathscr{D}$ is a functor. Then, for any cocomplete category $\mathscr{E}$ the functor* $F^\sharp : \mathrm{Funct}(\mathscr{D}, \mathscr{E}) \to \mathrm{Funct}(\mathscr{D}', \mathscr{E})$ *defined by $G \mapsto GF$ has a left adjoint $F_\sharp$.*

Proof. See [CT, 2.8]. □

If we look at the construction of this left adjoint for $F^\sharp = f_*$, then we'll see that, for a presheaf $\mathscr{F} \in \mathrm{Pre}(X, \mathscr{C})$, $F_\sharp\mathscr{F}(V) = \lim_{U \supset f(V)} \mathscr{F}(U)$. This gives us a definition:

irimage-invimage-adjoint

Definition 7.4. If $f : X \to Y$ is a continuous map, inducing a functor $F : \mathrm{Top}(Y)^{op} \to \mathrm{Top}(X)^{op}$, then the *inverse image functor* f^{-1} is the left adjoint to the functor $F^\sharp U = f_* : \mathrm{Shf}(X, \mathscr{C}) \to \mathrm{Shf}(Y, \mathscr{C})$ defined above.

More concretely, for a sheaf $\mathscr{F} \in \mathrm{Shf}(Y, \mathscr{C})$, the inverse image $f^{-1}\mathscr{F}$ is the sheafification of the presheaf that takes an open set $V \subset X$ to $\lim_{U \supset f(V)} \mathscr{F}(U)$.

Remark 7.5. That this is the right definition follows from the fact that Shf is left adjoint to the forgetful functor U, and so $f^{-1} = \mathrm{Shf}\, F_\sharp$ is left adjoint to the functor $f_* = F^\sharp U$.

Remark 7.6. One might wonder what the stalks of $f_*\mathscr{F}$ and $f^{-1}\mathscr{F}$ look like. In the latter case, it's easy to see that $(f^{-1}\mathscr{F})_x = \mathscr{F}_{f(x)}$. For the former, there is no such simple description.

There is another way of looking at the inverse image functor that in some ways is more natural. It uses the bundle of stalks view of sheaves.

If $f : X \to Y$ is a continuous map, and $\mathscr{F} \in \mathrm{Shf}(Y, \mathscr{C})$, then we have the corresponding bundle $\pi : \mathrm{Spc}(\mathscr{F}) \to Y$ over Y. Then we can consider the pullback $X \times_f \mathrm{Spc}(\mathscr{F}))$ in the following diagram:

$$\begin{array}{ccc} X \times_f \mathrm{Spc}(\mathscr{F}) & \longrightarrow & \mathrm{Spc}(\mathscr{F}) \\ \Big\downarrow \widetilde{\pi} & & \Big\downarrow \pi \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

Definition 7.7. If $f : X \to Y$ is a continuous map, then the *inverse image functor* is the functor f^{-1} that associates to each sheaf $\mathscr{F} \in \mathrm{Shf}(Y, \mathscr{C})$ the sheaf of sections $\Gamma(X \times_f \mathrm{Spc}(\mathscr{F}))$ of the pullback defined above.

Remark 7.8. In fact, this definition is equivalent (upto canonical isomorphism) to the one given above, but we won't really be using this construction, and so there's no reason to dwell too much upon it.

Now, let's collect the exactness properties of the two functors.

Proposition 7.9. *For a continuous map $f : X \to Y$, the direct image functor $f_* : \mathrm{Shf}(X, \mathscr{C}) \to \mathrm{Shf}(Y, \mathscr{C})$ is left exact, and the inverse image functor $f^{-1} : \mathrm{Shf}(Y, \mathscr{C}) \to \mathrm{Shf}(X, \mathscr{C})$ is exact.*

Proof. Most of the exactness properties follow from the fact that f_* and f^{-1} form an adjunction of functors. The only thing that still needs proof is the assertion that f^{-1} is left exact. So assume that $\phi : \mathscr{F} \to \mathscr{G}$ is an injective morphism of sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{Shf}(Y, \mathscr{C})$. It suffices to show that $f^{-1}(\phi)$ induces injections on stalks. But observe that the map $(f^{-1}\mathscr{F})_x \to (f^{-1}\mathscr{G})_x$ is simply $\phi_{f(x)} : \mathscr{F}_{f(x)} \to \mathscr{G}_{f(x)}$. From this the statement follows. □

Here's a property of the inverse image functor that adjointness makes a cinch to prove.

invimg-composition

Proposition 7.10. *If we have continuous maps $f : X \to Y$, $g : Y \to Z$, then the functor $f^{-1}g^{-1}$ is naturally isomorphic to the functor $(g \circ f)^{-1}$.*

Proof. It follows directly from the definitions that $(g \circ f)_* = g_* f_*$. Now, we know that $(g \circ f)^{-1}$ is left adjoint to $(g \circ f)_*$. But since g^{-1} is left adjoint to g_* and f^{-1} is left adjoint to f_*, we see that $f^{-1}g^{-1}$ is left adjoint to $g_* f_* = (g \circ f)_*$. Since both $f^{-1}g^{-1}$ and $(g \circ f)^{-1}$ are left adjoints of the same functor, we see that they must be canonically isomorphic. □

8. Other operations on sheaves

Suppose we have a sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, and assume $Z \subset X$. Then it makes intuitive sense to talk about the restriction of $\mathscr{F}$ to Z. Conversely, if we had a sheaf $\mathscr{G} \in \mathrm{Shf}(Z, \mathscr{C})$, then we'd like to extend it to all of X, by setting it to zero outside of Z. Both these processes can be formalized using the direct and inverse image functors.

8.1. **Restrictions and Extensions by Zero.**

restriction-extzero

Definition 8.1. If $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, and $Z \subset X$, then the *restriction* of $\mathscr{F}$ to Z is the inverse image sheaf $i^{-1}\mathscr{F} \in \mathrm{Shf}(Z, \mathscr{C})$, where $i : Z \hookrightarrow X$ is the inclusion. It is usually denoted by $\mathscr{F}|_Z$.

Conversely, if $\mathscr{G} \in \mathrm{Shf}(Z, \mathscr{C})$, then the *extension by zero* of $\mathscr{G}$ to X is the direct image sheaf $i_*\mathscr{G} \in \mathrm{Shf}(X, \mathscr{C})$.

