# Homological Algebra

Keerthi Madapusi

# Contents

CHAPTER 1

# Chain Complexes

chap:chain

## 1. Basics

We use the cohomological representation of chain complexes. To translate to the homological picture, just lower all indices and change them to their negatives. We'll also use standard homological lemmas like the Snake Lemma or the 5-lemma without comment, because by now we don't give a fuck for their proofs.

DEFINITION 1.1.1. A *chain complex* $C^\bullet$ in an abelian category $\mathscr{C}$ is a $\mathbb{Z}$-indexed set $\{(C^n, d_C^n) : C^n \in \operatorname{Ob}\mathscr{C}, d_C^n \in \mathscr{C}(C^n, C^{n+1})\}$, satisfying the condition that $d_C^n \circ d_C^{n-1} = 0$.

We usually picture a chain complex as a chain of morphisms

$$C^\bullet : \ldots \xrightarrow{d_C^n} C^{n+1} \xrightarrow{d_C^{n+1}} C^{n+2} \to \ldots$$

The morphisms $d_C^n$ are called the *boundary maps* or the *differentials*.

NOTE ON NOTATION 1. We will usually use $d^n$ as a generic signifier for boundary map for any chain complex. We might also omit the subscript $n$ sometimes. Things should be clear from the context.

DEFINITION 1.1.2. A *morphism* or a *chain map* between two chain complexes $f^\bullet : C^\bullet \to D^\bullet$ in $\mathscr{C}$ is a collection of morphisms $\{f^n : C^n \to D^n\}$ such that, for every $n \in \mathbb{Z}$, $f^n \circ d_D^{n-1} = d_C^{n-1} \circ f^{n-1}$. In other words the following diagram commutes for every $n$:

$$\begin{tikzcd}
\ldots \arrow[r, "d^{n-2}"] & C^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & C^n \arrow[r, "d^n"] \arrow[d, "f^n"'] & \ldots \\
\ldots \arrow[r, "d^{n-2}"] & C^{n-1} \arrow[r, "d^{n-1}"] & C^n \arrow[r, "d^n"] & \ldots
\end{tikzcd}$$

It's easy to see that the composition of two chain maps is also a chain map. This gives us a *category* $\operatorname{Ch}\mathscr{C}$ of chain complexes and chain maps in $\mathscr{C}$. It's easy to see that this is an additive category. For example, the direct sum of two chain complexes $C^\bullet$ and $D^\bullet$ is just $\{C^n \oplus D^n, \begin{pmatrix} d_C^n & 0 \\ 0 & d_D^n \end{pmatrix}\}$.

chain-dual-isomorphic

REMARK 1.1.3. Observe that $\operatorname{Ch}\mathscr{C}$ is isomorphic to $\operatorname{Ch}\mathscr{C}^{op}$. Indeed, we can define a functor $F : \operatorname{Ch}\mathscr{C} \to \operatorname{Ch}\mathscr{C}^{op}$ that takes a complex and reverses all arrows and reindexes the complex by setting $F(C)^n = C^{-n}$. We see immediately that this gives us an isomorphism of categories

Now, suppose we had a chain map $f^\bullet : C^\bullet \to D^\bullet$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
K^{n-1} \arrow[r, dotted, "\widetilde{d}^{n-1}"] \arrow[d, "\ker f^{n-1}"'] & K^n \arrow[r, dotted, "\widetilde{d}^{n}"] \arrow[d, "\ker f^{n}"'] & K^{n+1} \arrow[d, "\ker f^{n+1}"'] \\
C^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & C^n \arrow[r, "d^{n}"] \arrow[d, "f^{n}"'] & C^{n+1} \arrow[d, "f^{n+1}"'] \\
D^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "\operatorname{coker} f^{n-1}"'] & D^n \arrow[r, "d^{n}"] \arrow[d, "\operatorname{coker} f^{n}"'] & D^{n+1} \arrow[d, "\operatorname{coker} f^{n+1}"'] \\
L^{n-1} \arrow[r, dotted, "\widetilde{d}^{n-1}"] \arrow[d] & L^n \arrow[r, dotted, "\widetilde{d}^{n}"] \arrow[d] & L^{n+1} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

where we get the dotted maps among the kernels and the cokernels because of their universal properties. For example $d^n_C \circ d^{n-1}_C \circ \ker f^{n-1} = 0$ and so $d^n_C \circ \ker f^{n-1} : K^{n-1} \to C^n$ factors through $\ker f^n$ via the dotted map $\widetilde{d}^{n-1}$. We see also from this that

$$\ker f^{n+1} \circ \widetilde{d}^n \circ \widetilde{d}^{n-1} = d^n_C \circ \ker f^n \circ \widetilde{d}^{n-1} = d^n_C \circ d^{n-1}_C \circ \ker f^{n-1} = 0.$$

Since $\ker f^{n+1}$ is monic, we see that $\widetilde{d} \circ \widetilde{d} = 0$. A similar computation works for the cokernels.

So we have chain complexes $K^\bullet$ and $L^\bullet$, with chain maps $K^\bullet \to C^\bullet$ and $D^\bullet \to L^\bullet$ given by $\{\ker f^n\}$ and $\{\operatorname{coker} f^n\}$ respectively.

PROPOSITION 1.1.4. *With the notation as above, the chain map $K^\bullet \to C^\bullet$ is the kernel of $f$. Similarly, the map $D^\bullet \to L^\bullet$ is the cokernel of $f$.*

PROOF. Easy. $\square$

DEFINITION 1.1.5. A chain complex $C^\bullet$ is *bounded below* if there exists $s \in \mathbb{Z}$ such that $C^n = 0$, for $n \leq s$. It is *bounded above* if there exists $s \in \mathbb{Z}$ such that $C^n = 0$, for $n \geq s$. It is *bounded* if it is bounded both above and below.

We denote the full subcategory of $\mathrm{Ch}\,\mathscr{C}$ that consists of the bounded below (resp. bounded above, resp. bounded) chain complexes by $\mathrm{Ch}^{\leq}\,\mathscr{C}$ (resp. $\mathrm{Ch}^{\geq}\,\mathscr{C}$, resp. $\mathrm{Ch}^b\mathscr{C}$). For $s \in \mathbb{Z}$, we denote by $\mathrm{Ch}^{\geq c}\mathscr{C}$ (resp. $\mathrm{Ch}^{\leq c}\mathscr{C}$) the category of chain complexes over $\mathscr{C}$ with $C^n = 0$, for $n < c$ (resp. $n > c$).

Proposition 1.1.6. *The category* $\mathrm{Ch}\,\mathscr{C}$ *(or* $\mathrm{Ch}^{\leq}\mathscr{C}$, $\mathrm{Ch}^{\geq}\mathscr{C}$, $\mathrm{Ch}^b\mathscr{C}$, $\mathrm{Ch}^{\geq s}\mathscr{C}$, *or* $\mathrm{Ch}^{\leq s}\mathscr{C}$*), is an abelian category.*

Proof. It only remains to show that for a monic map $f$, we have $\ker(\operatorname{coker} f) = f$, and for an epi $g$, we have $\operatorname{coker}(\ker g) = g$. From Proposition above, we see that a map $f$ is monic iff each $f^n$ is monic, and similarly $g$ is epi iff each $g^n$ is epi. Now, the statement follows immediately from the corresponding one for the abelian category $\mathscr{C}$, since $\ker(\operatorname{coker} f)^n = \ker(\operatorname{coker} f^n) = f^n$, and similarly for $\operatorname{coker}(\ker g)^n$. $\square$

Remark 1.1.7. Observe that, for every $n \in \mathbb{Z}$ we have a natural embedding of $\mathscr{C}$ into $\mathrm{Ch}\,\mathscr{C}$ that takes every object $A$ to the complex $A[n]^\bullet$, with $A[n]^r = 0$, for $r \neq n$, and $A[n]^{=}A$, with 0 boundary maps everywhere. In a sense, to be made precise later, complexes are like generalized objects.

## 2. Cohomology and the Long Exact Sequence

Consider, for a chain complex, $C^\bullet$, the morphisms $\ker d^n : Z^n \to C^n$ and $\operatorname{im} d^{n-1} : B^n \to C^n$. Since $d^n \circ d^{n-1} = 0$, we see that $\operatorname{im} d^{n-1}$ factors through $Z^n$, giving us an exact sequence

$$0 \to B^n \xrightarrow{\operatorname{im} d^{n-1}} Z^n \xrightarrow{\operatorname{coker}(\operatorname{im} d^{n-1})} H^n \to 0.$$

Definition 1.2.1. The $n^{th}$ *cohomology* of a chain complex $C^\bullet$ is the codomain of $\operatorname{coker}(\operatorname{im} d^{n-1})$ in the exact sequence above, and it is denoted by $H^n(C)$.

If $f : C^\bullet \to D^\bullet$ is a chain map, then it's not hard to see that $f$ induces maps $B^n(C) \to B^n(D)$ and $Z^n(C) \to Z^n(D)$, giving us the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & B^n(C) \arrow[r] \arrow[d] & Z^n(C) \arrow[r] \arrow[d] & H^n(C) \arrow[r] \arrow[d, dotted, "H^n(f)"'] & 0 \\
0 \arrow[r] & B^n(D) \arrow[r] & Z^n(D) \arrow[r] & H^n(D) \arrow[r] & 0
\end{tikzcd}$$

This tells us that the $n^{th}$ cohomology gives us a *functor* $H^n : \mathrm{Ch}\,\mathscr{C} \to \mathscr{C}$, for every $n \in \mathbb{Z}$.

Definition 1.2.2. A morphism $f : C^\bullet \to D^\bullet$ of chain complexes is a *quasi-isomorphism* if $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

Here's a result that will prove very useful.

exact-preserves-homology

Proposition 1.2.3. *Let* $F : \mathscr{C} \to \mathscr{D}$ *be a functor between abelian categories; then* $F$ *induces functors* $\mathrm{Ch}\,\mathscr{F} : \mathrm{Ch}\,\mathscr{C} \to \mathrm{Ch}\,\mathscr{D}$*. If* $F$ *is exact, then, for every* $n \in \mathbb{Z}$, *the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Ch}\,\mathscr{C} \arrow[r, "\mathrm{Ch}\,F"] \arrow[d, "H^n"'] & \mathrm{Ch}\,\mathscr{D} \arrow[d, "H^n"'] \\ \mathscr{C} \arrow[r, "F"] & \mathscr{D} \end{tikzcd}$$

*In other words, $F$ preserves cohomology.*

Proof. Follows immediately from the fact that $F$ preserves kernels and cokernels. □

The next Theorem is the most important elementary result on the cohomology of complexes.

hain-long-exact-sequence

Theorem 1.2.4. *Let $\mathscr{A} = \mathrm{Exact}(\mathrm{Ch}\,\mathscr{C})$ be the category of exact sequences of complexes*

$$0 \to D \xrightarrow{f} C \xrightarrow{g} E \to 0.$$

*over an abelian category $\mathscr{C}$. Then we have a functor $H^\bullet : \mathscr{A} \to \mathrm{Ch}(\mathscr{C})$ that assigns to every short exact sequence of complexes a long exact sequence of cohomology of the form*

$$\ldots H^{n-1}(E) \xrightarrow{\delta^{n-1}} H^n(D) \xrightarrow{H^n(f)} H^n(C) \xrightarrow{H^n(g)} H^n(E) \xrightarrow{\delta^n} \ldots$$

Proof. We'll first construct a long exact sequence for each short exact sequence of complexes using the Snake Lemma, and then show functoriality by using Freyd's Embedding Theorem and chasing diagrams.

We find from the Snake Lemma that, for every $n$, the rows of the following diagram are exact:

$$\begin{tikzcd} & D^n/B^n(D) \arrow[r] \arrow[d, "d^n_D"'] & C^n/B^n(C) \arrow[r] \arrow[d, "d^n_C"'] & E^n/B^n(E) \arrow[r] \arrow[d, "d^n_E"'] & 0 \\ 0 \arrow[r] & Z^{n+1}(D) \arrow[r] & Z^{n+1}(C) \arrow[r] & Z^{n+1}(E) \end{tikzcd}$$

Applying the Snake Lemma once again, we find an exact sequence

$$H^n(D) \to H^n(C) \to H^n(E) \xrightarrow{\delta^n} H^{n+1}(D) \to H^{n+1}(C) \to H^{n+1}(E)$$

Putting all these exact sequences together gives us the long exact sequence of cohomology associated to the short exact sequence of complexes.

To show functoriality, it suffices to show that, given another short exact sequence $0 \to D' \to C' \to E' \to 0$, and a morphism $(\alpha, \beta, \gamma)$ from the original exact sequence to this one, the following square commutes:

$$\begin{tikzcd} H^n(E) \arrow[r, "\delta^n"] \arrow[d, "H^n(\gamma)"'] & H^{n+1}(D) \arrow[d, "H^n(\alpha)"'] \\ H^n(E') \arrow[r, "\delta^n"] & H^{n+1}(D') \end{tikzcd}$$

For this we'll assume that we're in $R$-mod for some ring $R$, and chase diagrams. Let $w \in E$ be a cycle, and $e$ its image in $H^n(E)$. Then $\delta^n(e)$ is represented by $z \in Z^{n+1}(D)$ such that $f^{n+1}(z) = d^n y$, for $y \in C^{n+1}$ such that $g^n(y) = e$. And $\delta^n H^n(\gamma)(e)$ is represented by an element $z' \in Z^{n+1}(D')$ such that $f'^{n+1}(z') = d^n y'$, where $y' \in C'^{n+1}$ is such that $g'^n(y') = \gamma^n(w)$. Consider the element $x = z' - \alpha^{n+1}(z) \in Z^{n+1}(D')$. Then we have

$$\begin{aligned} f'^{n+1}(x) &= d^n y' - \beta^{n+1} d^n y \\ &= d^n(y' - \beta^n y). \end{aligned}$$

Now we also have

$$g'^n(y' - \beta^n y) = \gamma^n(w - g^n(y)) = 0.$$

Hence $y' - \beta^n y \in \operatorname{im} f'^n$, which, since $f^{n+1}$ is injective, implies that $x \in \operatorname{im} d^n$ and thus that $z'$ and $\alpha^{n+1}(z)$ represent the same element in $H^{n+1}(D')$, which is precisely what we wanted to show. $\square$

## 3. Chain Homotopies

DEFINITION 1.3.1. Let $f, g : C^\bullet \to D^\bullet$ be two morphisms of chain complexes. A *chain homotopy* from $f$ to $g$ is a collection of morphisms $\{k^n : C^n \to D^{n-1}\}$ such that $k^{n+1} d^n_C + d^{n-1}_D k^n = f^n - g^n$. In this case, we say that $f$ and $g$ are *chain homotopic* and we denote this by $f \sim g$. If $f$ is chain homotopic to 0, then we say that $f$ is *null-homotopic*.

If we have morphisms $f : C^\bullet \to D^\bullet$ and $g : D^\bullet \to C^\bullet$ such that $gf$ is chain homotopic to $1_C$ and $fg$ is chain homotopic $1_D$, then we say that $C^\bullet$ and $D^\bullet$ are *chain homotopic*.

A chain complex $C^\bullet$ is *exact* if $H^n(C) = 0$, for all $n \in \mathbb{Z}$. It's *split exact* if it exact, and if, for every $n$, the short exact sequence:

$$0 \to \ker d^n \to C^n \to \operatorname{im} d^n \to 0$$

splits. In other words, if $C^n \cong \operatorname{im} d^n \oplus \operatorname{im} d^{n-1}$, for all $n$.

hain-homotopic-quasi-iso

PROPOSITION 1.3.2. *Let $f, g : C^\bullet \to D^\bullet$ be two chain homotopic morphisms. Then $H^\bullet(f) = H^\bullet(g)$. In particular, if $C^\bullet$ and $D^\bullet$ are chain homotopic, then they are in fact quasi-isomorphic.*

PROOF. Replacing $f$ with $f - g$ and $g$ with 0, it suffices to consider the case where $f$ is null-homotopic, and show that $H^\bullet(f) = 0$. Indeed, let $a \in Z^n(C)$ be a cycle; then we find:

$$f^n(a) = d^{n-1}_D k^n(a) \in B^n(D),$$

which shows that the induced morphism on cohomology is trivial.

From the definitions, and the first assertion of the Proposition, we see that we have morphisms $f : C^\bullet \to D^\bullet$ and $g : C^\bullet \to D^\bullet$ such that $H^\bullet(fg) = H^\bullet(1_D) = 1_{H^\bullet(D)}$ and $H^\bullet(gf) = 1_{H^\bullet(C)}$. This shows that $H^\bullet(f) : H^\bullet(C) \to H^\bullet(D)$ is an isomorphism, and thus that $f : C^\bullet \to D^\bullet$ is a quasi-isomorphism. $\square$

chain-basic-split-exact

LEMMA 1.3.3. *Pick an integer $n \in \mathbb{Z}$, and let $C^\bullet$ be a chain complex with $C^r = 0$, for $r \neq n, n+1$. Then the following are equivalent:*

(1) *$d^n : C^n \to C^{n+1}$ is an isomorphism.*
(2) *$C^\bullet$ is split exact.*
(3) *$C^\bullet$ is exact.*

(4) $1_{C^\bullet}$ *is null-homotopic.*

Proof. Before we start the rounds, observe that we have

$$H^r(C) = \begin{cases} \ker d^n & \text{if } r = n \\ \operatorname{coker} d^n & \text{if } r = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

It's clear now that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and that (4) $\Rightarrow$ (1). It remains to show that (1) $\Rightarrow$ (4): define morphisms $k^r : C^r \to C^{r-1}$ by setting $k^{n+1} = (d^n)^{-1}$ and $k^r = 0$, for $r \neq n+1$. We see immediately that this gives us a null-homotopy of the identity morphism. $\square$

Definition 1.3.4. A complex that satisfies the hypotheses of the lemma is called a *fundamental split exact complex.*

The next Proposition characterizes split exact complexes.

n-split-exact-direct-sum

Proposition 1.3.5. *The following are equivalent for a chain complex $C^\bullet$:*

(1) $C^\bullet$ *is a direct sum of fundamental split exact complexes.*
(2) $C^\bullet$ *is split exact.*
(3) $1_{C^\bullet}$ *is null-homotopic.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. We'll first show (2) $\Rightarrow$ (3): for this let $i^n : \operatorname{im} d^n \to C^n$ be the embedding given by the splitting on $C^n$, and let $\pi^{n+1} : C^{n+1} \to \operatorname{im} d^n$ be the projection given by the splitting on $C^{n+1}$. Define $k^{n+1} = i^n \pi^{n+1} : C^{n+1} \to C^n$. Then we find

$$\begin{aligned} k^{n+1} d^n + d^{n-1} k^n &= i^n \pi n + 1 d^n + d^{n-1} i^{n-1} \pi^n \\ &= i^n p^n + j^n \pi^n = 1_{C^n}. \end{aligned}$$

where $p^n : C^n \to \operatorname{im} d^n$ is the natural projection and $j^n : \operatorname{im} d^{n-1} \to C^n$ is the natural embedding. Thus we see that $1_{C^\bullet}$ is null-homotopic.