Remark 8.2. Note that if $Z \subset X$ is open, then $\mathscr{F}|_Z$ is simply the sheaf assigning to every open set $V \subset Z$, the object $\mathscr{F}(V)$. In particular, the presheaf obtained from the functor adjoint to $I^\sharp = i_*$, corresponding to the functor $I : \mathrm{Top}(X) \to \mathrm{Top}(Z)$, $I(V) = Z \cap V$, does not need to be sheafified.

The term 'extension by zero' is justified by the following proposition.

extzero-stalks

Proposition 8.3. *With all the notation as in the definition above, for $x \in X$,*

$$(i_*\mathscr{G})_x = \begin{cases} \mathscr{G}_x & \text{if } x \in Z \\ 0 & \text{if } x \notin \overline{Z}. \end{cases}$$

where $\overline{Z}$ is the closure of Z.

Proof. Suppose $x \notin \overline{Z}$; then there is a neighborhood U of x such that $U \cap Z = \phi$. But then $(i_*\mathscr{G})(U) = \mathscr{G}(\phi) = 0$, and so $(i_*\mathscr{G})_x = 0$.

Now, suppose $x \in Z$. Then, for any open set $U \ni x$, $(i_*\mathscr{G})(U) = \mathscr{G}(U \cap Z)$. From this, we see that $(i_*\mathscr{G})_x = \lim_{U \ni x}(i_*\mathscr{G})(U) = \lim_{U \ni x} \mathscr{G}(U \cap Z) = \mathscr{G}_x$, since Z has the subspace topology. □

Note on Notation 3. From now on, $\mathscr{C}$ will only denote a concrete, complete, abelian category.

Suppose now that Z is closed. Then we can consider the sheaf $i_*(\mathscr{F}|_Z)$, and the natural morphism $\eta : \mathscr{F} \to i_*(\mathscr{F}|_Z)$ given by the unit for the adjoint pair i_*, i^{-1}. What is the kernel of this morphism? Let's consider instead the natural morphism $\varpi : \mathscr{F} \to I^\sharp I_\sharp(\mathscr{F})$, where $I : \mathrm{Top}(X) \to \mathrm{Top}(Z)$ is such that $I(U) = U \cap Z$. We have, of course, that $\mathrm{Shf}\,\varpi = \eta$. Observe, from the proof of [CT, 2.8], that ϖ_U is simply the natural morphism

$$\varpi_U : \mathscr{F}(U) \longrightarrow \lim_{U \cap Z \subset V \cap Z} \mathscr{F}(V).$$

Therefore, a section $s \in \mathscr{F}(U)$ is in the kernel of this natural morphism iff either $s = 0$, or $U \cap Z = \phi$. In particular, the kernel presheaf corresponds to the assignment

$$U \mapsto \begin{cases} \mathscr{F}(U) & \text{if } U \subset X \setminus Z \\ 0 & \text{otherwise.} \end{cases}$$

This leads to a definition.

ext-zero-closed

Definition 8.4. If $V \subset X$ is open and $\mathscr{G} \in \mathrm{Shf}(V, \mathscr{C})$, then the *extension by zero* of $\mathscr{G}$, is the sheaf $j_!\mathscr{G} \in \mathrm{Shf}(X, \mathscr{C})$ that is the sheafification of the presheaf that corresponds to the assignment $U \mapsto \mathscr{G}(U)$, if $U \subset V$, and $U \mapsto 0$, otherwise, with the restriction maps being inherited from $\mathscr{G}$.

extzero-open-exctseq

Proposition 8.5. *If $Z \subset X$ is closed, and $V = X \setminus Z$, then for $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, we have an exact sequence*

$$0 \to j_!(\mathscr{F}|_V) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0.$$

where $j : V \hookrightarrow X$, and $i : Z \hookrightarrow X$ are the inclusion maps.

Proof. Most of the proof is contained in the discussion right before Definition 8.4. The rest follows from the fact that sheafification is an exact functor. One thing that might need checking still is that the sequence is exact on the right: but this is easy, because the morphism on the right induces surjective maps of stalks. To see that the last statement is true, simply observe that from Proposition 8.3 we have

$$(i_*\mathscr{F}|_Z)_x = \begin{cases} \mathscr{F}_x & \text{if } x \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

□

In fact, extension by zero outside V can be made into a *functor* $j_! : \mathrm{Shf}(V, \mathscr{C}) \to \mathrm{Shf}(X, \mathscr{C})$. It has some natural exactness properties as we will show in the next proposition.

extzero-open-exact

Proposition 8.6. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of $\mathscr{C}$-valued sheaves over V, then we have an exact sequence*

$$0 \to j_!\mathscr{F}' \to j_!\mathscr{F} \to j_!\mathscr{F}'' \to 0$$

Proof. On stalks, the sequence consists of 0s, when $x \notin V$, and is simply

$$0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x \to 0$$

when $x \in V$. This last sequence is exact because the sequence of sheaves was exact. The statement follows, since exactness of maps induced on stalks is a sufficient condition for exactness of maps on sheaves. □

In similar fashion, one can show:

extzero-closed-exact

Proposition 8.7. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of $\mathscr{C}$-valued sheaves over Z, then we have an exact sequence*

$$0 \to i_*\mathscr{F}' \to i_*\mathscr{F} \to i_*\mathscr{F}'' \to 0$$

In fact, the functor $j_! : \mathrm{Shf}(V, \mathscr{C}) \to \mathrm{Shf}(X, \mathscr{C})$ has a more interesting property.

xtzero-open-restrict-adj

Proposition 8.8. *The functor $j_!$ is a left adjoint to the functor i_V^{-1}, where $i_V : V \hookrightarrow X$ is the inclusion map.*

Proof. Given sheaves $\mathscr{F} \in \mathrm{Shf}(V, \mathscr{C})$ and $\mathscr{G} \in \mathrm{Shf}(X, \mathscr{C})$, we want to construct a natural isomorphism $\mathrm{Hom}(j_!\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, \mathscr{G}|_V)$. If $\mathscr{F}'$ is the presheaf that $j_!(\mathscr{F}) = \mathrm{Shf}(\mathscr{F}')$ (see Definition 8.4), then it suffices to construct an isomorphism $\mathrm{Hom}(\mathscr{F}', U\mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, \mathscr{G}|_V)$, since Shf is left adjoint to U. But this is easy, since a morphism ϕ on the left is by necessity 0 on $\mathscr{F}'(W)$, for $W \nsubseteq V$, and so corresponds to a unique morphism on the right. □

8.2. **Sections with local support.**

Definition 8.9. If $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, and $s \in \mathscr{F}(U)$ is a section over an open set U, then the *support* of s, $\mathrm{Supp}(s) = \{x \in U : s_x \neq 0\}$.