Now we show (3) $\Rightarrow$ (2): suppose $\{k^n : C^n \to C^{n+1}\}$ is a null-homotopy for $1_{C^\bullet}$. Then we see that, for each $n \in \mathbb{Z}$, we have

$$C^n = k^{n+1}(\operatorname{im} d^n) + d^{n-1}(\operatorname{im} k^n).$$

Now we have

$$\operatorname{im} d^n = d^n(k^{n+1}(\operatorname{im} d^n)).$$

Therefore, $k^{n+1}|_{\operatorname{im} d^n}$ gives a splitting morphism for the epimorphism $C^n \to \operatorname{im} d^n$. Since $1_{C^\bullet}$ is nullhomotopic, we also see that $C^\bullet$ is exact, which shows that it is in fact split exact.

We'll now finish by showing (2) $\Rightarrow$ (1). For every $m \in \mathbb{Z}$, define $C^\bullet_{(m)}$ to be the fundamental split exact complex given by

$$C^r_{(m)} = \begin{cases} i^m(\operatorname{im} d^m) & \text{if } r = m \\ \operatorname{im} d^m & \text{if } r = m+1 \\ 0 & \text{otherwise} \end{cases}$$

Here $i^m : \operatorname{im} d^m \to C^m$ is the splitting morphism, and the morphism $d^m_{C_{(m)}} : C^m_{(m)} \to C^{m+1}_{(m)}$ induced by $d^m_C$ is an isomorphism. It is now evident that $C^\bullet = \bigoplus_{m \in \mathbb{Z}} C^\bullet_{(m)}$. $\square$

## 4. Resolutions

DEFINITION 1.4.1. Let $C^\bullet$ be a bounded below (resp. a bounded above) chain complex; then $C^\bullet$ is *acyclic* if the following conditions hold:

(1) $C^n = 0$, for $n < 0$ (resp. for $n > 0$).
(2) $H^n(C) = 0$, for all $n > 0$ (resp. for all $n < 0$).

DEFINITION 1.4.2. Let $\mathscr{K}$ be a class of objects in $\mathscr{C}$, and let $A$ be any object in $\mathscr{C}$. Then, a *right (resp. left) $\mathscr{K}$-resolution* of $A$, denoted $A \hookrightarrow C^\bullet$ (resp. $C^\bullet \twoheadrightarrow A$) is an acyclic bounded below (resp. bounded above) chain complex $C^\bullet$ with $C^n \in \mathscr{K}$, for all $n > 0$ (resp. for all $n < 0$), such that $H^0(C) = A$. In other words, it's a bounded below (resp. bounded above) chain complex $C^\bullet$ quasi-isomorphic to $A[0]$.

DEFINITION 1.4.3. If $\mathscr{K}$ is the class of projective objects in $\mathscr{C}$, then a left $\mathscr{K}$-resolution is called a *projective resolution.*

If $\mathscr{K}$ is the class of injective objects in $\mathscr{C}$, then a right $\mathscr{K}$-resolution is called a *injective resolution.*

If $\mathscr{C} = R$-mod, for some ring $R$, and $\mathscr{K}$ is the class of flat $R$-modules, then a left $\mathscr{K}$-resolution is a called a *flat resolution.*

If $\mathscr{C} = R$-mod, for some ring $R$, and $\mathscr{K}$ is the class of finitely generated free $R$-modules, then a left $\mathscr{K}$-resolution is called a *finite free resolution.*

DEFINITION 1.4.4. An abelian category $\mathscr{C}$ has *enough injectives* if, for every object $A$ in $\mathscr{C}$, there is a monomorphism $u : A \to I$, with $I$ injective. It has *enough projectives* if, for every object $A$ in $\mathscr{C}$, there is an epimorphism $u : P \to A$ with $P$ projective.

It's easy to see inductively that if a category $\mathscr{C}$ has enough injectives (resp. enough projectives) then every object has an injective (resp. projective) resolution.

in-acyclic-inj-extension

LEMMA 1.4.5. *Let $N^\bullet$ be a bounded below acyclic complex. Then $H^0N = \ker d_N^0[0]$ injects into $N^\bullet$. Suppose $I^\bullet$ is a bounded below chain complex with $I^r = 0$, for $r < 0$, and with $I^n$ an injective object, for $n \geq 0$. Then, for every morphism $f : H^0N \to I^\bullet$, there exists an extension $F : N^\bullet \to I^\bullet$ of $f$ to $N^\bullet$. Moreover, any two such extensions of $f$ are chain homotopic.*

$$\begin{tikzcd} N^\bullet \arrow[r, dotted, "\exists F"] & I^\bullet \\ H^0N \arrow[u] \arrow[ur, "f"] & \end{tikzcd}$$

PROOF. We will construct the chain morphism $F$ inductively. For $n = 0$, observe that we have:

$$\begin{tikzcd} N^0 \arrow[r, dotted, "\exists F^0"] & I^0 \\ \ker d^0 \arrow[u, hook] \arrow[ur, "f^0"] & \end{tikzcd}$$

We obtain the extension $F^0$ immediately from the injectivity of $I^0$. Now suppose, for $r < n$, we've constructed morphisms $F^r : N^r \to I^r$ compatible with the boundary morphisms. Let $G^{n-1} : N^{n-1} \to I^n$ be the composition $d_I^{n-1}F^{n-1}$. Observe that $G^{n-1}d_N^{n-1} = 0$, and so $G^{n-1}$ factors through $N^{n-1}/\operatorname{im} d^{n-1} = N^{n-1}/\ker d^n$. In other words, we have the following picture, where the extension $F^n$ is obtained by the injectivity of $I^n$.

$$\begin{tikzcd}
N^n \arrow[r, dotted, "F^n"] & I^n \\
N^{n-1}/\ker d^n \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

Now suppose $F : N^\bullet \to I^\bullet$ is an extension of the zero morphism from $H^0N$ to $I^\bullet$. We will construct a chain homotopy from $F$ to 0 by induction on $n$. Define $k^n = 0$, for $n \leq 0$. When $n = 1$, we have the following picture:

$$\begin{tikzcd}
N^1 \arrow[r, "F^1"] \arrow[dr, dotted, "k^1"] & I^1 \\
N^0 \arrow[u, "d_N^0"] \arrow[r, "F^0"] & I^0 \arrow[u, "d_I^0"'] \\
\ker d_N^0 \arrow[u, hook] \arrow[ur, "0"] &
\end{tikzcd}$$

We get the morphism $k^1$, by observing that $F^0$ factors through $N^0/\ker d_N^0$, a subobject of $N^1$, and by using the injectivity of $I^0$. Suppose now that we have define morphisms $k^r : N^r \to I^{r-1}$ such that $F^{r-1} = k^r d_N^{r-1} + d_N^{r-2}k^{r-1}$, for $r < n$; then we have a diagram

$$\begin{tikzcd}
N^n \arrow[r, "F^n"] \arrow[dr, dotted, "k^n"] & I^n \\
N^{n-1} \arrow[u, "d_N^{n-1}"] \arrow[r, "F^{n-1}"] \arrow[dr, "k^{n-1}"] & I^{n-1} \arrow[u, "d_I^{n-1}"'] \\
N^{n-2} \arrow[u, "d_N^{n-2}"] \arrow[r, "F^{n-2}"] & I^{n-2} \arrow[u, "d_I^{n-2}"']
\end{tikzcd}$$

where we get the morphism $k^n$ in the following fashion: Consider the morphism $G^{n-1} = F^{n-1} - d_I^{n-2}k^{n-1}$ from $N^{n-1}$ to $I^{n-1}$, and observe that we have:

$$\begin{aligned} G^{n-1}d_N^{n-2} &= d_I^{n-2}F^{n-2} - d_I^{n-2}k^{n-1}d_N^{n-2} \\ &= d_I^{n-2}F^{n-2} - d_I^{n-2}(F^{n-2} - d_N^{n-3}k^{n-2}) = 0. \end{aligned}$$

Therefore, $G^{n-1}$ factors through $N^{n-1}/\operatorname{im} d^{n-2} = N^{n-1}/\ker d^{n-1}$, and we can therefore extend it to a morphism $k^n : N^n \to I^{n-1}$ that satisfies our requirements. □

Here's the dual statement for acyclic bounded above resolutions:

ain-acyclic-proj-lifting

LEMMA 1.4.6. *Let $N^\bullet$ be a bounded above acyclic complex. Then $N^\bullet$ surjects onto $H^0N = (N^0/\operatorname{im} d_N^{-1})[0]$. Suppose $P^\bullet$ is a bounded above chain complex with $P^r = 0$, for $r > 0$, and with $P^n$ an projective object, for $n \leq 0$. Then, for every morphism $f : P^\bullet \to H^0N$, there exists a lift $F : P^\bullet \to N^\bullet$ of $f$ to $N^\bullet$. Moreover, any two such liftings of $f$ are chain homotopic.*

$$\begin{tikzcd} P^\bullet \arrow[r, dotted, "\exists F"] \arrow[dr, "f"'] & N^\bullet \\ & H^0N \arrow[u] \end{tikzcd}$$

niqueness-of-resolutions

THEOREM 1.4.7 (Uniqueness of Resolutions). *Suppose $I^\bullet$ and $J^\bullet$ are two injective resolutions of an object $A$. Then $I^\bullet$ and $J^\bullet$ are chain homotopic and thus quasi-isomorphic. In fact, any quasi-isomorphism $f : I^\bullet \to J^\bullet$ is determined uniquely upto chain homotopy.*

*Dually, if $P^\bullet$ and $Q^\bullet$ are two projective resolutions of an object $A$, then $P^\bullet$ and $Q^\bullet$ are chain homotopic and thus quasi-isomorphic. In fact, any quasi-isomorphism $f : P^\bullet \to Q^\bullet$ is determined uniquely upto chain homotopy.*

PROOF. We prove the statement about injective resolutions; the one about projective resolutions will follow dually, using (1.4.6) instead of (1.4.5).

Observe that we have $H^0I = H^0J = A[0]$. So we have the following diagram:

$$\begin{tikzcd} & I^\bullet \arrow[dr, "g"] & \\ J^\bullet \arrow[ur, "f"] & H^0J = H^0I \arrow[l, hook'] \arrow[u, hook] \arrow[r, hook] & J^\bullet \end{tikzcd}$$

Since both $I^\bullet$ and $J^\bullet$ are acyclic, we have used lemma (1.4.5) to find extensions (unique upto chain homotopy) $f : J^\bullet \to I^\bullet$ and $g : I^\bullet \to J^\bullet$ of the inclusions of $A$ into $I^\bullet$ and $J^\bullet$, respectively. Since these extensions are unique upto chain homotopy, we find that $gf \sim 1_J$ and $fg \sim 1_I$, which shows that $I^\bullet$ and $J^\bullet$ are chain homotopic. □

chain-horseshoe-lemma

PROPOSITION 1.4.8 (Horseshoe Lemma).

## 5. Mapping Cones and Cylinders

Definition 1.5.1. The *translation functor* $T : \mathrm{Ch}\mathscr{C} \to \mathrm{Ch}\mathscr{C}$ is the functor that sends a chain complex $\{C^n, d^n\}$ to the chain complex $\{TC^n, Td^n\}$, with $TC^n = C^{n-1}$ and $Td^n = d^{n-1}$. Given a chain map $f^\bullet : C^\bullet \to D^\bullet$, $Tf^\bullet : TC^\bullet \to TD^\bullet$ is just the chain map with $(Tf)^n = f^{n-1}$.

$T$ is in fact an automorphism of $\mathrm{Ch}\mathscr{C}$, with inverse $T^{-1}$ acting via the assignments $T^{-1}C^n = C^{n+1}$ and $T^{-1}d^n = d^{n+1}$.

For any $p \in \mathbb{Z}$, we denote $T^pC^\bullet$ by $C[p]^\bullet$.

It's easy to check that $H^n(C[p]) = H^{n-p}(C)$.

Now, we consider two important operations that arise from chain maps.

chain-mapping-cone

Definition 1.5.2. The *mapping cone* $\mathrm{cone}(f)$ of a chain map $f : C^\bullet \to D^\bullet$ is the chain complex with chain objects $\mathrm{cone}(f)^n = C^n \oplus D^{n-1}$ and chain maps

$$d^n = \begin{pmatrix} -d_C^n & 0 \\ f^n & d_D^{n-1} \end{pmatrix} : \mathrm{cone}(f)^n \to \mathrm{cone}(f)^{n+1}.$$

This indeed gives us chain maps, because we have the composition

$$d^{n+1}d^n = \begin{pmatrix} d_C^{n+1}d_C^n & 0 \\ -f^{n+1}d_C^n + d_D^n f^n & d_D^n d_D^{n-1} \end{pmatrix} = 0.$$

We can visualize this in the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & C^n \arrow[r, "-d^n"] \arrow[ddr, "f^{n+1}"] & C^{n+1} \arrow[r] \arrow[ddr, "f^{n+1}"] & \dots \\
& \bigoplus & \bigoplus & \\
\dots \arrow[r] & D^{n-1} \arrow[r, "d^{n-1}"] & D^n \arrow[r] & D^{n+1}
\end{tikzcd}$$

There is a natural inclusion $D[1]^\bullet \to \mathrm{cone}(f)$, with cokernel $C^\bullet$. So we get an exact sequence of complexes

$$0 \to D[1]^\bullet \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{\pi} C^\bullet \to 0,$$

where $\pi^n(c, d) = (-1)^n c$. The sign is to ensure that it is a chain map.

This gives rise to a long exact sequence of cohomology.

$$\dots \xrightarrow{H}{}^{n}(C) \xrightarrow{\delta^n} H^n(D) \to H^{n+1}(\mathrm{cone}(f)) \to H^{n+1}(C) \to \dots$$

in-cone-connecting-morph

Proposition 1.5.3. *The connecting morphism $\delta^n$ in the sequence above is simply $H^n(f)$.*

Proof. We'll do a diagram chase for the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & D^{n-1} \arrow[r, "i^n"] \arrow[d, "d_D^n"'] & \mathrm{cone}(f)^n \arrow[r, "\pi^n"] \arrow[d, "d^n"'] & C^n \arrow[r] \arrow[d, "d_C^{n+1}"'] & 0 \\
0 \arrow[r] & D^n \arrow[r, "i^{n+1}"] & \mathrm{cone}(f)^{n+1} \arrow[r, "\pi^{n+1}"] & C^{n+1} \arrow[r] & 0
\end{tikzcd}$$

Pick $z \in Z^n(C)$; so $d_C^n z = 0$. Then $(-z, 0) \in \mathrm{cone}(f)^n$ projects onto $z$. Since $d^n(-z, 0) = (d_C^n z, f^n(z))$, we see that $\delta^n[z] = [f^n z] = H^n(f)[z]$. $\square$

With the mapping cone in hand, we are ready to describe injectives and projectives in the category of chain complexes.

chain-proj-inj-complexes

Proposition 1.5.4. *A chain complex $P^\bullet$ is projective in the category of chain complexes if and only if $P^\bullet$ is split exact with $P^n$ projective, for all $n \in \mathbb{N}$.*

*Dually, a chain complex $I^\bullet$ is injective in the category of chain complexes if and only if $I^\bullet$ is split exact with $I^n$ injective, for all $n \in \mathbb{N}$.*

Proof. Suppose the first assertion about projectives is proven; then observe that the injectives in $\mathrm{Ch}\,\mathscr{C}$ are also injectives in $\mathrm{Ch}\,\mathscr{C}^{op}$, which, by the first assertion, consist of split exact complexes of projectives in $\mathscr{C}^{op}$ and thus of injectives in $\mathscr{C}$ under the canonical isomorphism from $\mathrm{Ch}\,\mathscr{C}^{op}$ to $\mathrm{Ch}\,\mathscr{C}$. Thus it suffices to prove the first assertion.

For this, note that it's easy to see right away that if $P^\bullet$ is projective, then it must be a complex of projectives. To see that it must in fact be split exact, consider the exact sequence:

$$0 \to P[1] \to \mathrm{cone}(1_P) \to P \to 0.$$

Since $P^\bullet$ is projective, this sequence splits. And so we get a morphism $s = \binom{\pm 1_P}{s_2}$ : $P^\bullet \to \mathrm{cone}(1_P)$. Writing out the condition for it to be a morphism of complexes, we find, for each $n \in \mathbb{Z}$,

$$\begin{pmatrix} \pm d^n \\ 1_{P^n} + d^{n-1} s_2^n \end{pmatrix} = \begin{pmatrix} \pm d^n \\ s_2^{n+1} d^n \end{pmatrix}.$$

So we get $1_{P^n} = s_2^{n+1} d^n - d^{n-1} s_2^n$. Taking $k^n = (-1)^n s_2^n : P^n \to P^{n-1}$, for each $n \in \mathbb{Z}$, gives us a null-homotopy for $1_{P^n}$, which, by (1.3.5), means that $P^\bullet$ is split exact.

Now for the converse assume $P^\bullet$ is a split exact complex of projectives. By (1.3.5), it's a direct sum of fundamental split exact complexes. Therefore it suffices to show that a fundamental split exact complex of projectives is projective in $\mathrm{Ch}\,\mathscr{C}$. So let $P^\bullet$ be such a fundamental split exact complex with $P^r = 0$, for $r \neq n, n+1$, and suppose we have a diagram

$$\begin{tikzcd} & C^\bullet \arrow[d, two heads, "\pi"'] \\ P^\bullet \arrow[r, "f"] & C'^\bullet \end{tikzcd}$$

Now, observe that giving a morphism $F : P^\bullet \to C^\bullet$ is equivalent to giving a morphism $F^n : P^n \to C^n$, since we can define $F^{n+1} : P^{n+1} \to C^{n+1}$ by setting $F^{n+1} = d_C^n F^n {d_P^n}^{-1}$. To see that this defines a morphism of complexes, all we need to check is that $d_C^{n+1} F^n = 0$; but this follows immediately from its definition.

Since $P^n$ is projective, we can find a lifting $F^n : P^n \to C^n$ of $f^n$. This as noted above gives us a morphism $F : P^\bullet \to C^\bullet$. It remains to check that $\pi F = f$. We find:

$$\begin{aligned}\pi^{n+1}F^{n+1} &= \pi^{n+1}d_C^n F^n d_P^{n-1}\\ &= d_{C'}^n \pi^n F^n d_P^{n-1}\\ &= d_{C'}^n f^n d_P^{n-1}\\ &= f^{n+1}d_P^n d_P^{n-1} = f,\end{aligned}$$

as we had wanted to show. ☐

chain-enough-proj-inj

COROLLARY 1.5.5. *Let $\mathscr{C}$ be an abelian category with enough projectives (resp. enough injectives); then* $\operatorname{Ch}\mathscr{C}$ *also has enough projectives (resp. injectives).*

PROOF. We prove the result about projectives; the one about injectives is obtained by formal duality.