It is clear that $\mathrm{Supp}(s)$ is a closed set.

If we now return to the example of $\mathscr{C}_X$, then, for a continuous functions f on an open set U, $\mathrm{Supp}(f)$ is just the usual support: the closure of the set $\{x \in X : f(x) \neq x\}$. To see this, just observe that the complement of the set-theoretic support consists of those points for which there is a neighborhood on which f vanishes identically, which is precisely the collection of points at which the germ of f is 0.

Now, suppose that $U \subset X$ is open, with the inclusion map $j : U \hookrightarrow X$. For a sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, consider again, just as in the case for a closed subset, the natural morphism $\eta : \mathscr{F} \to j_*(\mathscr{F}|_U)$. In this case, by the comment after Definition 8.1, we see that this is just the natural transformation given by the natural morphisms

$$\eta_V : \mathscr{F}(V) \longrightarrow \mathscr{F}(V \cap U).$$

Again, we can ask what the kernel of this morphism is. The answer is simple: $s \in \mathscr{F}(V)$ is in the kernel of this map iff $\mathrm{res}_{V, V \cap U}(s) = 0$. This is equivalent to saying that $\mathrm{Supp}\, s \subset X \setminus U$. This, as always, leads to a definition.

Definition 8.10. If $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$ and $Z \subset X$ is closed, then, for every open set $V \subset X$, $\Gamma_Z(V, \mathscr{F})$ is the subobject of $\Gamma(V, \mathscr{F})$, which consists of all the sections $s \in \Gamma(V, \mathscr{F})$ such that $\operatorname{Supp} s \subset Z$.

Observe with this definition that $\ker \eta_V = \Gamma_Z(V, \mathscr{F})$, where $Z = X \setminus U$. Since the kernel of a sheaf morphism is already a sheaf, the next definition should not be a surprise.

Definition 8.11. If $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, and $Z \subset X$ is closed, then the *subsheaf with supports in* Z, $\mathscr{H}^0_Z(\mathscr{F})$, is the sheaf that assigns to an open set $V \subset X$, the object $\Gamma_Z(V, \mathscr{F})$.

Remark 8.12. That this is a sheaf follows from the remark right before the definition, since we can look at the inclusion map $\mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F}$ as being the kernel of the map $\mathscr{F} \to j_*(\mathscr{F}|_{X \setminus Z})$, where $j : X \setminus Z \hookrightarrow X$ is the inclusion.

local-support-exctseq

Proposition 8.13. *If $Z \subset X$ is a closed subset, $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, and $j : U := X \setminus Z \hookrightarrow X$ is the inclusion map, then we have an exact sequence:*

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U).$$

Proof. Done above. □

There is more that can be said about the object $\Gamma_Z(U, \_\_)$. Suppose we have a morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, and suppose $s \in \mathscr{F}(V)$ is a section; then, since $(\phi(s))_x = \phi_x(s_x)$, for any $x \in X$, we see that $\mathrm{Supp}(\phi(s)) \subset \mathrm{Supp}(s)$. So $\Gamma_Z(U, \_\_)$ is a subfunctor of $\Gamma(U, \_\_)$. We have a statement analogous to Proposition 5.7 for sections with supports in Z.

local-support-leftexact

Proposition 8.14. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, then we have an exact sequence:*

$$0 \to \Gamma_Z(U, \mathscr{F}') \to \Gamma_Z(U, \mathscr{F}) \to \Gamma_Z(U, \mathscr{F}'')$$

Proof. Let $V = X \setminus Z$. Then, by Proposition 8.13 and the discussion above, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma_Z(U, \mathscr{F}') & \longrightarrow & \Gamma(U, \mathscr{F}') & \longrightarrow & \Gamma(U, j_*(\mathscr{F}'|_V)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma_Z(U, \mathscr{F}) & \longrightarrow & \Gamma(U, \mathscr{F}) & \longrightarrow & \Gamma(U, j_*(\mathscr{F}|_V)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma_Z(U, \mathscr{F}'') & \longrightarrow & \Gamma(U, \mathscr{F}'') & \longrightarrow & \Gamma(U, j_*(\mathscr{F}''|_V))
\end{array}$$

By Proposition 5.7, the two columns in the right and the middle are exact. From this, it follows easily that the column on the left must also be exact. □

9. Sheaf Hom

If we have two sheaves $\mathscr{M}, \mathscr{N} \in \mathrm{Shf}(X, \mathrm{Ab})$, then we can consider the presheaf $\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N}) \in \mathrm{Pre}(X, \mathrm{Ab})$ that assigns to each open set U, the abelian group $\mathrm{Hom}(\mathscr{M}|_U, \mathscr{N}|_U)$, with the obvious restriction maps. That is, if $\phi \in \mathrm{Hom}(\mathscr{M}|_U, \mathscr{N}|_U)$, then $\mathrm{res}_{U,V}(\phi)$ is such that for any open set $W \subset V \subset U$, $\mathrm{res}_{U,V}(\phi)_W = \phi_W$.

This is in fact a sheaf, as we show in the following proposition.

Proposition 9.1. *With the notation as above,* $\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N})$ *is in fact a sheaf.*

Proof. We need to show that for any open set U and any weak covering sieve $\mathcal{V} = \{V_i\}$ of U, the natural map $\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N})(U) \to \mathcal{V}(\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N}))$ is an isomorphism. Suppose we have a sequence $(\phi_i) \in \mathcal{V}(\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N}))$. Note that we then have, for any open set $V \subset U$, a natural map $\mathscr{M}(V) \to (\mathcal{V} \cap V)(\mathscr{N})$ given by $s \mapsto ((\phi_i)_{V_i \cap V}(\mathrm{res}_{V, V\cap V_i}(s))) = (t_i)$. To show that this sequence is indeed coherent, all we have to show is that $\mathrm{res}_{V_i\cap V, V_i \cap V_j \cap V}(t_i) = \mathrm{res}_{V_j \cap V, V_i \cap V_j \cap V}(t_j)$. Suppose $V_i \cap V_j = V_k$; then the left hand side equals $(\phi_i)_{V_k \cap V}(\mathrm{res}_{V, V\cap V_k}(s)) = (\phi_k)_{V_k \cap V}(\mathrm{res}_{V, V \cap V_k}(s))$, since $\mathrm{res}_{V_i, V_k}(\phi_i) = \phi_k$. Clearly, the right hand side will also equal the same thing.

Now, all we have to do is to check that for open sets $W \subset V \subset U$, the following diagram is commutative:

$$\begin{array}{ccccc} \mathscr{M}(V) & \longrightarrow & (\mathcal{V} \cap V)(\mathscr{N}) & \xrightarrow{\cong} & \mathscr{N}(V) \\ {\scriptstyle \mathrm{res}_{V,W}} \downarrow & & {\scriptstyle \mathrm{res}_{V,W}} \downarrow & & {\scriptstyle \mathrm{res}_{V,W}} \downarrow \\ \mathscr{M}(W) & \longrightarrow & (\mathcal{V} \cap W)(\mathscr{N}) & \xrightarrow{\cong} & \mathscr{N}(W) \end{array}$$

The right hand square commutes by Remark 2.7, and the left hand square commutes, well, more or less, by definition. So if we define for every open set $V \subset U$, ϕ_V to be the composition $\mathscr{M}(V) \to (\mathcal{V} \cap V)(\mathscr{N}) \to \mathscr{N}(V)$, then we see immediately that this gives us an element of $\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N})(U)$ which restricts to ϕ_i on each V_i. To verify the last statement, just observe that for any $V \subset V_i$, and $s \in \mathscr{M}(V)$, we have $\phi_V(s) = \mathrm{res}_{V, V\cap V_i}(\phi(s)) = (\phi_i)_{V_i}(s)$.

Conversely, if we had any other element $\widetilde{\phi} \in \underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N})(U)$ restricting to each of the ϕ_i, then, for any $V \subset U$, the composite map $\mathscr{M}(V) \to \mathscr{N}(V) \to (\mathcal{V} \cap V)(\mathscr{N})$ will be the same as the map defined earlier from $\mathscr{M}(V) \to (\mathcal{V} \cap V)(\mathscr{N})$, which of course means that $\phi_V = \widetilde{\phi}_V$.

Thus, we've built a bijection $\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N})(U) \to \mathcal{V}(\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N}))$. To show that this is an isomorphism of abelian groups, all we have to do is to show that if we have two elements $(\phi_i), (\psi_i) \in \mathcal{V}(\underline{\mathrm{Hom}}(\mathscr{M}, \mathscr{N}))$, then the natural map corresponding to the sum $(\phi_i + \psi_i)$ is the same as the sum of the natural maps $\mathscr{M}(V) \to (\mathcal{V} \cap V)(N)$ corresponding to ϕ_i and ψ_i. But this is obvious from the definitions of these natural maps, and so we're done. □

Remark 9.2. Observe that we did not use the fact that $\mathscr{M}$ and $\mathscr{N}$ were sheaves of abelian groups right till the last paragraph. So it's clear that we can do the same thing for any two sheaves over the same concrete category.

Definition 9.3. Suppose X is a topological space, and $\mathscr{M}, \mathscr{N} \in \mathrm{Shf}(X, \mathrm{Ab})$, then the *sheaf of local homomorphisms* or, more simply, the *sheaf hom* between

$\mathscr{M}$ and $\mathscr{N}$ is the sheaf $\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})$. If $\mathscr{M}=\mathscr{N}$, then we write $\underline{\mathrm{End}}(\mathscr{M})$ for $\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{M})$. Observe that $\underline{\mathrm{End}}(\mathscr{M})$ is naturally a sheaf of rings (it's actually a ring in the category of sheaves!).

Remark 9.4. If we're given morphisms $\phi:\mathscr{M}\to\mathscr{M}'$, $\psi:\mathscr{N}\to\mathscr{N}'$ of sheaves of abelian groups, then we have a natural morphism $\underline{\mathrm{Hom}}(\phi,\psi):\underline{\mathrm{Hom}}(\mathscr{M}',\mathscr{N})\to\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N}')$, defined on $\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})(U)$ by $\alpha\mapsto(\psi|_U)\circ\alpha\circ(\phi|_U)$. In fact, this gives us a functor $\underline{\mathrm{Hom}}(\_\_,\_\_):\mathrm{Shf}(X,\mathrm{Ab})^{op}\times\mathrm{Shf}(X,\mathrm{Ab})\to\mathrm{Shf}(X,\mathrm{Ab})$, just as we're used to for regular Hom.

Now that we have a functor in $\underline{\mathrm{Hom}}$, we'd like to investigate its exactness properties, which follow quite trivially from the exactness properties of regular Hom on abelian categories.

Proposition 9.5. *The functor* $\underline{\mathrm{Hom}}$ *is left exact in both variables.*

Proof. This follows immediately from the fact that this is true for the functor $\mathrm{Hom}(\_\_,\_\_):\mathrm{Shf}(X,\mathscr{C})\to\mathrm{Ab}$, and the fact that for a sequence of sheaves $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''$ is exact iff the sequence $0\to\mathscr{F}'(U)\to\mathscr{F}(U)\to\mathscr{F}''(U)$ is exact for every open set $U\subset X$. □

There's also a nice connection between $\underline{\mathrm{Hom}}$ and the direct image functor.

shom-dirimg-commute

Proposition 9.6. *If $f:X\to Y$ is a continuous map, and $\mathscr{M},\mathscr{N}\in\mathrm{Shf}(X,\mathrm{Ab})$ and $\mathscr{M}',\mathscr{N}'\in\mathrm{Shf}(Y,\mathrm{Ab})$, then $f_*\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})=\underline{\mathrm{Hom}}(f_*\mathscr{M},f_*\mathscr{N})$.*

Proof. Note that for an open set $U\subset Y$, we have

$$\begin{aligned}\underline{\mathrm{Hom}}(f_*\mathscr{M},f_*\mathscr{N})(U)&=\mathrm{Hom}((f_*\mathscr{M})|_U,(f_*\mathscr{N})|_U)\\&=\mathrm{Hom}(\mathscr{M}|_{f^{-1}(U)},\mathscr{N}|_{f^{-1}(U)})\\&=\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N})(f^{-1}(U))\\&=(f_*\underline{\mathrm{Hom}}(\mathscr{M},\mathscr{N}))(U)\end{aligned}$$