By the Proposition above, it's enough to show that, for every chain complex $C^\bullet$, there is an epimorphism $P^\bullet \twoheadrightarrow C^\bullet$ with $P^\bullet$ a split exact complex of projectives. For this, choose, for each $n \in \mathbb{Z}$, epimorphisms $u^n : Q^n \twoheadrightarrow C^n$ with $Q^n$ projective. Set $P^n = Q^n \oplus Q^{n-1}$, and let $\pi^n : P^n \to Q^n$ be the natural projection, and let $i^{n+1} : Q^n \to P^{n+1}$ be the natural embedding. Set $d_P^n = i^{n+1}\pi^n$; it's easy to see that this gives us a split exact complex $P^\bullet$ of projectives. Take the epimorphism $u\pi : P^\bullet \to C^\bullet$ to finish the proof. ☐

## 6. Differential Graded Algebras

## 7. Double Complexes

Put simply, double complexes are complexes over the category of chain complexes. They will prove important when we get to Chapter 3 on spectral sequences. The formal definition follows.

DEFINITION 1.7.1 (Double Complexes). A *double complex* $C^{\bullet,\bullet}$ over an abelian category $\mathscr{C}$ consists of the following data:

(1) For every pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, an object $C^{p,q} \in \mathscr{C}$,
(2) For every pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, morphisms $d_{II}^{p,q} : C^{p,q} \to C^{p,q+1}$ and $d_I^{p,q} : C^{p,q} \to C^{p+1,q}$ satisfying the condition:

$$d_I^{p,q} d_I^{p-1,q} = 0 \qquad d_{II}^{p,q} d_{II}^{p,q-1} = 0$$
$$d_I^{p,q+1} d_{II}^{p,q} = d_{II}^{p+1,q} d_I^{p,q}.$$

Observe that this means that $\{C^{p,\bullet} : q \in \mathbb{Z}\}$ and $\{C^{\bullet,q} : p \in \mathbb{Z}\}$ are chain complexes for each fixed $p, q \in \mathbb{Z}$. The last equality can be restated as saying that $d_{II}^{\bullet,q}$ is a chain map from $C^{\bullet,q}$ to $C^{\bullet,q+1}$.

A *morphism* $f : C^{\bullet,\bullet} \to D^{\bullet,\bullet}$ between double complexes is a collection of morphisms $f^{p,q} : C^{p,q} \to D^{p,q}$ such that

$$d_I^{p,q} f^{p,q} = f^{p+1,q} d_I^{p,q} d_{II}^{p,q} f^{p,q} = f^{p,q+1} d_{II}^{p,q}$$

DEFINITION 1.7.2. Let $S \subset \mathbb{Z} \times \mathbb{Z}$ be a subset of the plane lattice. Then $S$ is *bounded below*, if, for each $n \in \mathbb{Z}$, there is $s \in \mathbb{Z}$ such that $(p, n - p) \notin S$, for $p \geq s$. It is *bounded above*, if, for each $n \in \mathbb{Z}$, there is $s \in \mathbb{Z}$ such that $(p, n - p) \notin S$, for $p \leq s$. It is *bounded* if it is bounded both below and above. It is *first (resp. second, third, fourth) quadrant* if $(p, q) \notin S$, for $p < 0$ or $q < 0$ (resp. $p > 0$ or $q < 0$,

$p > 0$ or $q > 0$, $p < 0$ or $q > 0$). It is *upper (resp. lower, right, left) half plane* if $(p, q) \notin S$, for $p < 0$ (resp. $p > 0$, $q < 0$, $q > 0$).

DEFINITION 1.7.3 (Boundedness conditions). Let $C^{\bullet,\bullet}$ be a double complex. To such a double complex we can associate a subset $S(C)$ of the plane lattice by: $(p, q) \in S(C) \Leftrightarrow C^{p,q} \neq 0$. Now, $C^{\bullet,\bullet}$ is bounded below (resp. bounded above, bounded, first quadrant, etc.) if $S(C)$ is bounded below (resp. bounded above, bounded, first quadrant, etc).

DEFINITION 1.7.4. If $\mathscr{C}$ is complete and $C^{\bullet,\bullet}$ is a double complex over $\mathscr{C}$, then the *total complex* of $C^{\bullet,\bullet}$, which we denote by $\mathrm{Tot}^\bullet_\Pi(C)$ is given by

$$\mathrm{Tot}^n_\Pi(C) = \prod_{p+q=n} C^{p,q} d^n = \prod_{p+q}^{n} \left(d_I^{p,q} + (-1)^{p+q} d_{II}^{p,q}\right).$$

Observe that we have

$$\begin{aligned} d^{n+1}d^n = &\prod_{p+q=n+1} \left(d_I^{p,q} d_{II}^{p-1,q} + (-1)^{p+q-1} d_I^{p,q} d_{II}^{p,q-1} + (-1)^{p+q} d_{II}^{p,q} d_I^{p-1,q} - d_{II}^{p,q} d_{II}^{p,q-1}\right) \\ &= 0, \end{aligned}$$

by our conditions on $d_{II}$ and $d_I$.

If $\mathscr{C}$ is instead cocomplete, then we analogously define $\mathrm{Tot}^\bullet_\oplus(C)$, the *restricted total complex* of $C^{\bullet,\bullet}$, by replacing products everywhere with sums.

REMARK 1.7.5. It's clear that a morphism between double complexes induces a morphism between their total complexes. That is, the assignment $C^{\bullet,\bullet} \to \mathrm{Tot}^\bullet(C)$ is functorial.

REMARK 1.7.6. Observe that if $C^{\bullet,\bullet}$ is bounded then both these total complexes are isomorphic. In general we can refer to either of these as the total complex of $C^{\bullet,\bullet}$.

chain-hom-double-complex

EXAMPLE 1.7.7 (Hom Double Complex). Let $P^\bullet$ and $J^\bullet$ be two complexes over $\mathscr{C}$ and consider the double complex obtained in the following fashion:

$$\mathrm{Hom}(P, J)^{p,q} = \mathrm{Hom}(P^{-p}, J^q)$$

$$d_I^{p,q} = \mathrm{Hom}(d_P^{-(p+1)}, J^q) \qquad d_{II}^{p,q} = \mathrm{Hom}(P^{-p}, d_J^q)$$

This is called the *Hom complex* of $P^\bullet$ and $J^\bullet$. We refer to $\mathrm{Tot}^\bullet_\Pi(\mathrm{Hom}(P, J))$ as the *total Hom complex* of $P^\bullet$ and $J^\bullet$.

chain-tensor-product

EXAMPLE 1.7.8 (Tensor Product of Complexes). Fix a ring $R$ and suppose $P^\bullet$ and $Q^\bullet$ are complexes of right $R$-modules and left $R$-modules, respectively. Now consider the double complex with $(P \otimes_R Q)^{p,q} = P^p \otimes_R Q^q$, and with $d_{II}^{p,q} = 1 \otimes d_Q^q$ and $d_I^{p,q} = d_P^p \otimes 1$. We then have:

$$d_{II}^{p+1,q} d_I^{p,q} - d_I^{p,q+1} d_{II}^{p,q} = d_P^p \otimes d_Q^q - d_P^p \otimes d_Q^q = 0.$$

This is the *tensor product* of $P^\bullet$ with $Q^\bullet$ over $R$. We denote the total complex $\mathrm{Tot}^\bullet_\oplus(P \otimes_R Q)$ as the *total tensor product* of $P$ and $Q$. It's not hard to see that, for $P^\bullet$ a complex of right $R$-modules, $Q^\bullet$ a complex of left $R$-modules, and $I^\bullet$ a complex of abelian groups, we have a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathrm{Tot}^\bullet_\oplus(P \otimes_R Q), I^\bullet) \cong \mathrm{Hom}_{R\text{-mod}}(P, \mathrm{Tot}^\bullet_\Pi(\mathrm{Hom}(Q, I))).$$

Thus the total Hom and the total tensor product are adjoints to each other, as one would expect.

DEFINITION 1.7.9. A *chain homotopy* between two morphisms $f, g : C^{\bullet,\bullet} \to D^{\bullet,\bullet}$ consists of two collections of maps: $\{k_I^{p,q} : C^{p,q} \to D^{p-1,q}\}$ and $\{k_{II}^{p,q} : C^{p,q} \to D^{p,q-1}\}$ such that

$$g^{p,q} - f^{p,q} = d_I^{p-1,q} h_I^{p,q} + (-1)^{q-1} d_{II}^{p,q-1} h_{II}^{p,q} + h_I^{p+1,q} d_I^{p,q} + (-1)^q h_{II}^{p,q+1} d_{II}^{p,q}$$

We see immediately that a chain homotopy between morphisms of double complexes induces a chain homotopy between the induced morphisms on the total complex.

DEFINITION 1.7.10 (Cohomology of a Double Complex). Let $C^{\bullet,\bullet}$ be a double complex. For every pair $(p, q)$ of integers, we define the following objects:

$$\begin{aligned} Z_I^{p,q}(C) &= \ker(d_I^{p,q} : C^{p,q} \to C^{p+1,q}) \\ Z_{II}^{p,q}(C) &= \ker(d_{II}^{p,q} : C^{p,q} \to C^{p,q+1}) \\ B_I^{p,q}(C) &= \operatorname{im}(d_I^{p-1,q} : C^{p-1,q} \to C^{p,q}) \\ B_{II}^{p,q}(C) &= \operatorname{im}(d_{II}^{p,q-1} : C^{p,q-1} \to C^{p,q}) \\ H_I^{p,q}(C) &= Z_I^{p,q}(C)/B_I^{p,q}(C) \\ H_{II}^{p,q}(C) &= Z_{II}^{p,q}(C)/B_{II}^{p,q}(C). \end{aligned}$$

That is $H_I^*$ takes the cohomology of the rows and $H_{II}$ the cohomology of the columns. Now, observe that $H_I^{p,\bullet}$ is a complex with the boundary morphisms induced by $d_{II}$. Taking its cohomology we get new objects $H_{II}^q(H_I^p(C))$. Similarly, taking the cohomology of the complex $H_{II}^{p,\bullet}$, we get objects $H_I^q(H_{II}^p(C))$.

CHAPTER 2

# Derived and $\delta$-functors

chap:delta

## 1. $\delta$-functors

DEFINITION 2.1.1. Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories, and let $\mathrm{Exact}(\mathscr{C})$ be the category of short exact sequences in $\mathscr{C}$ with the obvious morphisms. We have three forgetful functors $O^i : \mathrm{Exact}(\mathscr{C}) \to \mathscr{C}$, for $i = 1, 2, 3$, that extract first, second and third objects, respectively, of a short exact sequence over $\mathscr{C}$.

A *cohomological (resp. homological) $\delta$-functor* from $\mathscr{C}$ to $\mathscr{D}$, denoted $T^\bullet : \mathscr{C} \to \mathscr{D}$, is a collection of additive functors $\{T^n : \mathscr{C} \to \mathscr{D}\}$, for each $n \geq 0$ (resp. for $n \leq 0$), along with a collection of natural transformations $\{\delta^n : T^n O^3 \to T^{n+1} O^1\}$, so that for every exact sequence

$$0 \to C^1 \xrightarrow{f} C^2 \xrightarrow{g} C^3 \to 0$$

we have a long exact sequence

$$\cdots \to T^n C^1 \xrightarrow{T^n(f)} T^n C^2 \xrightarrow{T^n(g)} T^n C^3 \xrightarrow{\delta^n} T^{n+1} C^1 \xrightarrow{T^{n+1}(f)} T^{n+1} C^2 \to \cdots$$

REMARK 2.1.2. We will reserve the use of the unqualified term $\delta$-functor for cohomological $\delta$-functors.

REMARK 2.1.3. Note that if $T^\bullet$ is a $\delta$-functor, then $T^0$ is a left exact functor (right exact, if $T^\bullet$ is homological).

delta-cohom-delta

PROPOSITION 2.1.4. *Let $\mathscr{C}$ be an abelian category. The functors $\{H^n : \mathrm{Ch}\,\mathscr{C} \to \mathscr{C}\}$, for $n \geq 0$, (resp. for $n \leq 0$) define a cohomological (resp. homological) $\delta$-functor from $\mathrm{Ch}^{\geq 0}\mathscr{C}$ (resp. $\mathrm{Ch}^{\leq 0}\mathscr{C}$) to $\mathscr{C}$.*

PROOF. Immediate from (1.2.4) $\square$

DEFINITION 2.1.5. A *morphism* $\varphi : T^\bullet \to T'^\bullet$ between two (cohomological or homological) $\delta$-functors from $\mathscr{C}$ to $\mathscr{D}$ is a collection of natural transformations $\varphi^n : T^n \to T'^n$ such that the following diagram commutes, for all $n$:

$$\begin{tikzcd} T^n O^3 \arrow[r, "\delta^n"] \arrow[d, "\varphi^n O^3"'] & T^{n+1} O^1 \arrow[d, "\varphi^{n+1} O^1"] \\ T'^n O^3 \arrow[r, "\delta^n"'] & T'^{n+1} O^1 \end{tikzcd}$$

A cohomological (resp. homological) $\delta$-functor $T^\bullet : \mathscr{C} \to \mathscr{D}$ is *universal* if, for every other cohomological (homological) $\delta$-functor $T'\bullet : \mathscr{C} \to \mathscr{D}$ equipped with a natural transformation $\eta : T^0 \to T'^0$ (resp. a natural transformation $\eta : T'^0 \to T^0$), there is a unique morphism $\varphi : T^\bullet \to T'^\bullet$ (resp. $\varphi : T'^\bullet \to T^\bullet$) such that $\varphi^0 = \eta$.

REMARK 2.1.6. By its definition a universal $\delta$-functor $S^\bullet$ is an initial (or terminal) object in the subcategory of the category of $\delta$-functors that consists of $\delta$-functors $T^\bullet : \mathscr{C} \to \mathscr{D}$ with $T^0 \cong S^0$. Hence, given a functor $F : \mathscr{C} \to \mathscr{D}$, upto isomorphism of $\delta$-functors, there exists a unique $\delta$-functor $T^\bullet : \mathscr{C} \to \mathscr{D}$ with $T^0 \cong F$.

Here's our first example of a universal $\delta$-functor:

elta-exact-trivial-delta

PROPOSITION 2.1.7. *Let $F : \mathscr{C} \to \mathscr{D}$ be an exact functor between abelian categories. Then the cohomological (resp. homological) $\delta$-functor $T^\bullet : \mathscr{C} \to \mathscr{D}$ defined by $T^0 = F$, $T^n = 0$, for $n \geq 0$ (resp $n \leq 0$), is a universal cohomological (resp. homological) $\delta$-functor.*

PROOF. We'll only do the cohomological case. It follows from the exactness of $F$ that $T^\bullet$ is indeed a $\delta$-functor (with $\delta^n = 0$, for all $n$).

Assume $S^\bullet : \mathscr{C} \to \mathscr{D}$ is a $\delta$-functor and suppose we have a natural transformation $\eta : F \to S^0$. To check that the obvious choice for a morphism from $T^\bullet$ to $S^\bullet$ is indeed a morphism of $\delta$-functors, we only have to check that the following diagram commutes:

$$\begin{tikzcd} FO^2 \arrow[r, "Ff^{2,3}"] \arrow[d, "\eta O^1"'] & FO^3 \arrow[r] \arrow[d, "\eta O^3"] & 0 \arrow[d] \\ S^0O^2 \arrow[r, "S^0f^{2,3}"] & S^0O^3 \arrow[r, "\delta^0"] & S^1O^1 \end{tikzcd}$$

So it suffices to show that $\delta^0(\eta O^3) = 0$. But observe that we have

$$\delta^0(\eta O^3)Ff^{2,3} = \delta^0 S^0 f^{2,3}(\eta O^1) = 0,$$

where $f^{2,3} : O^2 \to O^3$ is the obvious natural transformation. Since $f^{2,3}$ is an epimorphism and $F$ is exact, we see that $\delta^0(\eta O^3) = 0$. $\square$

This will be generalized to left and right exact functors in the next section.

DEFINITION 2.1.8. Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor. We say that $F$ is *effaceable* (resp. *coeffaceable*) if, for every object $A$ of $\mathscr{C}$, there is a monomorphism $u : A \to I$ (resp. an epimorphism $u : P \to A$) with $I$ injective (resp. with $P$ projective) such that $F(u) = 0$.

The next Theorem is the most important general nonsense result about $\delta$-functors.

elta-criterion-universal

THEOREM 2.1.9 (Grothendieck). *Let $T^\bullet : \mathscr{C} \to \mathscr{D}$ be a cohomological (resp. homological) $\delta$-functor such that $T^n$ is effaceable for all $n \geq 0$ (resp. coeffaceable for all $n \leq 0$). Then $T^\bullet$ is universal.*

PROOF. Again, we'll only prove the cohomological version. Suppose $T^\bullet$ is an effaceable $\delta$-functor and suppose $S^\bullet : \mathscr{C} \to \mathscr{D}$ is another $\delta$-functor equipped with a natural transformation $\eta : T^0 \to S^0$. We'll construct a morphism $\varphi : T^\bullet \to S^\bullet$ inductively. The base case is our hypothesis, so assume that we've constructed natural transformations $\varphi^r : T^r \to S^r$ for $r < n$, which satisfy the required commutativity conditions.

Choose an object $A$ in $\mathscr{C}$, and let $u : A \to I$ be a monomorphism into an injective object $I$ such that $T^n(u) = 0$. Let $C = \operatorname{coker} u$; then we have the following diagram:

$$\begin{tikzcd}
T^{n-1}I \arrow[r] \arrow[d, "\varphi_I^{n-1}"'] & T^{n-1}C \arrow[r, "\delta^{n-1}"] \arrow[d, "\varphi_C^{n-1}"'] & T^nA \arrow[r] \arrow[d, dotted, "\varphi_A^n"'] & 0 \\
S^{n-1}I \arrow[r] & S^{n-1}C \arrow[r, "\delta^{n-1}"] & S^nA &
\end{tikzcd}$$

where the dotted morphism $\varphi_A^n$ is obtained from the universal property of $T^nA$ as the cokernel of the morphism from $T^{n-1}I$ to $T^{n-1}C$.

Of course, *a priori*, $\varphi_A^n$ is dependent on the choice of the monomorphism $u$, and so it's not clear how natural it is. As it turns out, it's very natural indeed. To see this, we'll do something more general. Choose another object $A'$ in $\mathscr{C}$ and let $u' : A' \to I'$ be a monomorphism into an injective object $I'$ such that $T^n(u') = 0$, and let $C' = \operatorname{coker} u'$. Suppose we have a morphism $f : A \to A'$; then, by a baby version of (1.4.5), we can extend $f$ to a morphism $g : I \to I'$, thus obtaining the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "f"'] & I \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] & I' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

where of course $h$ is the morphism induced by $f$ and $g$.