All the equalities here, except for the second, are quite clear. For the second, observe that for $V\subset U$, $((f_*M)|_U)(V)=M(f^{-1}(V))=(M|_{f^{-1}(U)})(f^{-1}(V))$. This gives us a natural map

$$\mathrm{Hom}(\mathscr{M}|_{f^{-1}(U)},\mathscr{N}|_{f^{-1}(U)})\to\mathrm{Hom}((f_*\mathscr{M})|_U,(f_*\mathscr{N})|_U)$$

On the other hand, if we're given an element in $\mathrm{Hom}((f_*\mathscr{M})|_U,(f_*\mathscr{N})|_U)$, then we get maps $\mathscr{M}(f^{-1}(V))\to\mathscr{N}(f^{-1}(V)$, for all open subsets $V\subset U$, satisfying the commutativity criterion expressed in this diagram for inclusions $V'\subset V\subset U$:

$$\begin{array}{ccc}\mathscr{M}(f^{-1}(V)) & \longrightarrow & \mathscr{N}(f^{-1}(V)\\ {\scriptstyle\mathrm{res}_{V,V'}}\downarrow & & \downarrow{\scriptstyle\mathrm{res}_{V,V'}}\\ \mathscr{M}(f^{-1}(V')) & \longrightarrow & \mathscr{N}(f^{-1}(V')\end{array}$$

The collection $\mathcal{V} = \{f^{-1}(V) : V \subset U\}$ forms a weak covering sieve for $f^{-1}(U)$, and so we have the following diagram, for open subsets $W' \subset W \subset f^{-1}(U)$

$$\begin{array}{ccccccc}
\mathscr{M}(W) & \longrightarrow & (\mathcal{V} \cap W)(\mathscr{M}) & \longrightarrow & (\mathcal{V} \cap W)(\mathscr{N}) & \longrightarrow & \mathscr{N}(W) \\
\downarrow \mathrm{res}_{W,W'} & & \downarrow \mathrm{res}_{W,W'} & & \downarrow \mathrm{res}_{W,W'} & & \downarrow \mathrm{res}_{W,W'} \\
\mathscr{M}(W') & \longrightarrow & (\mathcal{V} \cap W')(\mathscr{M}) & \longrightarrow & (\mathcal{V} \cap W')(\mathscr{N}) & \longrightarrow & \mathscr{N}(W')
\end{array}$$

So we get a map $\mathscr{M}|_{f^{-1}(U)} \to \mathscr{N}|_{f^{-1}(U)}$ in a natural fashion. It is easily checked that the map

$$\mathrm{Hom}((f_*\mathscr{M})|_U, (f_*\mathscr{N})|_U) \to \mathrm{Hom}(\mathscr{M}|_{f^{-1}(U)}, \mathscr{N}|_{f^{-1}(U)})$$

that we get from this, is an inverse to the natural map defined earlier in the other direction. □

10. Examples

10.1. **Locally constant sheaves.** The simplest presheaf that I can think of is the constant presheaf.

Definition 10.1. If $A \in \mathrm{Ob}\,\mathscr{C}$, then the *constant presheaf* $\mathrm{CShf}_A \in \mathrm{Pre}(X, \mathscr{C})$ is the presheaf that assigns to each open set $U \subset X$ the object A, with the exception, deferential to our convention, that $\mathrm{CShf}_A(\phi) = 0$, the final object in $\mathscr{C}$. For $U \subset V$, the restriction map is

$$\mathrm{res}_{V,U} = \begin{cases} 1_A & \text{if } U \neq \phi, \\ 0 & \text{otherwise.} \end{cases}$$

Note on Notation 4. Remember that our $\mathscr{C}$ is always concrete, complete and abelian!

Definition 10.2. If $A \in \mathrm{Ob}\,\mathscr{C}$, then the *locally constant sheaf* $\mathrm{LCShf}_A \in \mathrm{Shf}(X, \mathscr{C})$ is the sheafification $\mathrm{Shf}\,\mathrm{CShf}_A$ of the constant presheaf CShf_A

What does the sheafification look like? If $\mathscr{C}$ is a category of modules over a ring R, then this will be easy to see from the bundle of stalks point of view. The underlying topological space of $\mathrm{Spc}(\mathrm{CShf}_A)$ is just $\coprod_{x \in X} A$, and the open sets of $\mathrm{Spc}(\mathrm{CShf}_A)$ are simply disjoint unions of the form $\coprod_{x \in U}\{a\} = (U, a)$, for open subsets $U \subset X$ and elements $a \in A$. Suppose that we had a continuous bundle section $t : U \to \mathrm{Spc}(\mathrm{CShf}_A)$. Then, for $x \in U$, we have a neighborhood $V \ni x$ contained in U, such that $t(V) = (V, a)$. This gives us a correspondence between bundle sections and continuous maps $U \to A$, where A is equipped with the discrete topology. So we have:

Proposition 10.3. *If $\mathscr{C}$ is the category of modules over a ring R, then* $\mathrm{LCShf}_A \in \mathrm{Shf}(X, \mathscr{C})$ *is the sheaf that takes an open set U to the set of continuous functions $U \to A$, with A equipped with the discrete topology.*

Proof. Contained in the discussion above. □

Note on Notation 5. We will usually denote LCShf_A by $\underline{A}$.

10.2. **Skyscraper sheaves.** Another simple and useful example of sheaves is given by the skyscraper sheaves.

Definition 10.4. If $A \in \mathrm{Ob}\,\mathscr{C}$, and $x \in X$, then the *skyscraper sheaf* $\mathrm{Sky}_x(A)$ is the direct image $i_*(\underline{A})$ of the locally constant (and hence constant) sheaf $\underline{A} \in \mathrm{Shf}(\{x\}, \mathscr{C})$, where $i : \{x\} \hookrightarrow X$ is the inclusion map.

The stalks of $\mathrm{Sky}_x(A)$ can be described as follows:

$$\mathrm{Sky}_x(A)_y = \begin{cases} A & \text{if } y \in \overline{\{x\}} \\ 0 & \text{otherwise.} \end{cases}$$

Most of this follows from Proposition 8.3. The only thing we have to show is that for $y \in \overline{\{x\}}$, we have $\mathrm{Sky}_x(A)_y = A$. But this is easy, since for every neighborhood $U \ni y$, $x \in U$, and so $\mathrm{Sky}_x(A)(U) = A$.

10.3. **Sheaf of holomorphic functions.** For any family of functions on a space, whose defining properties are local, in the sense that they're all happening in open neighborhoods of points, we can define the sheaf of such functions on the space in the obvious fashion, by assigning to an open set U all the functions which are defined on U. In this way, we can get the sheaf of continuous functions on a topological space, or the sheaf of smooth functions on a smooth manifold, or the sheaf of regular functions on an algebraic variety.

We can define the sheaf $\mathscr{H}$ of holomorphic functions on $\mathbb{C}$, and the sheaf $\mathscr{H}^*$ of nowhere zero holomorphic functions on $\mathbb{C}$, in similar fashion. Observe that for every open set $U \subset \mathbb{C}$, we have the exponential map

$$\begin{aligned} \exp_U : \mathscr{H}(U) &\to \mathscr{H}^*(U) \\ f &\mapsto e^f \end{aligned}$$

It's easy to see that the kernel of exp is the sheaf $\underline{\mathbb{Z}}$. It's also clear that the map $\exp_U$ is not surjective, since, for example, if U is not simply connected and $0 \notin U$, then $\log z$ is not everywhere defined, and so the function z will not be in the image of $\exp_U$. But note that the map exp induces surjections on stalks, since for every $x \in \mathbb{C}$, there is a simply connected neighborhood U_x of x, and so the map $\exp_{U_x}$ will be surjective. Therefore, the sequence

$$0 \to \underline{\mathbb{Z}} \longrightarrow \mathscr{H} \xrightarrow{\exp} \mathscr{H}^* \to 0$$

is exact, although, the sequence

$$0 \to \Gamma(\mathbb{C}, \underline{\mathbb{Z}}) \to \Gamma(\mathbb{C}, \mathscr{H}) \to \Gamma(\mathbb{C}, \mathscr{H}^*)$$

is only left exact.

Moreover, if $\mathscr{G}$ is the cokernel presheaf of exp, then we see that $\mathrm{Shf}\,\mathscr{G} = 0$, even though $\mathscr{G} \neq 0$. Thus, we also have an example of the unfaithfulness of Shf.

10.4. **Sheaves on varieties.** This example and the next are based on Problem II.1.21 in Hartshorne. In what follows, X will denote an algebraic variety over an algebraically closed field, k, and $\mathscr{O}_X$ will be its sheaf of regular functions.

Let Y be a subvariety of X, with $i : Y \hookrightarrow X$ being the inclusion map. We have a natural map $\mathscr{O}_X|_Y \to \mathscr{O}_Y$. Through the adjunction between restriction and i_*, we have a natural map $\mathscr{O}_X \to i_*\mathscr{O}_Y$. This is surjective on stalks: for $y \in Y$, we can find an affine open U around y, and an ideal $I_y \subset \Gamma(U, \mathscr{O}_X)$ such that the map on stalks is simply the quotient map $\mathscr{O}_X(U)_{\mathfrak{m}_y} \to (\mathscr{O}_X(U)/I_y)_{\mathfrak{m}_y}$, where $\mathfrak{m}_y$ is the maximal ideal in $\Gamma(U, \mathscr{O}_X)$ associated to y. If $y \notin Y$, then by Proposition 8.3, $(i_* \mathscr{O}_Y)_y = 0$, and so the map is trivially surjective on stalks. So we see that the natural map $\mathscr{O}_X \to i_* \mathscr{O}_Y$ is surjective.

What is the kernel of this map? We should consider the map $\mathscr{O}_X(U) \to i_* \mathscr{O}_Y(U) = \mathscr{O}_Y(U \cap Y)$, for an open set $U \subset X$. The kernel consists precisely of the regular functions on U that vanish on $U \cap Y$. These functions clearly form an ideal $\mathscr{I}_Y(U)$ of $\mathscr{O}_X(U)$. This leads to a definition:

Definition 10.5. For a subvariety $Y \subset X$, the *ideal of sheaves for* Y is the subsheaf $\mathscr{I}_Y \hookrightarrow \mathscr{O}_X$ whose sections over an open set U consist of the regular functions that vanish on $U \cap Y$

Remark 10.6. This is indeed a sheaf, since it is the kernel of a morphism of sheaves.

Let's collect what we have in the following proposition.

Proposition 10.7. *With all the notation as in the preceding discussion, we have the following short exact sequence*

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to i_* \mathscr{O}_Y \to 0.$$

Proof. Already done. □

Now, suppose Y is the one point set $\{x\}$. Then $\mathscr{O}_Y$ is the constant sheaf $\underline{k}$, and so $i_* \mathscr{O}_Y$ is simply the skyscraper sheaf $\mathrm{Sky}_x(k) \in \mathrm{Shf}(X, k\text{-mod})$. If, instead, Y were the two point set $\{x, y\}$, then it's easy to see that $i_* \mathscr{O}_Y$ should be the direct sum $\mathrm{Sky}_x(k) \oplus \mathrm{Sky}_y(k)$. So we have a short exact sequence:

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \mathrm{Sky}_x(k) \oplus \mathrm{Sky}_y(k) \to 0$$

If now, we specialize to $X = \mathbb{P}^1$, then $\Gamma(X, \mathscr{O}_X) = k$, while $\Gamma(X, \mathrm{Sky}_x(k) \oplus \mathrm{Sky}_y(k)) = k \oplus k$. So the corresponding sequence for the global section functor is not right exact, giving another example of the failure of its exactness.

10.5. **The First Cousin Problem.** Suppose now that $X \subset \mathbb{P}^n$ is an irreducible, projective curve. Let $S(X)$ be its homogeneous co-ordinate ring, and let $K = S(X)_{((0))}$ be its function field. Then, for any point $x \in X$, we have a natural map $K \to K/\mathscr{O}_x = M_x$, where $\mathscr{O}_x$ is the local ring $S(X)_{(\mathfrak{p}_x)}$, the degree 0 component of $S(X)$ localized at the prime corresponding to x. What this map basically does is that, if we're given the formal Laurent expansion of an element $f_i \in K$ around x, it picks out the principal part at x. If we put together all these maps, we get a map $K \to \oplus_{x \in X} M_x$. The First Cousin Problem wonders if this map is surjective. That is, if we're given a bunch of purported principal parts at each $x \in X$, can we put them together to find a rational function on X, which has precisely those principal parts at each x?