We then have the following cube diagram, all of whose faces except the vertical one facing east are known to be commutative:

$$\begin{tikzcd}
T^{n-1}C \arrow[rr, "\delta^{n-1}"] \arrow[dr, "T^{n-1}h"] \arrow[dd, "\varphi_C^{n-1}"'] & & T^nA \arrow[dr, "T^nf"] \arrow[dd, "\varphi_A^n"' near start] & \\
& T^{n-1}C \arrow[rr, "\delta^{n-1}" near start] \arrow[dd, "\varphi_{C'}^{n-1}"' near start] & & T^nA' \arrow[dd, "\varphi_{A'}^n"] \\
S^{n-1}C \arrow[rr, "\delta^{n-1}" near end] \arrow[dr, "S^{n-1}h"'] & & S^nA \arrow[dr, "S^nf"'] & \\
& S^{n-1}C' \arrow[rr, "\delta^{n-1}"] & & S^nA
\end{tikzcd}$$

Chasing this commutative diagram, it's not hard to see that we have

$$\varphi_{A'}^n(T^nf)\delta^{n-1} = (S^nf)\varphi_A^n\delta^{n-1}.$$

Since $\delta^{n-1}$ is an epimorphism, this shows both that $\varphi^n_A$ is independent of the choice of the monomorphism $u$ (take $f$ to be $1_A$), and that it actually gives us a natural transformation from $T^n$ to $S^n$.

So we've inductively constructed natural transformations $\varphi^n_A : T^n \to S^n$, for all $n \geq 0$. There is just a little more work to go before we can be sure that this is a morphism of $\delta$-functors. Fix $n \geq 0$ and suppose we have an exact sequence $0 \to C^1 \to C^2 \to C^3 \to 0$. Choose a monomorphism $u : C^1 \to I$ with $I$ injective so that $T^n(u) = 0$, and let $C = \operatorname{coker} u$. We then obtain the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C^1 \arrow[r] \arrow[d, equal] & C^2 \arrow[r] \arrow[d, dotted, "g"'] & C^3 \arrow[r] \arrow[d, dotted, "h"'] & 0 \\
0 \arrow[r] & C^1 \arrow[r, "u"] & I \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

where $g$ is obtained by the injectivity of $I$ and $h$ by the universal property of $C^3$ as cokernel.

We obtain another cube diagram; this time all the faces except the top face are commutative.

$$\begin{tikzcd}
T^nC^3 \arrow[rr, "\delta^n"] \arrow[dr, "\varphi^n_{C^3}"'] \arrow[dd, "T^nh"'] & & T^{n+1}C^1 \arrow[dr, "\varphi^{n+1}_{C^1}"'] \arrow[dd] & \\
& S^nC^3 \arrow[rr, "\delta^n"] \arrow[dd, "S^nh"'] & & S^{n+1}C^1 \arrow[dd] \\
T^nC \arrow[rr, "\delta^n"] \arrow[dr, "\varphi^n_C"'] & & T^{n+1}C^1 \arrow[dr, "\varphi^{n+1}_{C^1}"'] & \\
& S^nC \arrow[rr, "\delta^n"] & & S^{n+1}C^1
\end{tikzcd}$$

By a similar argument, using again the fact that $\delta^n : T^nC \to T^{n+1}C^1$ is an epimorphism, we find that the top face does indeed commute, thus showing that $\varphi^\bullet : T^\bullet \to S^\bullet$ does indeed define a morphism of $\delta$-functors. $\square$

delta-cohom-universal

COROLLARY 2.1.10. *Let $\mathscr{C}$ be an abelian category with enough injectives (resp. enough projectives); then $H^\bullet : \mathrm{Ch}^{\geq 0}\mathscr{C} \to \mathscr{C}$ (resp. $H^\bullet : \mathrm{Ch}^{\leq 0}\mathscr{C} \to \mathscr{C}$) is a universal $\delta$-functor.*

PROOF. As always, we give a proof only of the cohomological version. By the Theorem above, it suffices to prove that $H^\bullet$ is effaceable. Let $C^\bullet$ be a chain complex; then, by (1.5.5), there is a monomorphism $u : C^\bullet \hookrightarrow I^\bullet$ with $I^\bullet$ a split exact complex of injectives. Then, since $H^n(I) = 0$, for all $n \in \mathbb{Z}$, we see immediately that $H^\bullet$ is indeed effaceable and is thus universal. $\square$

delta-contravariant

REMARK 2.1.11 (Contravariant $\delta$-functors). There is also the notion of a contravariant $\delta$-functor. A *homological (resp. cohomological) contravariant $\delta$-functor* $T^\bullet : \mathscr{C} \to \mathscr{D}$ between abelian categories is simply a cohomological (resp. homological) $\delta$-functor $T^\bullet : \mathscr{C}^{op} \to \mathscr{D}$. Put more concretely, it is a collection of additive contravariant functors $T^n : \mathscr{C} \to \mathscr{D}$ and natural transformations $\delta^n : T^n O^1 \to T^{n-1} O^3$, which associate the appropriate long exact sequence to each short exact sequence in $\mathscr{C}$. Universality of such $\delta$-functors also has an analogous definition.

## 2. Derived Functors

The main result of this section is the following theorem.

e-derived-functors-right

THEOREM 2.2.1. *Let $F : \mathscr{C} \to \mathscr{D}$ be a left exact functor between abelian categories, and suppose $\mathscr{C}$ has enough injectives. Then there exists a universal $\delta$-functor $R^\bullet F : \mathscr{C} \to \mathscr{D}$ such that the following conditions hold:*

(1) *$F \cong R^0 F$.*
(2) *For $n > 0$ and for any injective object $I$, we have $R^n F(I) = 0$.*

PROOF. We will proceed in steps:

**Construction**: Given an object $A$ in $\mathscr{C}$, take any injective resolution $I^\bullet$ of $A$ and define $R^n F(A) = H^n(F(I^\bullet))$. Given any other injective resolution $J^\bullet$ of any other object $A'$, and a morphism there is a chain homotopy $f : I^\bullet \to J^\bullet$ (1.4.7) unique again upto chain homotopy. Now we see that $\mathrm{Ch}\, Ff : \mathrm{Ch}\, FI^\bullet \to \mathrm{Ch}\, FJ^\bullet$ is also a chain homotopy that is unique upto chain homotopy. Hence $R^n F(A)$ is independent of choice of injective resolution upto canonical isomorphism.

Now, given any other injective resolution $J^\bullet$ of any other object $A'$, and a morphism $g : A \to A'$, there is, according to (1.4.5), an extension $g' : I^\bullet \to J^\bullet$ that is unique up to chain homotopy. But then $R^n F(g) = H^n(\mathrm{Ch}\, F(g')) : R^n F(A) \to R^n F(A')$ is determined again upto unique isomorphism. Moreover, if $f : A \to A'$ and $g : A' \to A''$ are two morphisms and $\widetilde{f}$ is an extension of $f$ between injective resolutions $I^\bullet$ and $J^\bullet$ of $A$ and $A'$, respectively, and $\widetilde{g}$ is an extension of $g$ between injective resolutions $J^\bullet$ and $K^\bullet$ of $A'$ and $A''$, respectively, then $\widetilde{g}\widetilde{f}$ is an extension of $gf : A \to A''$ between injective resolutions, and so we find that $R^n F(gf) = R^n F(g) R^n F(f)$. Hence $R^n F$ is indeed a functor, for all $n \geq 0$.

By construction, it's evident that, for an injective object $I$, $R^n F(I) = 0$, for $n > 0$. It remains to show that $R^0 F \cong F$. Suppose $f : A \to A'$ is a morphism in $\mathscr{C}$; then let $I^\bullet$ and $J^\bullet$ be injective resolutions of $A$ and $A'$, and let $g : I^\bullet \to J^\bullet$ be an extension of $f$. We then see that $R^0 F(f) = H^0(Fg) \cong Fg^0 = Ff$.

**Additivity**: Suppose $0 : A \to A'$ is the 0 morphism; then, an extension of it between injective resolutions of $A$ and $A'$ is of course again the 0 morphism, which then induces trivial maps on cohomology. Thus we find that $R^n F(0) = 0$. If $f, g : A \to A'$ are two morphisms, and if $\widetilde{f}, \widetilde{g}$ are extensions of $f$ and $g$, respectively, to injective resolutions of $A$ and $A'$, then $\widetilde{f} + \widetilde{g}$ is an extension of $f + g$, and so we find that $R^n F$ preserves sums of morphisms.

**$\delta$-functoriality**: Suppose we have an exact sequence in $\mathscr{C}$: $0 \to C_1 \to C_2 \to C_3 \to 0$. Let $I_1^\bullet$ and $I_3^\bullet$ be injective resolutions of $C_1$ and $C_3$ respectively. Then, by the Horseshoe Lemma (**??**), there is an injective resolution $I_2^\bullet$ for $C_2$ such that the sequence

$$0 \to I_1^\bullet \to I_2^\bullet \to I_3^\bullet \to 0$$

is split exact. In this case, since $F$ is additive, we find that we have an exact sequence

$$0 \to \operatorname{Ch} F(I_1^\bullet) \to \operatorname{Ch} F(I_2^\bullet) \to \operatorname{Ch} F(I_3^\bullet) \to 0$$

of complexes over $\mathscr{D}$. Taking the long exact sequence of cohomology of this sequence gives us the morphisms $\delta^n : R^nF(C_3) \to R^{n+1}F(C_1)$, for all $n \in \mathbb{N}$.

It remains to check that these morphisms satisfy the required naturality conditions. So suppose we have another short exact sequence: $0 \to D_1 \to D_2 \to D_3 \to 0$, and another split exact sequence of injective resolutions $0 \to J_1^\bullet \to J_2^\bullet \to J_3^\bullet \to 0$ attached to this sequence as above. Suppose we have a morphism $(\alpha, \beta, \gamma)$ in $\operatorname{Exact}(\mathscr{C})$ from the first short exact sequence to this one. Then we can extend $\alpha$ and $\gamma$ to morphisms $\widetilde{\alpha} : I_1^\bullet \to J_1^\bullet$ and $\widetilde{\gamma} : I_3^\bullet \to J_3^\bullet$. Now, using (**??**), we can find a morphism $\widetilde{\beta} : I_2^\bullet \to J_2^\bullet$, such that the following diagram commutes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I_1^\bullet \arrow[r] \arrow[d, "\widetilde{\alpha}"'] & I_2^\bullet \arrow[r] \arrow[d, "\widetilde{\beta}"'] & I_3^\bullet \arrow[r] \arrow[d, "\widetilde{\gamma}"'] & 0 \\
0 \arrow[r] & J_1^\bullet \arrow[r] & J_2^\bullet \arrow[r] & J_3^\bullet \arrow[r] & 0
\end{tikzcd}$$

Now showing that $RF^\bullet$ is a $\delta$-functor reduces to the showing that $H^\bullet : \operatorname{Ch}^{\geq 0} \mathscr{D} \to \mathscr{D}$ is a $\delta$-functor, which is what we did in (2.1.4).

**Universality**: We will show that $R^nF$ is effaceable for every $n > 0$; universality will then follow from (2.1.9). Given any object $A$ in $\mathscr{C}$ take any monomorphism $u : A \to I$, with $I$ injective. Since $R^nF(I) = 0$, for $n > 0$, we have $R^nF(u) = 0$, for $n > 0$, and thus $R^nF$ is effaceable, for all $n > 0$. □

DEFINITION 2.2.2. The functors $R^nF$ associated to $F$ as in the Theorem above are called the *right derived functors* of $F$.

The dual statement is the following, which we will not prove.

ce-derived-functors-left

THEOREM 2.2.3. *Let $F : \mathscr{C} \to \mathscr{D}$ be a right exact functor between abelian categories, and suppose $\mathscr{C}$ has enough projectives. Then there exists a universal homological $\delta$-functor $L^\bullet F : \mathscr{C} \to \mathscr{D}$ such that the following conditions hold:*

(1) *$F \cong L^0F$.*
(2) *For $n < 0$ and for any projective object $P$, we have $L^nF(P) = 0$.*

DEFINITION 2.2.4. The functors $L^nF$ associated to $F$ as in the Theorem above are called the *left derived functors* of $F$.

iversal-derived-functors

COROLLARY 2.2.5. *Let $T^\bullet : \mathscr{C} \to \mathscr{D}$ be a universal cohomological (resp. homological) $\delta$-functor, and suppose $\mathscr{C}$ has enough injectives (resp. projectives). Then $T^\bullet \cong R^\bullet T^0$ (resp. $T^\bullet \cong L^\bullet T^0$).*

PROOF. First note that $T^0$ is a left exact functor, and so $R^\bullet T^0$ is defined and is moreover a universal $\delta$-functor with $R^0T^0 \cong T^0$. Since upto isomorphism there is a unique universal $\delta$-functor $S^\bullet : \mathscr{C} \to \mathscr{D}$ with $S^0 \cong T^0$, we find $R^\bullet T^0 \cong T^0$.

The homological case is formally the same. □

delta-derived-exact

COROLLARY 2.2.6. *Let $F : \mathscr{C} \to \mathscr{D}$ be an exact functor, and suppose that $\mathscr{C}$ has enough injectives (resp. enough projectives); then $R^nF = 0$, for $n > 0$ (resp. $L^nF = 0$, for $n < 0$).*

PROOF. Follows from (2.2.5) and (2.1.7). $\square$

REMARK 2.2.7 (The Contravariant Case). Given a left exact contravariant functor $F : \mathscr{C}^{op} \to \mathscr{D}$, and supposing $\mathscr{C}$ has enough projectives (and thus $\mathscr{C}^{op}$ has enough injectives) we can define in exactly the same fashion the right derived functors $R^n F$ of $F$. These will then form a contravariant homological universal $\delta$-functor from $\mathscr{C}$ to $\mathscr{D}$.

The next result gives us a useful way to obtain long exact sequences.

elta-long-exact-sequence

PROPOSITION 2.2.8. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories, and suppose $F_1 \to F_2 \to F_3$ is a complex of left exact functors in* $\operatorname{Funct}(\mathscr{C}, \mathscr{D})$. *Now, suppose that $\mathscr{C}$ has enough injectives, and that, for every injective object $I$ in $\mathscr{C}$, the sequence*

$$0 \to F_1(I) \to F_2(I) \to F_3(I) \to 0$$

*is exact. Then, for every object $C$ in $\mathscr{C}$, we have a long exact sequence:*

$$0 \to F_1(C) \to F_2(C) \to F_3(C) \to R^1 F_1(C) \to R^1 F_2(C) \to R^1 F_3(C) \to \dots$$

PROOF. Given an object $C$ in $\mathscr{C}$, let $I^\bullet$ be an injective resolution of $C$. Now, we have an exact sequence of chain complexes in $\mathscr{D}$:

$$0 \to \operatorname{Ch} F_1(I^\bullet) \to \operatorname{Ch} F_2(I^\bullet) \to \operatorname{Ch} F_3(I^\bullet) \to 0.$$

Taking the long exact sequence of cohomology associated to this sequence gives us the result. $\square$

## 3. $F$-acyclicity, $F$-syzygies and $F$-dimension

NOTE ON NOTATION 2. From now on we will assume that all our domain categories have enough injectives (or projectives, as the case may be).

DEFINITION 2.3.1 ($F$-acyclicity). Let $F : \mathscr{C} \to \mathscr{D}$ be a left (resp. right) exact functor between abelian categories. Then an object $A$ in $\mathscr{C}$ is said to be *$F$-acyclic* if $R^n F(A) = 0$, for all $n > 0$ (resp. $L^n F(A) = 0$, for all $n < 0$).

Given a left (resp. right) exact functor $F : \mathscr{C} \to \mathscr{D}$, and an object $A$ in $\mathscr{C}$, an *$F$-acyclic resolution* of $A$ is a right (resp. left) $\mathscr{K}$-resolution of $A$, where $\mathscr{K}$ is the class of $F$-acyclic objects.

REMARK 2.3.2. Observe that, given a left (resp. right) exact functor $F : \mathscr{C} \to \mathscr{D}$, every injective (resp. projective) object is $F$-acyclic.

We now present a criterion for a class of objects to be $F$-acyclic, for some left exact functor $F$.

a-f-acyclicity-criterion

PROPOSITION 2.3.3. *Let $F : \mathscr{C} \to \mathscr{D}$ be a left exact functor between abelian categories, and suppose $\mathscr{K}$ is a class of objects in $\mathscr{C}$ satisfying the following properties:*

(1) *For every object $C \in \mathscr{K}$, there exists a monomorphism $u : C \to I$, such that $I$ is $F$-acyclic, and such that* $\operatorname{coker} u$ *is again in $\mathscr{K}$.*
(2) *For every exact sequence*

$$0 \to C_1 \to C_2 \to C_3 \to 0,$$

*in $\mathscr{C}$ with $C_1 \in \mathscr{K}$, the sequence*

$$0 \to F(C_1) \to F(C_2) \to F(C_3) \to 0$$

*is also exact in $\mathscr{D}$.*

*Then every object in $\mathscr{K}$ is $F$-acyclic.*

REMARK 2.3.4. There is the evident dual version for right exact functors (begin by replacing monomorphism with epimorphism in (1) and $C_1$ with $C_3$ in condition (2)). We will use it without comment when the need arises (which is unlikely).

PROOF. We will do this by induction. That is, we will show that $R^1F(C) = 0$, for all $C \in \mathscr{K}$ and then we will show that if, for $n > 1$, $R^{n-1}F(C) = 0$, for all $C \in \mathscr{K}$, then in fact $R^nF(C) = 0$, for all $C \in \mathscr{K}$.

Pick an object $C \in \mathscr{K}$, and let $u : C \to I$ be the monomorphism into an $F$-acyclic object guaranteed to us by condition (1), and set $C' = \operatorname{coker} u$. By condition (2), we have a short exact sequence

$$0 \to F(C) \to F(I) \to F(C') \to 0,$$

which, since $I$ is $F$-acyclic, implies that $R^1F(C) = 0$ (using the long exact sequence of derived functors of $F$). Since $C$ was arbitrary, this finishes the base step of the induction.

Maintaining the notation of the previous paragraph, observe now that we have (again from the long exact sequence of derived functors and the $F$-acyclicity of $I$):

$$R^{n-1}F(C') \cong R^nF(C), \quad \text{for all } n > 2.$$

By condition (2), $C'$ is also in $\mathscr{K}$, and so by the induction step, we find that $R^nF(C) = 0$, for all $n > 0$. This finishes our proof. □

Here's another situation where we get acyclicity of objects.

cyclic-finite-filtration

PROPOSITION 2.3.5. *Let $F : \mathscr{C} \to \mathscr{D}$ be a left exact functor between abelian categories, and let $A$ be an object in $\mathscr{C}$ equipped with a finite filtration*

$$A = F^0A \supset F^1A \supset \ldots \supset F^{n+1}A \supset F^nA = 0$$

*such that, for $1 \leq i \leq n$, $F^{i-1}A/F^iA$ is $F$-acyclic. Then $A$ is $F$-acyclic.*

PROOF. We'll do this by induction on the length $n$ of the filtration. If $n = 0$, then there is nothing to prove; so assume $n \geq 1$. By the induction step, $F^1A$ is $F$-acyclic. Moreover, we have a short exact sequence

$$0 \to F^1A \to A \to A/F^1A \to 0,$$

where $F^1A$ and $A/F^1A$ are both $F$-acyclic. It's easy to conclude now from the long exact sequence of derived functors of $F$ associated to this short exact sequence that $A$ is also $F$-acyclic. □

DEFINITION 2.3.6 (Syzygies). Suppose $\mathscr{K}$ is a class of objects in $\mathscr{C}$, and suppose we have an exact sequence

$$0 \to A \to I^0 \to I^1 \to \ldots \to I^{n-1} \to N \to 0,$$

with $I^j$ in $\mathscr{K}$, for $0 \leq j \leq n$. Then $N$ is called a $n^{th}$ *right $\mathscr{K}$-syzygy* of $A$.