This can be rephrased more fruitfully using the language of sheaves. The function field over every open set U is just the function field K over X. So, $\mathscr{K} = \underline{K} \in \mathrm{Shf}(X, \mathrm{Ring})$ defines the sheaf of function fields on X. Now, consider the quotient sheaf $\mathscr{K}/\mathscr{O}_X$: if we look at the description of a section of a quotient sheaf in 4.9, we see that a section $s \in \Gamma(U, \mathscr{K}/\mathscr{O}_X)$ is described by an open covering $\{U_i : 1 \leq i \leq n\}$ (finite, in this case, since X is quasicompact), and rational functions $f_i \in \mathscr{K}(U_i)$ such that

$$\mathrm{res}_{U_i, U_i \cap U_j}(f_i) - \mathrm{res}_{U_j, U_i \cap U_j}(f_j) \in \Gamma(U_i \cap U_j, \mathscr{O}_X).$$

Since X is a curve, any rational function can only have finitely many poles, that is, points at which it's not regular. So, by making the U_i smaller, if necessary, we can assume that each U_i contains at most one pole a_i, at which f_i has the principal part G_i. Now, this means that $f_i - G_i \in \Gamma(\mathscr{O}_x, U_i)$, and so as sections of the quotient sheaf the collection (f_i, U_i) equals the collection (G_i, U_i). In other words, the quotient sheaf is $\mathscr{G} = \oplus_{x \in X} \mathrm{Sky}_x(M_x) \in \mathrm{Shf}(X, \mathrm{Ab})$. So we have a short exact sequence of sheaves

$$0 \to \mathscr{O}_X \to \mathscr{H} \to \mathscr{G} \to 0.$$

The First Cousin Problem can now be rephrased in the following fashion: When is the following sequence of global sections

$$0 \to \Gamma(X, \mathscr{O}_X) \to \Gamma(X, \mathscr{H}) \to \Gamma(X, \mathscr{G}) \to 0.$$

exact?

It's evident now that we've reduced the question to a cohomological one, since we're measuring the failure of right exactness of the global section functor. In particular, once we've defined sheaf cohomology, we'll see that the exactness of the sequence above is equivalent to the vanishing of the first cohomology group $H^1(X, \mathscr{O}_X)$, which should make sense once one realizes that this cohomology is nothing but the right derived functor of the global section functor.

Since we don't know anything about cohomology at this point, let's solve this problem by hand for the case $X = \mathbb{P}^1$. So we have a finite collection of points $\{x_i\} \in X$, and, for each i, we're given an element $f_i \in M_{x_i}$. By a change of coordinates, if necessary, we can assume that $x_i \in U_1 = \{z_1 \neq 0\}$, for all i. Putting $z = z_2/z_1$, we see that for each i, $M_{x_i} = k(z)/k[z]_{\mathfrak{m}_{x_i}}$, where $\mathfrak{m}_{x_i} \subset k[z]$ is the ideal associated to x_i. In this case, it's easy to see that if we take $f = \sum_i \widetilde{f}_i$, where the $\widetilde{f}_i$ are representatives of f_i in $k(z)$, then that maps to $\sum_i f_i \in \Gamma(X, \mathscr{G})$. So the sequence of global sections is indeed exact for $X = \mathbb{P}^1$.

If one believes the discussion before this simple calculation, then this implies that $H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 0$. This, and more, is in fact true. We'll get there much later.

11. Constructing Sheaves

gluing-sheaves

11.1. **Gluing Sheaves.** So far, we've sort of taken the existence of sheaves for granted. But in real life it seldom happens that someone comes along and gives us a nice functor defined on the *whole* of $\mathrm{Top}(X)$. More often than not, we're given a bunch of sheaves defined on an open cover (for example, on a basis for the topology), with some compatibility conditions. Then we can piece them all together to form a sheaf on the entire space. Let's formalize this now.

Proposition 11.1. *Suppose $\mathcal{V} = \{V_i\}$ is an open cover for X, and suppose that, for each i, we have a sheaf $\mathscr{F}_i \in \mathrm{Shf}(V_i, \mathscr{C})$, and, for every pair i, j, we have isomorphisms $\phi_{ij} : \mathscr{F}_i|_{V_i \cap V_j} \to \mathscr{F}_j|_{V_i \cap V_j}$ such that two conditions hold:*

(1) $\phi_{ii} = 1_{\mathscr{F}_i}$.
(2) *For each triple i, j, k, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $V_i \cap V_j \cap V_k$.*

Then, there is, upto isomorphism, a unique sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$, and isomorphisms $\psi_i : \mathscr{F}|_{V_i} \to \mathscr{F}_i$, such that for each pair i, j, we have $\psi_j = \phi_{ij} \circ \psi_i$.

Proof. If we had a sheaf $\mathscr{F}$ satisfying these conditions, then consider the weak covering sieve $\mathcal{W}$ of X generated by $\mathcal{V}$. We should have, for every $U \subset X$ open,$\mathscr{F}(U) = (\mathcal{W} \cap U)(\mathscr{F})$. But on the right we have a limit of objects of the type $(\psi_i)^{-1}_{U\cap W_i}(\mathscr{F}_i(U\cap W_i))$, for some open sets $W_i \cap V_i$, which is determined upto isomorphism by the $\mathscr{F}_i$. Hence such a sheaf $\mathscr{F}$ must be unique.

We now build $\mathscr{F}$. This construction is a close cousin of the inverse limit construction. For $V \subset X$ open, consider the following object:

$$\mathscr{F}(V) = \{(s_i) \in \prod_i \mathscr{F}_i(V \cap V_i) : (\phi_{ij})_{V\cap V_i\cap V_j}(\mathrm{res}_{V\cap V_i, V\cap V_i\cap V_j}(s_i)) \\ = \mathrm{res}_{V\cap V_j, V\cap V_i \cap V_j}(s_j), \forall i,j\}.$$

For $U \subset V$, we have the obvious restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ working on each co-ordinate. What about the isomorphisms $\psi_i : \mathscr{F}|_{V_i} \to \mathscr{F}_i$? We have a ready candidate for that: if $W \subset V_i$ consider for each i, the projection $(\psi_i)_W : \mathscr{F}(W) \to \mathscr{F}_i(W)$. The map $(\psi_i)_W$ is immediately seen to be injective. For surjectivity, suppose $s \in \mathscr{F}_i(W)$, and consider the sequence

$$(\phi_{ij}(\mathrm{res}\, W, W \cap V_j(s))) \in \prod_i \mathscr{F}_j(W \cap V_j)$$

We see that this sequence is in fact in $\mathscr{F}(V)$, since we have the compatibility condition $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$. Hence ψ_i is also surjective. By construction the isomorphisms ψ_i satisfy the condition $\psi_j = \phi_{ij} \circ \psi_i$.