If instead we have an exact sequence

$$0 \to M \to P^{-n+1} \to P^{-n+2} \to \ldots \to P^0 \to A \to 0,$$

with $P^j$ in $\mathscr{K}$, for $-n \leq j \leq 0$. Then $M$ is called a $n^{th}$ *left $\mathscr{K}$-syzygy* of $A$.

If $F$ is a left (resp. right) exact functor from $\mathscr{C}$ to $\mathscr{D}$ and $\mathscr{K}$ is the class of $F$-acyclic objects in $\mathscr{C}$, then the $n^{th}$ right (resp. left) $\mathscr{K}$-syzygy of $A$ is called the $n^{th}$ $F$-syzygy of $A$.

ygies-dimension-shifting

Proposition 2.3.7. *Let $F : \mathscr{C} \to \mathscr{D}$ be a left exact functor, and let $A$ be an object in $\mathscr{C}$. Suppose $N$ is an $n^{th}$ $F$-syzygy of $A$; then we have:*

$$R^r F(A) \cong \begin{cases} R^{r-n}F(N) & \text{if } r \geq n+1 \\ \operatorname{coker}(F(I^{n-1}) \to F(N)) & \text{if } r = n. \end{cases}$$

*If instead $F : \mathscr{C} \to \mathscr{D}$ is a right exact functor, and if $M$ is an $n^{th}$ $F$-syzygy of $A$; then we have:*

$$L^r F(A) \cong \begin{cases} L^{r+n}F(N) & \text{if } r \leq -n-1 \\ \ker(F(M) \to F(P^{-n-1})) & \text{if } r = -n. \end{cases}$$

Proof. We'll only prove the first assertion; this will be done by induction on $n$. First suppose that $N$ is a first $F$-syzygy of $A$. In this case, we have a short exact sequence

$$0 \to A \to I^0 \to N \to 0.$$

If we now consider the long exact sequence in the derived functors $R^nF$ associated to this sequence, we obtain, using the $F$-acyclicity of $I^0$, isomorphisms

$$R^{r-1}F(N) \xrightarrow[\cong]{\delta^{r-1}} R^rF(A), \qquad \text{for } r \geq 2$$

For $r = 1$, we get an exact sequence

$$F(I^n) \to F(N) \to R^1F(A) \to 0.$$

Now the result follows by induction on $n$ and from the observation that if $N$ is an $n^{th}$ $F$-syzygy of $A$ and if $N'$ is the $(n-1)^{th}$ $F$-syzygy given by $\operatorname{im}(I^{n-2} \to I^{n-1})$, then $N$ is a first $F$-syzygy of $N'$. □

The next result is very useful for computations.

elta-acyclic-resolutions

Corollary 2.3.8. *Let $F : \mathscr{C} \to \mathscr{D}$ be a left exact functor, and let $A$ be an object in $\mathscr{C}$. Suppose $I^\bullet$ is an $F$-acyclic resolution of $A$; then we have $R^nF(A) = H^n(\operatorname{Ch} F(I^\bullet))$.*

*Dually, if $F : \mathscr{C} \to \mathscr{D}$ is a right exact functor, then we can compute $L^nF(A) = H^n(\operatorname{Ch} F(P^\bullet))$, for some $F$-acyclic resolution $P^\bullet$ of $A$.*

Proof. Let $N = \operatorname{im}(d^n : I^{n-1} \to I^n)$ be an $n^{th}$ $F$-syzygy of $A$. Then we find from the Proposition that, for $n \geq 1$, we have

$$\begin{aligned} R^nF(A) &= \operatorname{coker}(F(I^{n-1}) \to F(N)) \\ &= F(N)/\operatorname{im} F(d^{n-1}) \\ &= F(\ker d^n)/\operatorname{im} F(d^{n-1}) \\ &= \ker F(d^n)/\operatorname{im} F(d^{n-1}) \\ &= H^n(\operatorname{Ch} F(I^\bullet)). \end{aligned}$$

Since $F$ is left exact, we find $H^0(\operatorname{Ch} F(I^\bullet)) = F(A)$, as we want it to be. □

Definition 2.3.9. Let $F$ be a left exact functor $F : \mathscr{C} \to \mathscr{D}$, and let $A$ be an object in $\mathscr{C}$. The *$F$-dimension* of $A$, denote $F - \dim(A)$ is the quantity

$$\sup\{n : \ R^nF(A) = 0\}.$$

If $F$ is instead a right exact functor, then the $F$-dimension of $A$ is the quantity

$$\sup\{-n : \ L^nF(A) = 0\}.$$

If $F - \dim(A) < \infty$, we say that $A$ has *finite $F$-dimension*. Observe that $F - \dim(A) = 0$ if and only if $A$ is $F$-acyclic.

Let $\mathscr{K}$ be a class of objects in $\mathscr{C}$; then the *length* of a $\mathscr{K}$-resolution $I^\bullet$ is the quantity

$$\sup\{|n| : \ I^n \neq 0\}$$

Given an object $A$ and a class of objects $\mathscr{K}$ in $A$, we define the *right (resp. left) $\mathscr{K}$-dimension*, $\mathscr{K} - \mathrm{rdim}(A)$ (resp. $\mathscr{K} - \mathrm{ldim}(A)$), of $A$ as the supremum of the lengths of all right (resp. left) $\mathscr{K}$-resolutions of $A$.

`delta-f-dimension`

PROPOSITION 2.3.10. *Let $F : \mathscr{C} \to \mathscr{D}$ be a left (resp. right) exact functor, and let $\mathscr{K}$ be the class of $F$-acyclic objects. Then the following are equivalent for an object $A$ in $\mathscr{C}$:*

1. $F - \dim(A) = n < \infty$.
2. *For every integer $r \geq 0$, and every $r^{th}$ $F$-syzygy $N$ of $A$, we have*
$$F - \dim(N) = \max\{n - r, 0\}.$$
3. *Every $n^{th}$ $F$-syzygy of $A$ is $F$-acyclic and no $(n-1)^{th}$ $F$-syzygy of $A$ is $F$-acyclic.*
4. *Some $n^{th}$ $F$-syzygy of $A$ is $F$-acyclic and some $(n-1)^{th}$ $F$-syzygy of $A$ is not $F$-acyclic.*
5. $\mathscr{K} - \mathrm{rdim}(A) = n < \infty$ *(resp.* $\mathscr{K} - \mathrm{ldim}(A) = n < \infty$*).*

PROOF. We'll do the case where $F$ is left exact. Observe first that if there exists an $F$-acyclic resolution $I^\bullet$ of $A$ of length $r$, then we have:

$$R^s F(A) = H^s(\mathrm{Ch}\, F(I^\bullet)) = 0 \qquad \text{for } s \geq r+1$$

This shows $F - \dim(A) \leq \mathscr{K} - \mathrm{rdim}(A)$.

Next, observe that there exists an $F$-acyclic $r^{th}$ $F$-syzygy of $A$ if and only if there exists an $F$-acyclic resolution of $A$ of length $r$. Thus the existence of an $F$-acyclic $r^{th}$ $F$-syzygy implies

$$\mathscr{K} - \mathrm{rdim}(A) \leq r.$$

Now, let $N$ be an $r^{th}$ $F$-syzygy of $A$, and suppose $F - \dim(A) = n$; then by (2.3.7) we have

$$R^s F(N) = R^{s+r} F(A) \qquad \text{for } s \geq 1$$

So if $r \geq n$, then $R^s F(N) = 0$, for all $s \geq 1$, which means that $N$ is $F$-acyclic. On the other hand, if $r < n$, we have:

$$R^s F(N) = \begin{cases} 0 & \text{for } s \geq n - r + 1 \\ R^n F(A) \neq 0 & \text{for } s = n - r. \end{cases}$$

This shows $F - \dim(N) = n - r$. In particular, an $r^{th}$ $F$-syzygy of $A$ is $F$-acyclic if and only if $r \geq n$. Therefore, if $F - \dim(A) = n$, then there exists an $F$-acyclic $n^{th}$ $F$-syzygy of $A$, and so we find

$$\mathscr{K} - \mathrm{rdim}(A) \leq n = F - \dim(A) \leq \mathscr{K} - \mathrm{rdim}(A).$$

This shows the equivalence between all the statements in the Proposition. $\square$

CHAPTER 3

# Spectral Sequences

chap:spectral

Unless otherwise noted, all our categories will be abelian categories satisfying axioms Ab-4 and Ab-4*.

## 1. Lots of Definitions and a Proposition

DEFINITION 3.1.1. A *filtration* $F^\bullet A$ of an object $A$ in a category $\mathscr{C}$ is a collection $\{F^r A : r \in \mathbb{Z}\}$ of subobjects of $A$ such that $F^r A \supset F^{r+1} A$, for $r \in \mathbb{Z}$.

A filtration $F^\bullet A$ is *exhaustive* if $\bigcup_{r\in\mathbb{Z}} F^r A = A$.

It is *separated* if $\bigcap_{r\in\mathbb{Z}} F^r A = 0$.

It is *complete* if $A \cong \lim_{\longleftarrow} A/F^r A$; it's immediate that a complete filtration is separated.

DEFINITION 3.1.2. A *filtered object* over a category $\mathscr{C}$ is a pair $(A, F^\bullet A)$, where $F^\bullet A$ is a filtration on $A$.

We will say that $(A, F^\bullet A)$ is *separated (resp. complete, resp. exhaustive* if the filtration $F^\bullet A$ is separated (resp. complete, resp. exhaustive).

Rather abusively, we will in this context say that $A$ is separated (or complete or exhaustive), leaving the filtration implicit in our assertion.

DEFINITION 3.1.3. Given a filtered object $(A, F^\bullet A)$, we consider the object $A' = A/\bigcap_{r\in\mathbb{Z}} F^r A$. This has a natural filtration $F^\bullet A'$ on it induced by the filtration on $A$. We call the filtered object $(A', F^\bullet A')$ the *separation* of $A$.

Given a filtration $F^\bullet A$ on $A$, we define the *completion* of $A$ by $\hat{A} = \lim_{\longleftarrow} A/F^r A$, and equip it with the filtration given by $F^r \hat{A} = \widehat{F^r} A$. We have

$$A/F^r A \cong A'/F^r A' \cong \hat{A}/F^r \hat{A} \qquad \text{for } r \in \mathbb{Z}.$$

From this it follows that $\hat{A} = \widehat{A'}$ is complete with its equipped filtration.

From now on we will concentrate on filtrations of chain complexes over $\mathscr{C}$.

DEFINITION 3.1.4. A filtration $F^\bullet C^\bullet$ is *bounded below* if, for every $n \in \mathbb{Z}$, there exists $s \in \mathbb{Z}$ such that $F^r C^n = F^s C^n$, for all $r \geq s$. It is *bounded above* if, for every $n \in \mathbb{Z}$, there exists $s \in \mathbb{Z}$ such that $F^r C^n = F^s C^n$, for all $r \leq s$. A filtration that is bounded both above and below is said to be *bounded* or *finite.* A filtration is *canonically bounded* if $F^r C^n = 0$, for $r \geq n+1$, and $F^0 C^n = C^n$.

Again in this situation we may conflate the chain complex $C^\bullet$ and the filtered chain complex $(C^\bullet, F^\bullet C^\bullet)$, when the filtration is clear from the context.

DEFINITION 3.1.5. A *spectral sequence* over a category $\mathscr{C}$ is a collection

$$\{E_r^\bullet : r \geq a, \text{ for some } a \in \mathbb{Z}\}$$

of chain complexes over $\mathscr{C}$ satisfying the following conditions:

(1) For $r \geq a$, $n \in \mathbb{Z}$, there exists a direct sum decomposition $E_r^n = \oplus_{p+q=n} E_r^{p,q}$ such that $d_r^{p,q} = d_r^n|_{E_r^{p,q}}$ maps $E_r^{p,q}$ into $E_r^{p+r,q-r+1}$.
(2) There exist isomorphisms

$$\varphi_r^{p,q} : H^{p,q}(E_r) = \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q-r+1} \xrightarrow{\cong} E_{r+1}^{p,q}.$$

Remark 3.1.6. $E_r^\bullet$ is called the $r^{th}$ *page* of the spectral sequence, and condition (1) above says that, if we visualize $E_r^\bullet$ as given by a plane lattice with $E_r^{p,q}$ at the $(p,q)$-position, then every line of slope $-(r-1)/r$ forms a chain complex by itself. The object $H^{p,q}(E_r)$ is nothing but the cohomology of this chain complex at the $(p,q)$-position. Observe that these lines get steeper with each subsequent page.

Definition 3.1.7. A *morphism* between two spectral sequences $\{E_r^\bullet :\ r \geq a\}$ and $\{\widetilde{E}_r^\bullet :\ r \geq b\}$ is a collection $\{f_r :\ r \geq \max\{a,b\}\}$ of chain maps $f_r : E_r^\bullet \to \widetilde{E}_r^\bullet$ satisfying the following conditions:

(1) For $p,q \in \mathbb{Z}$, and $r$ large enough, $f_r^{p,q} = f_r^{p+q}|_{E_r^{p,q}}$ maps into $\widetilde{E}_r^{p,q}$.
(2) For $p,q \in \mathbb{Z}$, and $r$ large enough, the following diagram commutes:

$$\begin{tikzcd}
H^{p,q}(E_r) \arrow[r, "\varphi^{p,q}"] \arrow[d, "H^{p,q}(f_r)"'] & E_{r+1}^{p,q} \arrow[d, "f_{r+1}"] \\
H^{p,q}(\widetilde{E}_r) \arrow[r, "\varphi^{p,q}"] & \widetilde{E}_{r+1}^{p,q}.
\end{tikzcd}$$

It's easy to check now that this gives us a category of spectral sequences over $\mathscr{C}$, which we will denote by $\mathrm{Sp}(\mathscr{C})$.

Definition 3.1.8 (The Limit Page). Let $\{E_r^\bullet :\ r \geq a\}$ be a spectral sequence over $\mathscr{C}$. Then, for $s,t \in \mathbb{Z}$, $s \geq t$, $E_s^{p,q}$ is a subquotient of $E_t^{p,q}$. This lets us find collections $\{B_r^{p,q}\}$ and $\{Z_r^{p,q}\}$ of $E_a^{p,q}$ such that $E_r^{p,q} \cong Z_r^{p,q}/B_r^{p,q}$ and such that we have a filtration

$$0 = B_a^{p,q} \subset \ldots B_r^{p,q} \subset \ldots \subset Z_r^{p,q} \subset \ldots \subset Z_a^{p,q} = E_a^{p,q}.$$

We define

$$\left(B_\infty^{p,q} = \bigcup_{r\geq a} B_r^{p,q} \quad \text{and} \quad Z_\infty^{p,q} = \bigcap_{r \geq a} Z_r^{p,q},\right)$$

and we set $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$.

Remark 3.1.9. Suppose now that we have two spectral sequences $\{E_r^\bullet :\ r \geq a\}$ and $\{E_r'^\bullet :\ r \geq b\}$ and a morphism $f$ from the first to the second. Replacing $a,b$ with $\max\{a,b\}$, we can assume that $a = b$. Now, since $f_a$ is a chain morphism, we have that $Z_a^{p,q}(E)$ maps into $Z_a^{p,q}(E')$ and that $B_a^{p,q}(E)$ maps into $B_a^{p,q}(E')$. Inductively from here, using the commutativity condition on $f$, we show that $Z_r^{p,q}(E)$ maps into $Z_r^{p,q}(E')$, for all $r \geq a$, and analogously for $B_r^{p,q}$. This gives us an induced morphism $E_\infty^{p,q}(E) \to E_\infty'^{p,q}(E')$. Thus the assignment of the doubly graded object $\{E_\infty^{p,q}\}$ to the spectral sequence $\{E_r^\bullet\}$ is functorial.

Definition 3.1.10 (Boundedness conditions). Observe now that to each spectral sequence $\{E_r^\bullet : r \geq a\}$, we can associate a subset $S(E) \subset \mathbb{Z} \times \mathbb{Z}$ of the plane lattice by the formula $(p,q) \in S(E) \Leftrightarrow E_a^{p,q} \neq 0$. A spectral sequence $\{E_r^\bullet :\ r \geq a\}$ is *bounded below, bounded above, bounded, first quadrant, etc.* if $S(E)$ is bounded below, bounded above, bounded, etc.

A spectral sequence $\{E_r^\bullet : \ r \geq a\}$ is *regular* if, for all pairs $p, q \in \mathbb{Z}$, there is $s \in \mathbb{Z}$, such that $d_r^{p,q} = 0$, for all $r \geq s$. It is clear that this can occur if and only if $Z_\infty^{p,q} = Z_s^{p,q}$.

A spectral sequence $\{E_r^\bullet : \ r \geq a\}$ *collapses* if there exists $r \geq a$ such that $E_r^\bullet$ has only one non-zero row or only one non-zero column.

REMARK 3.1.11. We see immediately that a bounded below spectral sequence is regular. Essentially the differentials get steeper and steeper till they fall off the chart.

ltration-to-spectral-seq

PROPOSITION 3.1.12. *A filtered chain complex $(C^\bullet, F^\bullet C^\bullet)$ naturally determines a spectral sequence $\{E_r^\bullet(C) : \ r \geq 0\}$ starting with $E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}$. More precisely, the assignment of the spectral sequence $\{E_r^\bullet(C)\}$ to the filtered complex $(C^\bullet, F^\bullet C^\bullet)$ gives rise to a functor from the category of filtered chain complexes over $\mathscr{C}$ to the category $\mathrm{Sp}(\mathscr{C})$ of spectral sequences over $\mathscr{C}$.*

NOTE ON NOTATION 3. For the purposes of this proof, we will index all our objects by the double index $(p, n)$, where $n = p + q$, instead of the double index $(p, q)$.