Let's check that $\mathscr{F}$ is a sheaf. Suppose $U \subset X$ is open $\mathcal{W} = \{W_\alpha\}$ is a weak covering sieve for U. We want to show that the natural map $\mathscr{F}(U) \to \mathcal{W}(\mathscr{F})$ is an isomorphism. First, let's prove injectivity: if $(s_i) \in \mathscr{F}(U)$ is such that $\mathrm{res}_{U,W_\alpha}((s_i)) = 0$, for all α, then $\mathrm{res}_{U\cap V_i, W_\alpha\cap V_i}(s_i) = 0$, for all i, α. Since the $\mathscr{F}_i$ are separated, we see that $s_i = 0$, and so $(s_i) = 0$.

Now, we'll show surjectivity. An element of $\mathcal{W}(\mathscr{F})$ consists of sequences $(s_{\alpha,i})$, such that $s_{\alpha,i} \in \mathscr{F}_i(W_\alpha \cap V_i)$, with two levels of coherence: On one level, we have that if $W_\gamma = W_\alpha \cap W_\beta$, then

$$\mathrm{res}_{W_\alpha\cap V_i, W_\gamma \cap V_i}(s_{\alpha,i}) = s_{\gamma,i} = \mathrm{res}_{W_\beta \cap V_i, W_\gamma\cap V_i}(s_{\beta,i})$$

This tells us that $(s_{\alpha,i})$ for fixed i, is an element of $(\mathcal{W}\cap V_i)(\mathscr{F}_i)$. Since $\mathscr{F}_i$ is a sheaf, there is an element $s_i \in \mathscr{F}_i(U \cap V_i)$ that maps to $(s_{\alpha,i}) \in (\mathcal{W}\cap V_i)(\mathscr{F}_i)$

On another level, we have

$$(\phi_{ij})_{W_\alpha\cap V_i\cap V_j}(\mathrm{res}_{W_\alpha\cap V_i, W_\alpha\cap V_i\cap V_j}(s_{\alpha,i})) = \mathrm{res}_{W_\alpha\cap V_j, W_\alpha \cap V_i\cap V_j}(s_{\alpha,j})$$

This implies the following:

$$(\phi_{ij})_{W_\alpha\cap V_i\cap V_j}(\mathrm{res}_{U\cap V_i, W_\alpha\cap V_i\cap V_j}(s_i)) = \mathrm{res}_{U\cap V_j, W_\alpha\cap V_i\cap V_j}(s_j).$$

Since $\{W_\alpha \cap V_i \cap V_j\}$ is an open cover for $U \cap V_i \cap V_j$, we see that

$$(\phi_{ij})_{U\cap V_i\cap V_j}(\mathrm{res}_{U\cap V_i, U\cap V_i\cap V_j}(s_i)) = \mathrm{res}_{U\cap V_j, U\cap V_i\cap V_j}(s_j).$$

which implies that $(s_i) \in \mathscr{F}(U)$. It is clear that (s_i) maps to $(s_{\alpha,i})$ under the natural map, which shows surjectivity, thus finishing our proof. □

sheaves-on-open-base

11.2. **Sheaves on an open base.** Here's a situation that'll come up when we're constructing the Spec of a ring as an affine scheme.

Definition 11.2. Given an open base $\mathcal{V} = \{U_i\}$ for the topology on X, we can consider the subcategory (or sub-lattice) $\mathrm{Top}_{\mathcal{V}}(X)$ of $\mathrm{Top}(X)$ that consists of the open sets in $\mathcal{V}$. A *presheaf on the open base $\mathcal{V}$ with values in $\mathscr{C}$* is simply a functor

$$\mathscr{F} : \mathrm{Top}_V(X)^{op} \to \mathscr{C}.$$

This gives a *category* $\mathrm{Pre}(\mathcal{V}, \mathscr{C})$ of presheaves on the base $\mathcal{V}$.

Now, suppose our category $\mathscr{C}$ is complete; then we can take inverse limits. In particular, given an $\mathscr{F} \in \mathrm{Pre}(\mathcal{V}, \mathscr{C})$ and an open set $V \subset X$, we can define

$$\mathscr{F}'(V) = \varprojlim_{U_i \subset V} \mathscr{F}(U_i).$$

If we have $W \subset V$, then we have a natural map $\mathrm{res}_{V,W} : \mathscr{F}'(V) \to \mathscr{F}'(W)$, since every U_i that appears in the inverse system defining $\mathscr{F}'(W)$ also appears in the inverse system $\mathscr{F}'(V)$. Moreover, it's clear that if we had $W' \subset W \subset V$, then $\mathrm{res}_{W,W'} \circ \mathrm{res}_{V,W} = \mathrm{res}_{V,W}$. So $\mathscr{F}'$ defines a presheaf on X with values in $\mathscr{C}$, and we see that $\mathscr{F}'(U_i)$ is canonically isomorphic to $\mathscr{F}(U_i)$.

So we see that to each presheaf on a base, we can associate an honest presheaf on X. Now, we can ask when is this associated presheaf actually a sheaf? To make our life easier, we'll assume that our base is 'nice', in the sense that it contains all intersections of its elements; in other words, we'll assume that $\mathcal{V}$ is a weak covering sieve.

Now, if $\mathscr{F}'$ were a sheaf, then, if we're given any $U_i \in \mathcal{V}$ and any weak covering sieve $\mathcal{W} \subset \mathcal{V}$ of U_i, we should have

condition-for-sheaf

$$\mathscr{F}(U_i) \cong \mathcal{W}(\mathscr{F}'). \tag{11.1}$$

In other words, if we're given any U_i, and any weak covering sieve $\mathcal{W}$ of U_i, whose elements are also members of $\mathcal{V}$, then for any coherent sequence in $\prod_{U_j \in \mathcal{W}} \mathscr{F}(U_j)$, we should be able to find a unique element in $\mathscr{F}(U_i)$ restricting to this coherent sequence. As it happens, this condition is sufficient, and we'll show that now.