PROOF. We begin by setting $E_0^{p,n} = F^pC^n/F^{p+1}C^n$. Next, for every pair $(p, n)$ and every $r \geq 0$, we define:

$$A_r^{p,n} = d^{-1}\left(F^{p+r}C^{n+1}\right) \cap F^pC^n.$$

Observe that we have

$$A_r^{p,n} \cap F^{p+1}C^n = A_{r-1}^{p+1,n}.$$

We have the following useful relations:

$$A_r^{p,n} \subset A_{r+1}^{p,n}$$
$$d(A_r^{p,n}) \subset A_s^{p+r,n+1},$$

for all $p, n, r, s$. Now, let $\eta^{p,n} : \ F^pC^n \to E_0^{p,n}$ be the natural surjection, and define the following subobjects of $E_0^{p,n}$:

$$Z_r^{p,n} = \eta^{p,n}\left(A_r^{p,n}\right) \text{ and}$$
$$B_r^{p,n} = \eta^{p,n}\Big(d\left(A_{r-1}^{p-r+1,n-1}\right)\Big).$$

It's clear that we have a chain

$$0 = B_0^{p,n} \subset \ldots B_r^{p,n} \subset \ldots \subset Z_r^{p,n} \subset \ldots \subset Z_a^{p,n} = E_0^{p,n}.$$

We now set $E_r^{p,n} = Z_r^{p,n}/B_r^{p,n}$. Observe that we have:

$$E_r^{p,n} \cong \frac{A_r^{p,n} + F^{p+1}C^n}{d\left(A_{r-1}^{p-r+1,n-1}\right) + F^{p+1}C^n} \cong \frac{A_r^{p,n}}{d\left(A_{r-1}^{p-r+1,n-1}\right) + A_{r-1}^{p+1,n}}.$$

Now, using the differential $d$, we get a map $d_r^{p,n}$ from $E_r^{p,n}$ to $E_r^{p+r,n+1}$. What is the kernel of $d_r^{p,n}$? For this we note:

$$d^{-1}\left(d\left(A_{r-1}^{p+1,n}\right) + A_{r-1}^{p+r+1,n+1}\right) = A_{r-1}^{p+1,n} + A_{r+1}^{p,n}.$$

This tells us that we have

$$\ker d_r^{p,n} = \frac{A_{r-1}^{p+1,n} + A_{r+1}^{p,n}}{d\left(A_{r-1}^{p-r+1,n-1}\right) + A_{r-1}^{p+1,n}} \cong Z_{r+1}^{p,n}/B_r^{p,n}.$$

So the map $d_r^{p,n}$ factors through $Z_r^{p,n}/Z_{r+1}^{p,n}$. But we now have:

$$Z_r^{p,n}/Z_{r+1}^{p,n} \cong A_r^{p,n}/\left(A_{r+1}^{p,n} + A_{r-1}^{p+1,n}\right);$$
$$B_{r+1}^{p+r,n+1}/B_r^{p+r,n+1} \cong d\left(A_r^{p,n}\right)/d\left(A_{r+1}^{p,n} + A_{r-1}^{p+1,n}\right).$$

This shows that $\operatorname{im} d_r^{p,n} \cong B_{r+1}^{p+r,n+1}/B_r^{p+r,n+1}$, which shows that we have

$$H^{p,n}(E_r) = \ker d_r^{p,n}/\operatorname{im} d_r^{p-r,n-1} \cong Z_{r+1}^{p,n}/B_{r+1}^{p,n} = E_{r+1}^{p,n}.$$

Hence $\{E_r^\bullet : r \geq 0\}$ does give us a spectral sequence as claimed.

The functoriality of this construction is quite clear at this point. □

The next Proposition lists some properties of this construction. Since we would like as much as possible not to go back to the construction ever again, this collection of results will prove very useful.

pectral-filtrations-prps

PROPOSITION 3.1.13. *Let $(C^\bullet, F^\bullet C^\bullet)$ be a filtered chain complex, and let $E(C) = \{E_r^\bullet : r \geq 0\}$ be the spectral sequence associated to this filtered complex.*

(1) *If $F^\bullet C$ is bounded below (resp. bounded above, resp. bounded, canonically bounded) then $\{E_r^\bullet\}$ is bounded below (resp. bounded above, resp. bounded, resp. a first quadrant sequence).*
(2) *Fix $p \in \mathbb{Z}$ and $k \in \mathbb{N}$; let $C_{p,k} = C/F^{p+k}C$ and $C^{p,k} = F^{p-k}C/F^{p+k}C$. Then the natural maps $C \to C_{p,k} \leftarrow C^{p,k}$ induce isomorphisms*

$$E_r^{p,q}(C) \cong E_r^{p,q}(C_{p,k}) \cong E_r^{p,q}(C^{p,k}) \qquad 0 \leq r \leq k.$$

(3) *If $\widehat{C}$ is the completion of $C$, then $E(\widehat{C}) \cong E(C)$. The same statement is true if we replace $\widehat{C}$ with the separation $C'$ of $C$.*

PROOF. (1) This is quite obvious from the construction.
(2) Observe that we have, for $0 \leq r \leq k$,

$$A_r^{p,n}(C_{p,k}) = A_r^{p,n}(C^{p,k}) = A_r^{p,n}(C)/F^{p+k}C^n.$$

(3) Just observe that for all pairs $(p,k)$, we have

$$\left(C_{p,k} \cong \hat{C}_{p,k} \quad \text{and} \quad C^{p,k} \cong \hat{C}^{p,k}.\right)$$

A similar natural isomorphism holds with $\hat{C}$ replaced by $C'$. Now use part (2).

□

Now, consider the cohomology $H^\bullet(C)$ of the complex $C^\bullet$. Given a filtration $F^\bullet C^\bullet$ on $C^\bullet$, we get a natural filtration on $H^\bullet(C)$, given by $F^pH^n(C) = \operatorname{im}(H^n(F^pC) \to H^n(C))$. This filtration is exhaustive (resp. bounded below, resp. bounded above, resp. bounded, resp. separated), whenever the original filtration is exhaustive (resp. bounded below, resp. bounded above, resp. bounded, resp. separated). In particular this gives us a functor from the category of filtered chain complexes to the category of filtered graded objects over $\mathscr{C}$.

l-filtrations-cohomology

PROPOSITION 3.1.14. *Let $(C^\bullet, F^\bullet C^\bullet)$ be a separated and exhaustive filtered chain complex. In the notation of (3.1.12), set $A_\infty^{p,n} = \bigcap_{r\geq 0} A_r^{p,n}$, let $\eta^{p,n}$ be the natural map from $F^pC^n$ onto $E_0^{p,n}$, and set $e_\infty^{p,n} = \eta^{p,n}(A_\infty^{p,n})/B_\infty^{p,n}$.*

(1) $\ker(d : F^pC^n \to F^pC^{n+1}) = A_\infty^{p,n}$.

(2) $F^pH^n(C) \cong A_\infty^{p,n}/d\left(\bigcup_{r\geq 0} A_r^{p-r,n-1}\right)$.
(3) $F^pH^n(C)/F^{p+1}H^n(C) \cong e_\infty^{p,n}$.

PROOF. Since the filtration is separated,

$$\begin{aligned}\ker d|_{F^pC^n} &= d^{-1}\left(\bigcap_r F^rC^{n+1}\right) \cap F^pC^n \\ &= \bigcap_r \left(d^{-1}\left(F^rC^{n+1}\right) \cap F^pC^n\right) = A_\infty^{p,n}.\end{aligned}$$

This gives us the first assertion. Now observe that $F^pH^n(C)$ is just the image of $A_\infty^{p,n}$ in $H^n(C)$, so we'll be done if we compute the kernel of $A_\infty^{p,n} \to H^n(C)$. This is simply $\operatorname{im} d \cap F^pC^n$, which is of course $\bigcup_{r\geq 0} d\left(A_r^{p-r,n-1}\right)$. For the last assertion, observe that $F^pH^{n+1}(C)$ is just the image of $A_\infty^{p+1,n}$ in $H^n(C)$, and so we have

$$\begin{aligned}F^pH^n(C)/F^pH^{n+1}(C) &\cong A_\infty^{p,n}/\left(A_\infty^{p,n+1} + d\big(\bigcup_{r\geq 0} A_r^{p-r,n-1}\big)\right) \\ &\cong e_\infty^{p,n}.\end{aligned}$$

□

## 2. Convergence

We now come to the main purpose of spectral sequences as a computational tool.

DEFINITION 3.2.1 (Convergence). Let $\{E_r^\bullet : r \geq a\}$ be a spectral sequence over $\mathscr{C}$, and let $H^* = \{H^n : n \in \mathbb{Z}\}$ be a sequence of objects in $\mathscr{C}$. Let $F^\bullet H^*$ be a filtration of $H^*$ (treating $H^*$ as an object in the category $\prod_{n\in\mathbb{Z}} \mathscr{C}$). We now list the possible kinds of convergence in increasing order of niceness.

**Weak Convergence:** : $\{E_r^\bullet\}$ *weakly converges* to $H^*$ if, for every pair $(p,q)$, we have
$$E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}.$$

**Abutment:** : $\{E_r^\bullet\}$ *abuts to* or *approaches* $H^*$ if it weakly converges to $H^*$, and if the filtration $F^\bullet H^*$ is separated and exhaustive.

**Convergence:** : $\{E_r^\bullet\}$ *converges* to $H^*$ if it is regular, it approaches $H^*$, and if the filtration $F^\bullet H^*$ is complete.

**Bounded Convergence:** : $\{E_r^\bullet\}$ *boundedly converges* to $H^*$ if it is bounded, it converges to $H^*$, and if, for every $n \in \mathbb{Z}$, the filtration $F^\bullet H^n$ is finite. We denote this kind of convergence by $E_a^{p,q} \Rightarrow H^{p+q}$.

We make several remarks.

ounded-below-convergence

REMARK 3.2.2 (Bounded below sequences). Suppose $\{E_r^\bullet : r \geq a\}$ is a bounded below spectral sequence weakly converging to $H^*$. In this case, the filtration on $H^*$ is also bounded below and $\{E_r^\bullet\}$ is already regular. Therefore, for it to converge to $H^*$, it suffices to check that the filtration on $H^*$ is complete. But since the filtration is bounded below, for it to be complete, it in fact suffices for it to be separated. In other words, if $\{E_r^\bullet\}$ approaches $H^*$, then it in fact converges to $H^*$.

l-collapsing-convergence

Remark 3.2.3 (Collapsing sequences). Suppose $\{E_r^\bullet : \ r \geq a\}$ is a spectral sequence weakly converging to $H^*$, and suppose also that it collapses on page $r$, say on the row $q = s$. Then we have $H^n = E_r^{n-s,s}$, which allows us to completely recover $H^*$ from our spectral sequence.

rst-quadrant-convergence

Remark 3.2.4 (First quadrant sequences). Suppose $\{E_r^\bullet : \ r \geq a\}$ is a first quadrant spectral sequence converging to $H^*$. Then, since $E_\infty^{p,n-p} = 0$, for $p < 0$ and $p > n$, we find that the filtration on $H^*$ must be canonically bounded. That is we must have $F^r H^n = 0$, for $r \geq n+1$ and $F^n H^n = H^n$. Moreover we must have $F^n H^n \cong E_\infty^{n,0}$ and $H^n/F^1H^n \cong E_\infty^{0,n}$. Observe also that we have $Z^{0,n})_\infty \cong E_\infty^{0,n}$ and $E_a^{n,0}/B_\infty^{n,0} \cong E_\infty^{n,0}$.

Then the natural morphisms $E_a^{n,0} \to E_\infty^{n,0} \subset H^n$ and $H^n \to E_\infty^{0,n} \subset E_a^{0,n}$ are called the *edge morphisms.*

ral-euler-characteristic

Remark 3.2.5 (Euler Characteristic). Let $\{E_r^\bullet : \ r \geq a\}$ be a spectral sequence over $R$-mod, for some ring $R$, weakly converging to $H^*$, and suppose $E_s^\bullet$ is bounded for some $r \geq s$, thus giving us that $E_r^\bullet$ is bounded for all $r \geq s$. Moreover, suppose that $E_s^{p,q}$ has finite length, for all pairs $(p, q)$. Then we have:

$$\chi(E_r^\bullet) = \sum_{n\in\mathbb{Z}} (-1)^n l(H^n(E_r)) = \chi(E_{r+1}^\bullet) = \chi(E_\infty^\bullet).$$

We also have

$$l(H^n) = \sum_{p\in\mathbb{Z}} l(F^pH^n/F^{p+1}H^n) = \sum_{p\in\mathbb{Z}} l(E_\infty^{p,n-p}).$$

So we see that we have

$$\chi(E_r^\bullet) = \chi(E_\infty^\bullet) = \sum_{n\in\mathbb{Z}} (-1)^n l(H^n).$$

Definition 3.2.6. Let $\{E_r^\bullet : r \geq a\}$ and $\{E_r'^\bullet : \ r \geq a\}$ be two spectral sequences weakly converging respectively to $H^*$ and $H'^*$. Let $h : H^* \to H'^*$ be a morphism of filtered objects in $\prod_n \mathscr{C}$ and let $f : E \to E'$ be a morphism of spectral sequences. Let $\varphi^{p,n} : F^pH^n/F^{p+1}H^n \to F^pH'^n/F^{p+1}H'$ be the morphism induced by $h$ and let $\psi^{p,n} : E_\infty^{p,n} \to E_\infty'^{p,n}$ be the morphism induced by $f$. We say that $h$ is *compatible* with $f$ if the following diagram commutes for every pair $(p, n)$.

$$\begin{tikzcd}
E_\infty^{p,n} \arrow[r, "\psi^{p,n}"] \arrow[d, "\cong"'] & E_\infty'^{p,n} \arrow[d, "\cong"'] \\
F^pH^n/F^{p+1}H^n \arrow[r, "\varphi^{p,n}"] & F^pH'^n/F^{p+1}H'^n
\end{tikzcd}$$

After this somewhat lengthy prelude, we are ready to present our main result.

al-classical-convergence

Theorem 3.2.7 (Classical Convergence Theorem). *Let $(C^\bullet, F^\bullet C^\bullet)$ be an exhaustive, separated filtered complex over $\mathscr{C}$, and let $\{E_r^\bullet : \ r \geq 0\}$ be its associated spectral sequence. If $F^\bullet C^\bullet$ is bounded below (resp. bounded), then $\{E_r^\bullet\}$ converges (resp. boundedly converges) to $H^\bullet(C)$.*

Proof. As we observed earlier, if $F^\bullet C^\bullet$ is bounded below (resp. bounded), then the filtration induced on $H^\bullet(C)$ is also bounded below (resp. bounded). So, as observed in (3.2.2), it suffices to assume that $F^\bullet C$ is bounded below and to then show that it weakly converges to $H^\bullet(C)$. For this, from part (3) of (3.1.14), it's enough to show that $\eta^{p,n}(A^{p,n}_\infty) = Z^{p,n}_\infty$. But since $F^\bullet C^\bullet$ is bounded below, $A^{p,n}_\infty = A^{p,n}_r$, for $r$ large enough, and so $Z^{p,n}_\infty = Z^{p,n}_r$, for $r$ large enough, from which the result follows. □

EXAMPLE 3.2.8 (The Canonical Filtration). Let $C^\bullet$ be a complex over $\mathscr{C}$. The *canonical filtration* on $C^\bullet$ is given by $F^pC^\bullet = \tau_{\leq p}C^\bullet$. So we have

$$F^pC^n = \begin{cases} C^n & \text{if } n < p \\ \ker d^p & \text{if } n = p \\ 0 & \text{if } n > p. \end{cases}$$

We then have

$$H^n(F^pC^\bullet) = \begin{cases} H^n(C) & \text{if } n \leq p \\ 0 & \text{if } n > p. \end{cases}$$

Now, we have

$$F^pC^n/F^{p+1}C^n = \begin{cases} 0 & \text{if } n < p-1 \text{ or } n > p \\ C^{p-1}/\ker d^{p-1} & \text{if } n = p-1 \\ \ker d^p & \text{if } n = p \end{cases}$$

Hence we see that

$$E_1^{p,n-p} = \begin{cases} H^p(C) & \text{if } n = p \\ 0 & \text{otherwise.} \end{cases}$$

So the sequence collapses on page 1, and we get the cohomology of the chain complex, as we expected.

## 3. The Spectral Sequences associated to a Double Complex

Let $C^{\bullet,\bullet}$ be a double complex over $\mathscr{C}$. Then we have two natural filtrations on $C^{\bullet,\bullet}$, either by rows or by columns. These give rise to different spectral sequences. The aim of this section is to study the relationship between these two spectral sequences.

DEFINITION 3.3.1. Given a double complex $C^{\bullet,\bullet}$ we define, for all $p \in \mathbb{N}$, two subcomplexes of $\operatorname{Tot}^\bullet(C)$ by the formulas below (we speak here only of the restricted total complex).

$$(F_I^p \operatorname{Tot}^\bullet(C))^n = \bigoplus_{\substack{i+j=n \\ i \geq p}} C^{i,j}$$

$$(F_{II}^p \operatorname{Tot}^\bullet(C))^n = \bigoplus_{\substack{i+j=n \\ j \geq p}} C^{i,j}.$$

This gives us two natural filtrations of $\operatorname{Tot}^\bullet(C)$.

From these two natural filtrations, we get two natural spectral sequences, which we'll denote $\{{}^IE_r^\bullet(C)\}$ and $\{{}^{II}E_r^\bullet(C)\}$, respectively.

We make the following observations about these spectral sequences: On the $0^{th}$ page we have:

$$
\begin{aligned}
{}^{I}E_0^{p,q} &= C^{p,q} \\
{}^{II}E_0^{p,q} &= C^{q,p}.
\end{aligned}
$$

Observe, in particular, that the spectral sequences associated to $C^{\bullet,\bullet}$ satisfy the same boundedness conditions that $C^{\bullet,\bullet}$ (to be more precise, for $\{{}^{II}E_r^{\bullet}\}$, this is only true after a reflection through the origin).

Moreover, the differentials on the $0^{th}$ page are induced by the total differential on $\mathrm{Tot}^{\bullet}(C)$. Therefore, on the first page we have:

$$
\begin{aligned}
{}^{I}E_1^{p,q} &= H^q(C^{p,\bullet}) = H_{II}^{q,p}(C) \\
{}^{II}E_1^{p,q} &= H^q(C^{\bullet,p}) = H_{I}^{q,p}(C).
\end{aligned}
$$

The differentials here are again induced by the total differential on $\mathrm{Tot}^{\bullet}(C)$, and so we have on the second page:

$$
\begin{aligned}
{}^{I}E_2^{p,q} &= H_I^p(H_{II}^q(C)) \\
{}^{II}E_2^{p,q} &= H_{II}^p(H_I^q(C)).
\end{aligned}
$$

We record this in the next Proposition.

double-complex-sequences

PROPOSITION 3.3.2. *To each double complex $C^{\bullet,\bullet}$ over $\mathscr{C}$ we can naturally associate two spectral sequences $\{{}^{I}E_r^{\bullet}\}$ and $\{{}^{II}E_r^{\bullet}\}$ such that*

$$
\begin{aligned}
{}^{I}E_2^{p,q} &= H_I^p(H_{II}^q(C)) \\
{}^{II}E_2^{p,q} &= H_{II}^p(H_I^q(C)).
\end{aligned}
$$

PROOF. □

Now we consider some boundedness conditions on the double complex and how they affect the filtrations on it, and hence the spectral sequences arising from these filtrations. Observe first that both filtrations on $\mathrm{Tot}^{\bullet}(C)$ are exhaustive. Now suppose $C^{\bullet,\bullet}$ is 0 in the fourth quadrant. Then we see that $F_I^{\bullet}\,\mathrm{Tot}^{\bullet}(C)$ is bounded below; so we see that $\{{}^{I}E_r^{\bullet}\}$ must converge to $H^{\bullet}(\mathrm{Tot}(C))$, by (3.2.7). Instead, if $C^{\bullet,\bullet}$ were 0 in the second quadrant, then $F_{II}^{\bullet}\,\mathrm{Tot}^{\bullet}(C)$ would be bounded below, and so $\{{}^{II}E_r^{\bullet}\}$ will now converge to $H^{\bullet}(\mathrm{Tot}(C))$. If now, $C^{\bullet,\bullet}$ is either first or third quadrant, then we see that both spectral sequences arising from it converge to $H^{\bullet}(\mathrm{Tot}(C))$. This last observation is a wellspring for many standard results; so we record in the next Proposition.

plex-bounded-convergence

PROPOSITION 3.3.3. *Let $C^{\bullet,\bullet}$ be either a first or a third quadrant double complex over $\mathscr{C}$. Then we have*

$$
\begin{aligned}
{}^{I}E_2^{p,q} &\Rightarrow H^{\bullet}(\mathrm{Tot}(C)) \\
{}^{II}E_2^{p,q} &\Rightarrow H^{\bullet}(\mathrm{Tot}(C)).
\end{aligned}
$$

PROOF. □

## 4. Derived Functors of Multi-functors

There's a nice consequence of the spectral sequence of a double complex that lets us relate the different derived functors of a multi-functor. Before that a definition.

DEFINITION 3.4.1. A multi-functor $F : \mathscr{C}_1 \times \ldots \times \mathscr{C}_n \to \mathscr{D}$ is *right balanced* if the following conditions hold:

(1) It is left exact in each of its variables.
(2) For any covariant variable $\mathscr{C}_i$ and any injective object $I$ in $\mathscr{C}_i$, the multi-functor

$$\begin{aligned} F_I^i : \mathscr{C}_1 \times \ldots \times \mathscr{C}_{i-1} \times \mathscr{C}_{i+1} \times \ldots \times \mathscr{C}_n &\to \mathscr{D} \\ (C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n) &\mapsto F(C_1, \ldots, C_{i-1}, I, C_{i+1}, \ldots, C_n) \end{aligned}$$

is exact.
(3) For any contravariant variable $\mathscr{C}_j$ and any projective object $P$ in $\mathscr{C}_j$, the multi-functor

$$\begin{aligned} F_P^j : \mathscr{C}_1 \times \ldots \times \mathscr{C}_{j-1} \times \mathscr{C}_{j+1} \times \ldots \times \mathscr{C}_n &\to \mathscr{D} \\ (C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_n) &\mapsto F(C_1, \ldots, C_{j-1}, P, C_{j+1}, \ldots, C_n) \end{aligned}$$

is exact.

A multi-functor $F : \mathscr{C}_1 \times \ldots \times \mathscr{C}_n \to \mathscr{D}$ is *left balanced* if the following conditions hold:

(1) It is right exact in each of its variables.
(2) For any covariant variable $\mathscr{C}_i$ and any projective object $P$ in $\mathscr{C}_i$, the multi-functor

$$\begin{aligned} F_P^i : \mathscr{C}_1 \times \ldots \times \mathscr{C}_{i-1} \times \mathscr{C}_{i+1} \times \ldots \times \mathscr{C}_n &\to \mathscr{D} \\ (C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n) &\mapsto F(C_1, \ldots, C_{i-1}, P, C_{i+1}, \ldots, C_n) \end{aligned}$$

is exact.
(3) For any contravariant variable $\mathscr{C}_j$ and any injective object $I$ in $\mathscr{C}_j$, the multi-functor

$$\begin{aligned} F_I^j : \mathscr{C}_1 \times \ldots \times \mathscr{C}_{j-1} \times \mathscr{C}_{j+1} \times \ldots \times \mathscr{C}_n &\to \mathscr{D} \\ (C_1, \ldots, C_{j-1}, C_{j+1}, \ldots, C_n) &\mapsto F(C_1, \ldots, C_{j-1}, I, C_{j+1}, \ldots, C_n) \end{aligned}$$

is exact.

ctral-derived-bifunctors

PROPOSITION 3.4.2. *Let $F : \mathscr{C}_1 \times \ldots \times \mathscr{C}_n \to \mathscr{D}$ be a multi-functor between abelian categories, and suppose that $F$ is covariant in the first $r$ variables and that it is contravariant in the next $n-r$ variables. Suppose also that each $\mathscr{C}_i$ has enough injectives if $i \leq r$ and that it has enough projectives if $i > r$.*

*If $F$ is right balanced, then, for any $n$-tuple of objects $(A_1, \ldots, A_n)$, with $A_i \in \mathscr{C}_i$, and any pair $i, j$ with $1 \leq i < j \leq n$, and for all $p \geq 0$, we have a natural isomorphism:*

$$(R^p F(A_1, \ldots, \hat{A}_i, \ldots, A_n))(A_i) \cong (R^p F(A_1, \ldots, \hat{A}_j, \ldots, A_n))(A_j)$$

*If $F$ is left balanced, then, for any $n$-tuple of objects $(A_1, \ldots, A_n)$, with $A_i \in \mathscr{C}_i$, and any pair $i, j$ with $1 \leq i < j \leq n$, and for all $p \leq 0$, we have a natural* *isomorphism:*

$$(L^pF(A_1,\dots,\hat{A}_i,\dots,A_n))(A_i)\cong(L^pF(A_1,\dots,\hat{A}_j,\dots,A_n))(A_j)$$

PROOF. We'll only do the case where $F$ is right balanced. It clearly suffices to prove the natural isomorphism for the case where $i=1$ and $j$ is arbitrary. In doing so we reduce essentially to the case where $F$ is a bi-functor $F:\mathscr{C}_1\times\mathscr{C}_2\to\mathscr{D}$. There are now three possible scenarios: $F$ is covariant in both variables, or $F$ is contravariant in both variables, or $F$ is covariant in one variable and contravariant in the other. We'll consider the last; the proofs are almost identical in the other cases.

So suppose $F$ is covariant in the first variable and contravariant in the second. Let $I^\bullet$ be an injective resolution of $A_1$ in $\mathscr{C}_1$ and let $P^\bullet$ be a projective resolution of $A_2$ in $\mathscr{C}_2$. Let $K^{\bullet,\bullet}$ be the first quadrant double complex defined by

$$K^{p,q}=F(I^p,P^{-q})$$
$$d_I^{p,q}=F(d_I^p,1_{P^{-q}})d_{II}^{p,q}=F(1_{I^p},d_P^{-q-1}).$$

We now consider the two spectral sequences associated with this double complex to find:

$${}^{I}E_1^{p,q}=\begin{cases}F(I^p,A_2) & \text{if } q=0\\ 0 & \text{otherwise}\end{cases}\qquad {}^{II}E_1^{p,q}=\begin{cases}F(A_1,P^{-p}) & \text{if } q=0\\ 0 & \text{otherwise}\end{cases}$$

Here, we used the fact that $F^1_{I^p}$ and $F^2_{P^{-p}}$ are exact functors. Computing the second page of both spectral sequences we get:

$${}^{I}E_2^{p,q}=\begin{cases}R^p(F(\_\_,A_2))(A_1) & \text{if } q=0\\ 0 & \text{otherwise}\end{cases}\qquad {}^{II}E_2^{p,q}=\begin{cases}R^p(F(A_1,\_\_))(A_2) & \text{if } q=0\\ 0 & \text{otherwise}\end{cases}$$

So both sequences collapse on the second page on the $0^{th}$ row, and since they both must converge $H^\bullet(\mathrm{Tot}(K))$, we see that we must in fact have a natural isomorphism

$$R^p(F(\_\_,A_2))(A_1)\cong R^p(F(A_1,\_\_))(A_2)$$

$\square$

But in fact it is very common for an abelian category to not have enough projectives. The next result shows that we can still get some useful information out of the covariant variable even in this case, and although it has nothing to do with spectral sequences, this is probably the most appropriate place for it.

spectral-bi-functor

PROPOSITION 3.4.3. *Let $F:\mathscr{C}_1\to\mathscr{C}_2\to\mathscr{D}$ be a right balanced bi-functor, contravariant in the first variable and covariant in the second. Suppose also that $\mathscr{C}_2$ has enough injectives. Let $\mathscr{K}$ be the class of objects $P$ in $\mathscr{C}_1$, for which the functor $F(P,\_\_)$ is exact.*

(1) *For every short exact sequence*

$$0\to A_1\to A_2\to A_3\to 0$$

*in $\mathscr{C}_1$, and every object $C$ in $\mathscr{C}_2$, there is an associated long exact sequence*

$$0\to F(A_3,C)\to F(A_2,C)\to F(A_1,C)\to R^1F(A_3,\_\_)(C)\to R^1F(A_2,\_\_)(C)\to R^1F(A_1,\_\_)(C)\to\dots$$

*In other words, the sequence of functors $R^\bullet F(\_\_,\_\_)(C)$ from $\mathscr{C}_1$ to $\mathscr{D}$ is in fact a contravariant cohomological $\delta$-functor.*

(2) *Suppose that $P^\bullet$ is a left $\mathscr{K}$-resolution of $A$ in $\mathscr{C}_1$. Then, for every object $C$ in $\mathscr{C}_2$, we have a natural isomorphism*

$$R^nF(\_\_, C)(A) \cong H^{-n}(F(P^\bullet, C))$$

PROOF. For the first statement, use (2.2.8), and the fact that the sequence

$$0 \to F(A_3, I) \to F(A_2, I) \to F(A_1, I) \to 0$$

is exact for every injective object $I$ in $\mathscr{C}_2$. If we had two exact sequences $A$ and $A'$ in $\operatorname{Exact}(\mathscr{C}_1)$ and a morphism between them, and if $I^\bullet$ is an injective resolution of $C$, then we have a diagram:

$$\begin{tikzcd}
0 \arrow[r] & F(A_3, I^\bullet) \arrow[r] \arrow[d] & F(A_2, I^\bullet) \arrow[r] \arrow[d] & F(A_1, I^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(A_3', I^\bullet) \arrow[r] & F(A_2', I^\bullet) \arrow[r] & F(A_1', I^\bullet) \arrow[r] & 0
\end{tikzcd}$$

From this diagram and the $\delta$-functoriality of $H^\bullet : \operatorname{Ch}\mathscr{D} \to \mathscr{D}$, we obtain the $\delta$-functoriality of the sequence $R^\bullet F(\_\_, \_\_)(C)$.

For the second observe that $R^\bullet F(\_\_, \_\_)(A)$ and $H^{-\bullet}(F(P^\bullet, \_\_))$ are both contravariant cohomological $\delta$-functors from $\mathscr{C}_2$ to $\mathscr{D}$. That the first is a $\delta$-functor follows from part (1), and that the second is a $\delta$-functor follows from the fact that $F(P^\bullet, \_\_)$ is an exact functor into $\operatorname{Ch}^{\leq}\mathscr{D}$. Now, when $n = 0$, they are both simply $F(A, \_\_)$, and for $n > 0$ they are both effaceable, since they vanish on injective objects. Hence we see that they must be isomorphic. $\square$

REMARK 3.4.4. Many properties (right balancedness, contravariance, etc.) are naturally dualistic in nature, and so we can replace them by their duals in the Proposition to get the appropriate analogues. If the need arises in the future, we will use these analogues without comment.

## 5. Cartan-Eilenberg Resolutions

So far we've only discussed resolutions of *objects*, but if, as remarked earlier, we want to treat chain complexes as generalized objects, then we should also be willing to consider resolutions of complexes as well. We will do exactly that in this section, and in doing so, we will be able to define a

DEFINITION 3.5.1 (Cartan-Eilenberg Resolutions). An *injective Cartan-Eilenberg resolution* of a complex $C^\bullet$ over $\mathscr{C}$ consists of an upper plane double complex $I^{\bullet,\bullet}$ of injectives and a monomorphism $\epsilon : C^\bullet \to I^{0,\bullet}$ such that the following conditions hold:

(1) If $C^p = 0$, then the column $I^{p,\bullet}$ is also 0.
(2) For all $q \geq 0$, the complex $B_I^{\bullet,q}$ is an injective resolution of $B^q(C)$, and the complex $H_I^{\bullet,q}$ is an injective resolution of $H^q(C)$.

Dually, a *projective Cartan-Eilenberg resolution* of a complex $C^\bullet$ over $\mathscr{C}$ consists of a lower plane double complex $P^{\bullet,\bullet}$ of injectives and an epimorphism $\epsilon : P^{0,\bullet} \to C^\bullet$ such that the following conditions hold:

(1) If $C^p = 0$, then the column $I^{p,\bullet}$ is also 0.
(2) For all $p \in \mathbb{Z}$, the complex $B_I^{p,\bullet}(I)$ is an injective resolution of $B^p(C)$, and the complex $H_I^{p,\bullet}(I)$ is an injective resolution of $H^p(C)$.

From now on we will only state results for injective Cartan-Eilenberg resolution, but will use their dual statements for projective ones without comment.

REMARK 3.5.2. It's easy to see that we also have, for all $p \in \mathbb{Z}$, that the complex $I^{p,\bullet}$ is an injective resolution of $C^p$, and that $Z_I^{p,\bullet}(I)$ is an injective resolution of $Z^p(C)$.

The next Proposition is a generalization of (1.4.5).

rtan-eilenberg-extension

PROPOSITION 3.5.3. *Let $C^\bullet$ and $\mathscr{C}'^\bullet$ be complexes over $\mathscr{C}$ and let $I^{\bullet,\bullet}$ and $J^{\bullet,\bullet}$ be injective Cartan-Eilbenberg resolution of $C^\bullet$ and $C'^\bullet$, respectively.*

1. *Every morphism $C^\bullet \to C'^\bullet$ can be extended to a morphism of $I^{\bullet,\bullet}$ into $J^{\bullet,\bullet}$ of double complexes. Moreover, this extension is unique upto homotopy of double complexes.*
2. *If two morphisms $f, f' : C^\bullet \to C'^\bullet$ are chain homotopic, then their extensions from $I^{\bullet,\bullet}$ to $J^{\bullet,\bullet}$ are also chain homotopic. In particular, any two Cartan-Eilenberg resolutions of $C^\bullet$ are chain homotopy equivalent.*

PROOF. □

n-eilenberg-construction

PROPOSITION 3.5.4. *Let $\mathscr{C}$ be a category with enough injectives. Then every complex $C^\bullet$ over $\mathscr{C}$ has a Cartan-Eilenberg resolution.*

PROOF. □

## 6. Hypercohomology and the Grothendieck Spectral Sequence

truction-hypercohomology

THEOREM 3.6.1. *Let $F : \mathscr{C} \to \mathscr{D}$ be a left exact functor between abelian categories, and suppose $\mathscr{C}$ has enough injectives. Then there is a cohomological universal $\delta$-functor*

$$\mathbb{R}^\bullet F : \mathrm{Ch}^{\geq 0}\mathscr{C} \to \mathscr{D}$$

*such that the following conditions hold:*

1. $\mathbb{R}^0 F \cong H^0(\mathrm{Ch}\, F)$.
2. *For any complex $C^\bullet \in \mathrm{Ch}^{\geq 0}$ and $n \geq 0$, we have*
$$\mathbb{R}^p F(C[n]^\bullet) = \begin{cases} \mathbb{R}^{p-n}F(C^\bullet) & \text{if } p \geq n \\ 0 & \text{otherwise.} \end{cases}$$
3. *For any object $A \in \mathscr{C}$, and $n \geq 0$, we have*
$$\mathbb{R}^p F(A[n]) = \begin{cases} R^{p-n}F(A) & \text{if } p \geq n \\ 0 & \text{otherwise.} \end{cases}$$
4. *For every complex $C \in \mathrm{Ch}^{\geq 0}\mathscr{C}$, there is a first quadrant spectral sequence $\{E_r^\bullet : r \geq 0\}$ such that*
$$R^p(H^q(C)) \cong E_2^{p,q} \Rightarrow \mathbb{R}^{p+q}F(C^\bullet).$$

DEFINITION 3.6.2. With the notation of the Theorem, we call $\mathbb{R}^n F : \mathrm{Ch}^{\geq 0}\mathscr{C} \to \mathscr{D}$ the *right hyper-derived functors* of $F$.

We of course have the dual version of this Theorem, which we will record below without proof:

nstruction-hyperhomology

Theorem 3.6.3. *Let $F : \mathscr{C} \to \mathscr{D}$ be a right exact functor between abelian categories, and suppose $\mathscr{C}$ has enough projectives. Then there is a homological universal $\delta$-functor*

$$\mathbb{L}^\bullet F : \mathrm{Ch}^{\leq 0}\mathscr{C} \to \mathscr{D}$$

*such that the following conditions hold:*

(1) $\mathbb{L}^0 F \cong H^0(\mathrm{Ch}\, F)$.

(2) *For any complex $C^\bullet \in \mathrm{Ch}^{\leq 0}$ and $n \leq 0$, we have*

$$\mathbb{L}^p F(C[n]^\bullet) = \begin{cases} \mathbb{L}^{p-n} F(C^\bullet) & \text{if } p \leq n \\ 0 & \text{otherwise.} \end{cases}$$

(3) *For any object $A \in \mathscr{C}$, and $n \leq 0$, we have*

$$\mathbb{L}^p F(A[n]) = \begin{cases} L^{p-n} F(A) & \text{if } p \leq n \\ 0 & \text{otherwise.} \end{cases}$$

(4) *For every complex $C \in \mathrm{Ch}^{\leq 0}\mathscr{C}$, there is a third quadrant spectral sequence $\{E_r^\bullet :\ r \geq 0\}$ such that*

$$L^p(H^q(C)) \cong E_2^{p,q} \Rightarrow \mathbb{L}^{p+q} F(C^\bullet).$$

We now come to one of the most useful gadgets in homological algebra: the Grothendieck spectral sequence.

-grothendieck-left-exact

Theorem 3.6.4. *Let $G : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{D} \to \mathscr{E}$ be left exact functors between abelian categories; suppose $\mathscr{C}$ and $\mathscr{D}$ have enough injectives and suppose that $G$ takes injective objects in $\mathscr{C}$ to $F$-acyclic objects in $\mathscr{D}$. Then, for every object $A \in \mathscr{C}$, there exists a first quadrant spectral sequence $\{E_r^\bullet :\ r \geq 0\}$ such that*

$$(R^p F)(R^q G)(A) \cong E_2^{p,q} \Rightarrow R^{p+q}(FG)(A).$$

rothendieck-one-is-exact

Corollary 3.6.5. *If, with the notation and hypotheses of the theorem above, $G$ is in fact exact, then, for all $n \geq 0$, we have natural isomorphisms*

$$(R^n F)(GA) \cong R^n(FG)(A).$$

*If instead $F$ is exact, then we have natural isomorphisms:*

$$F(R^n G)(A) \cong R^n(FG)(A).$$

Proof. Indeed, the spectral sequence collapses on the $0^{th}$ row, since $R^q G = 0$, for $q \geq 1$ (2.2.6). For the second statement, we apply the same argument, but this time to $F$. □

Remark 3.6.6. Observe that if $F$ is exact then we do not need any additional hypotheses on $G$ apart from left exactness: *every* object in $\mathscr{D}$ is $F$-acyclic in this case. The content of the corollary is in the first identity.

endieck-two-compositions

Corollary 3.6.7. *Let $\mathscr{C}$, $\mathscr{D}$, $\mathscr{D}'$ and $\mathscr{E}$ be abelian categories, and suppose we have left exact functors $G : \mathscr{C} \to \mathscr{D}$, $G' : \mathscr{C} \to \mathscr{D}'$, $F : \mathscr{D} \to \mathscr{E}$ and $F' : \mathscr{D}' \to \mathscr{E}$*

*such that the following diagram commutes up to natural equivalence:*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "G"] \arrow[d, "G'"'] & \mathscr{D} \arrow[d, "F"] \\ \mathscr{D}' \arrow[r, "F'"] & \mathscr{E} \end{tikzcd}$$

*Suppose also that $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{D}'$ all have enough injectives, that $G$ and $F'$ are in fact exact, and, finally, that $G$ takes injective objects to $F$-acyclic ones. Then, for every object $A \in \mathscr{C}$, and every $n \geq 0$, we have natural isomorphisms*

$$(R^n F)(GA) \cong F'(R^n G')(A)$$

PROOF. Since $FG \cong F'G'$, this follows from (3.6.5). □

Of course there's also a dual version of the Grothendieck spectral sequence. We present it below.

grothendieck-right-exact

THEOREM 3.6.8. *Let $G : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{D} \to \mathscr{E}$ be right exact functors between abelian categories, and suppose that $G$ takes projective objects in $\mathscr{C}$ to $F$-acyclic objects in $\mathscr{D}$. Then, for every object $A \in \mathscr{C}$, there exists a third quadrant spectral sequence $\{E_r^\bullet : r \geq 0\}$ such that*

$$(L^p F)(L^q G)(A) \cong E_2^{p,q} \Rightarrow L^{p+q}(FG)(A).$$

CHAPTER 4

# Some Classical Results

chap:classical

## 1. Ext

## 2. Some Other Derived Functors

## 3. Künneth Formula and the Universal Coefficient Theorem

CHAPTER 5

# Applications to Module Theory

chap:module

CHAPTER 6

# Group Cohomology

$\boxed{\texttt{chap:group}}$

CHAPTER 7

# Sheaf Cohomology

chap:sheaf

We will be using the notation and the language of my notes on sheaf theory [NOS, **??** ]. All our ringed spaces will be equipped with sheaves of commutative rings.

## 1. Flabby and Injective Sheaves

In this section, we'll discuss a class of sheaves that's very important for the study of the cohomology of sheaves. For the remainder of this section $\mathscr{C}$ will denote an abelian category.

Definition 7.1.1. A sheaf $\mathscr{F}$ is *flabby* if, for every inclusion of open sets $V \hookrightarrow U$, $\mathrm{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

Remark 7.1.2. It's clear that if $\mathscr{F}$ is flabby, and $U \subset X$ is open, then $\mathscr{F}|_U$ is also flabby.

The main result is the following one:

eaf-section-flabby-exact

Proposition 7.1.3. *If $\mathscr{F}' \in \mathrm{Shf}(X, \mathscr{C})$ is flabby, then $\Gamma(U, \_\_)$ preserves any short exact sequence of the form:*

$$0 \to \mathscr{F}' \to \mathscr{F} \xrightarrow{\phi} \mathscr{F}'' \to 0$$

Proof. We have to show that

$$0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \xrightarrow{\phi_U} \mathscr{F}''(U) \to 0$$

is exact.

Suppose $s \in \mathscr{F}''(U)$. We want to find $\widetilde{s} \in \mathscr{F}(U)$ such that $\phi_U(\widetilde{s}) = s$. We consider the set $\mathscr{W} = \{(W, w) : W \subset U, w \in \mathscr{F}(W), \phi_W(w) = \mathrm{res}_{U,W}(s)\}$, in anticipation of the moment when we can bring down the sledgehammer of Zorn's Lemma.

Since $\phi$ is surjective, $\mathscr{W}$ is non-empty (and in fact quite large) by [NOS, 4.8 ]. Now, suppose $(W, w) \in \mathscr{W}$. Again, by [NOS, 4.8 ], if $W \neq U$, we can find another element $(V, v) \in \mathscr{W}$, with $V \not\subseteq W$. Then, the element $\widetilde{t} = \mathrm{res}_{W,V\cap W}(w) - \mathrm{res}_{V,V\cap W}(v)$ lies in $\ker \phi_{V\cap W} = \mathscr{F}'(V \cap W)$. Since $\mathscr{F}'$ is flabby, we can find $w' \in \mathscr{F}'(W)$ such that $\mathrm{res}_{W,V\cap W}(w') = \widetilde{t}$. In that case, if we consider $w - w' \in \mathscr{F}(W)$ and $v \in \mathscr{F}(V)$, then both have the same restrictions to $\mathscr{F}(V \cap W)$. So we can patch them together to find a section $t \in \mathscr{F}(V \cup W)$, which by the Identity Axiom has to satisfy $\phi_{V\cup W}(t) = \mathrm{res}_{U,V\cup W}(s)$. So $(V \cup W, t)$ is an extension of $(W, w)$.

Now it's time for the sledgehammer, which does the rest of the work for us. $\square$

f-cokernel-flabby-flabby

Corollary 7.1.4. *With the notation and hypotheses as in the Proposition above, $\mathscr{F}$ is flabby iff $\mathscr{F}''$ is flabby.*

Proof. Follows from the Proposition and the commutativity of the following diagram, for open sets $V \subset U$.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

The vertical arrow on the left is surjective. So by a diagram chase it's clear that one of the arrows on the right is surjective iff the other one is. □

The direct image functor preserves flabbiness.

sheaf-dirimg-flabby

Proposition 7.1.5. *If $f : X \to Y$ is continuous, and $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$ is flabby, then so is $f_*\mathscr{F} \in \mathrm{Shf}(Y, \mathscr{C})$.*

Proof. We have to show that if $V \subset U$ are open sets in $Y$, then the restriction from $(f_*\mathscr{F})(U)$ to $(f_*\mathscr{F})(V)$ is surjective, but this is just the restriction from $\mathscr{F}(f^{-1}(U))$ to $\mathscr{F}(f^{-1}(V))$, which is surjective, because $\mathscr{F}$ is flabby. □

d-space-injective-flabby

Proposition 7.1.6. *Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{I}$ be an injective $\mathscr{O}_X$-module, that is, an injective object in $\mathscr{O}_X$-mod. Then, for any $\mathscr{O}_X$-module $\mathscr{G}$, the sheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{I})$ is flabby. In particular, $\mathscr{I}$ is flabby.*

Proof. For any open subset $U \subset X$, consider the exact sequence

$$0 \to j_!(\mathscr{G}|_U) \to \mathscr{G} \to i_*(\mathscr{G}|_{X\setminus U}) \to 0,$$

where $j : U \to X$ and $i : X \setminus U \to X$ are the inclusion maps. Applying $\mathrm{Hom}_{\mathscr{O}_X}(\_\_, \mathscr{I})$ to this sequence, we get another exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{O}_X}(i_*(\mathscr{G}|_{X\setminus U}), \mathscr{I}) \to \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{I}) \to \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}|_U, \mathscr{I}|_U) \to 0.$$

Since the morphism on the right is surjective, we find that $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{I})$ is indeed flabby. If we take $\mathscr{G} = \mathscr{O}_X$, then we get our second assertion. □

Both injectivity and flabbiness are local conditions. Before we show that, we need a lemma, which we can think of as a local criterion for flabbiness.

-flabby-characterization

Lemma 7.1.7. *A sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$ is flabby if and only if, for every open subspace $U \subset X$, the natural morphism $\mathscr{F} \to j_!(\mathscr{F}|_U)$ is surjective, where $j : U \to X$ is the inclusion map.*

Proof. One direction is trivial; so assume $\mathscr{F} \to j_!(\mathscr{F}|_U)$ is surjective, for all open sets $U \subset X$. To show that $\mathscr{F}$ is flabby, it suffices to show that the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(U, \mathscr{F})$ is surjective, for all open sets $U \subset X$. Pick a section $s$ of $\mathscr{F}$ over $U$, and set

$$\mathscr{S} = \{(V, t) : U \subset V,\ V \subset X \text{ open},\ t \in \Gamma(V, \mathscr{F}),\ t|_U = s\}.$$

This is a non-empty set with a natural ordering, and it clearly satisfies the requirements for Zorn's lemma to work. So let $(V, t)$ be a maximal element of $\mathscr{S}$. We claim that $V = X$. Suppose otherwise, and pick $x \in X \setminus V$. Since $\mathscr{F}_x \to (j_!(\mathscr{F}|_U))_x$ is surjective, there is some open neighborhood $W$ of $x$ and a section $t'$ of $\mathscr{F}$ over $W$ such that $t'|_{V\cap W} = t|_{V\cap W}$. But then we can extend $t$ to a section of $\mathscr{F}$ on $W \cup V$, which is a contradiction of the maximality of $(V, t)$. □

f-flabby-local-condition

PROPOSITION 7.1.8. *Let* $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$ *be a sheaf, and suppose we have an open cover* $\{U_i : i \in I\}$ *of* $X$.

(1) $\mathscr{F}$ *is flabby if and only if* $\mathscr{F}|_{U_i}$ *is flabby, for all* $i \in I$

(2) *Suppose in addition that* $(X, \mathscr{O}_X)$ *is a ringed space and that* $\mathscr{F}$ *is an* $\mathscr{O}_X$*-module. Then* $\mathscr{F}$ *is injective in* $\mathscr{O}_X$*-mod if and only if* $\mathscr{F}|_{U_i}$ *is injective, for all* $i \in I$.

PROOF. (1) The criterion in the Lemma above is clearly a local one, so the assertion follows easily.

(2) First suppose that $\mathscr{F}$ is injective, and let $U \subset X$ be any open subset. Then we have isomorphisms

$$\mathrm{Hom}_{\mathscr{O}_U}(\_\_, \mathscr{F}|_U) = \mathrm{Hom}_{\mathscr{O}_X}(j_!(\_\_), \mathscr{F}),$$

where $j_! : U \to X$ is the inclusion map. Since $j_!$ is exact, this shows that $\mathscr{F}|_U$ must be injective. Conversely, if $\mathscr{F}|_{U_i}$ is injective, for all $i \in I$, then, for any $\mathscr{G} \in \mathscr{O}_X$-mod, the sheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_{U_i}}(\mathscr{G}|_{U_i}, \mathscr{F}|_{U_i})$ is flabby, for all $i$ (7.1.6). Hence, by part (1) $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})$ is flabby also. But then the functor $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})$ is the composition

$$\mathscr{O}_X\text{-mod} \xrightarrow{\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\_\_, \mathscr{F})} \mathscr{O}_X\text{-mod} \xrightarrow{\Gamma(X, \_\_)} \mathrm{Ab},$$

which is an exact functor, and so $\mathscr{F}$ must be injective.

□

## 2. Sections with Local Support

Sections with local support behave well in the presence of flabbiness.

l-support-flabby-exctseq

PROPOSITION 7.2.1. *If* $Z \subset X$ *is a closed subset,* $\mathscr{F} \in \mathrm{Shf}(X, \mathscr{C})$*, and* $j : U := X \setminus Z \hookrightarrow X$ *is the inclusion map, then, if* $\mathscr{F}$ *is flabby, we have an exact sequence:*

$$0 \to \underline{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U) \to 0.$$

*Moreover,* $\underline{H}^0_Z(\mathscr{F})$ *is flabby.*

PROOF. Most of the work was done in [NOS, 8.13 ]. So assume that $\mathscr{F}$ is flabby. Then the statement follows immediately from the fact that the morphism $\mathscr{F}(V) \longrightarrow \mathscr{F}(V \cap U)$ is surjective for every open set $V \subset X$. So in fact, the sequence is exact as a sequence of presheaves. Given this, for any pair of open sets $V \subset W$, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_Z(W, \mathscr{F}) \arrow[r] \arrow[d] & \Gamma(W, \mathscr{F}) \arrow[r] \arrow[d] & \Gamma(W, j_*(\mathscr{F}|_U)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_Z(V, \mathscr{F}) \arrow[r] & \Gamma(V, \mathscr{F}) \arrow[r] & \Gamma(V, j_*(\mathscr{F}|_U)) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact, and the two vertical arrows on the right and in the middle are surjective. This implies that the arrow on the left is also surjective, and hence $\underline{H}^0_Z(\mathscr{F})$ is flabby. □

cal-support-flabby-exact

PROPOSITION 7.2.2. *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and* $\mathscr{F}'$ *is flabby, then we have an exact sequence:*

$$0 \to \Gamma_Z(U, \mathscr{F}') \to \Gamma_Z(U, \mathscr{F}) \to \Gamma_Z(U, \mathscr{F}'') \to 0$$

Proof. Suppose $\mathscr{F}'$ is flabby; then by Propositions (7.1.3) and (7.1.5), we see that we have now the following diagram with exact rows, and with exact columns on the right and in the middle:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathit{\Gamma}_Z(U,\mathscr{F}') \arrow[r] \arrow[d] & \mathit{\Gamma}(U,\mathscr{F}') \arrow[r] \arrow[d] & \mathit{\Gamma}(U, j_*(\mathscr{F}'|_V)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathit{\Gamma}_Z(U,\mathscr{F}) \arrow[r] \arrow[d] & \mathit{\Gamma}(U,\mathscr{F}) \arrow[r] \arrow[d] & \mathit{\Gamma}(U, j_*(\mathscr{F}|_V)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathit{\Gamma}_Z(U,\mathscr{F}'') \arrow[r] \arrow[d] & \mathit{\Gamma}(U,\mathscr{F}'') \arrow[r] \arrow[d] & \mathit{\Gamma}(U, j_*(\mathscr{F}''|_V)) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then, it follows that the column on the left must be exact. □

## 3. Cohomology of Sheaves

We fix a topological space $X$ for this section.

Definition 7.3.1. For any closed subset $Z \subset X$, the functor $\mathit{\Gamma}_Z(X, \_\_) : \mathrm{Shf}(X, \mathrm{Ab}) \to \mathrm{Ab}$ is a left exact functor, and so has right derived functors $R^n\mathit{\Gamma}_Z(X, \_\_)$. For a sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ we define the $n^{th}$ *cohomology of* $X$ *with support* $Z$ *and coefficients in* $\mathscr{F}$ to be

$$H^n_Z(X, \mathscr{F}) = R^n\mathit{\Gamma}_Z(X, ()\mathscr{F}).$$

If $Z = X$, then $\mathit{\Gamma}_X(X, \_\_)$ is simply $\mathit{\Gamma}(X, \_\_)$, and in this case we denote $H^n_X(X, \mathscr{F})$ simply by $H^n(X, \mathscr{F})$ and call it the $n^{th}$ *cohomology of* $X$ *with coefficients in* $\mathscr{F}$.

Before we do anything else, let's prove something trivial, but very fundamental. It will be used repeatedly without comment.

f-cohomology-open-subset

Proposition 7.3.2. *Let* $Z \subset X$ *be a closed subset and let* $U \subset X$ *be an open one. Then, for all* $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$*, we have a natural isomorphism:*

$$H^\bullet_Z(U, \mathscr{F}|_U) \cong R^\bullet(\mathit{\Gamma}_{Z\cap U}(U, \_\_))(\mathscr{F})$$

Proof. Follows from (3.6.5), since restriction to $U$ is an exact functor, and takes injective sheaves to flasque sheaves, by (7.1.6). □

Cohomology is functorial in $X$.

cohomology-functoriality PROPOSITION 7.3.3. *Let $f : X \to Y$ be a continuous map. Then, for every sheaf $\mathscr{F} \in \mathrm{Shf}(Y, \mathrm{Ab})$, we have a natural morphism*

$$H^\bullet(f) : H^\bullet(Y, \mathscr{F}) \to H^\bullet(X, f^{-1}\mathscr{F}).$$

*In particular, for every open subset $U \subset Y$, we have a natural morphism*

$$H^\bullet(Y, \mathscr{F}) \to H^\bullet(U, \mathscr{F}|_U).$$

PROOF. For the existence of $H^\bullet(f)$, since both $H^\bullet(Y, \_\_)$ and $H^\bullet(X, f^{-1}\_\_)$ are $\delta$-functors, with the first one universal, it suffices to build a natural map $\Gamma(Y, \mathscr{F}) \to \Gamma(X, f^{-1}\mathscr{F})$. This is easy, since by definition, $f^{-1}\mathscr{F}$ is the sheafification of the presheaf that assigns to $U \subset X$, the group

$$\lim_{\substack{longrightarrow \\ V \supset f(U)}} \Gamma(V, \mathscr{F}),$$

and so carries a natural map from $\Gamma(Y, \mathscr{F})$ into it. $\square$

f-abelian-flabby-acyclic PROPOSITION 7.3.4. *For every closed subset $Z \subset X$, every flabby sheaf in* $\mathrm{Shf}(X, \mathrm{Ab})$ *is $\Gamma_Z(X, \_\_)$-acyclic.*

PROOF. We'll show that the class $\mathscr{K}$ of flabby sheaves satisfies the two conditions of (2.3.3). Condition (1) there follows for flabby sheaves from (7.1.6) and (7.1.4), and condition (2) follows from (7.2.2). Hence we're done. $\square$

REMARK 7.3.5. Observe that the flabbiness of an $\mathscr{O}_X$-module $\mathscr{F}$ over a ringed space $(X, \mathscr{O}_X)$ is *independent* of whether we are considering it an object in $\mathscr{O}_X$-mod or as a sheaf with values in Ab.

inged-space-abelian-same COROLLARY 7.3.6. *Let $X$ be a topological space and let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be a sheaf of abelian groups over $X$. Let $Z \subset X$ be a closed subspace.*

(1) *For any right flabby resolution $\mathscr{G}^\bullet$ of $\mathscr{F}$ we have*

$$H^n_Z(X, \mathscr{F}) = H^n(\Gamma_Z(X, \mathscr{G}^\bullet)).$$

(2) *Suppose that $(X, \mathscr{O}_X)$ is a ringed space and that $\mathscr{F} \in \mathscr{O}_X$-mod. Consider $\Gamma_Z(X, \_\_)$ now as a functor from $X$-mod to* Ab. *Then we have*

$$R^n\Gamma_Z(X, \_\_)(\mathscr{F}) = H^n_Z(X, \mathscr{F}).$$

PROOF. (1) follows immediately from (2.3.8) and (7.3.4). (2) follows from (1) and the fact that any injective resolution of $\mathscr{F}$ in $X$-mod is a flabby resolution of $\mathscr{F}$ in $\mathrm{Shf}(X, \mathrm{Ab})$ (7.1.6). $\square$

EXAMPLE 7.3.7 (Cohomology of $S^1$ with coefficients in $\mathbb{Z}$).

## 4. Cohomology with Supports

In this section, we specifically consider cohomology with support in some closed subspace of a topological space $X$. We'll find that it satisfies many properties reminiscent of relative cohomology groups in topology, including excision and the Mayer-Vietoris.

al-support-section-exact LEMMA 7.4.1. *Let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be a flabby sheaf and let $Z \subset X$ be a closed subspace. Then the following sequence is exact:*

$$0 \to \Gamma_Z(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X \setminus Z, \mathscr{F}) \to 0.$$

Proof. The sequence is obtained simply by applying the global sections functor to the exact sequence (7.2.2)

$$0 \to \underline{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_{X\setminus Z}) \to 0,$$

where $j : X \setminus Z \to X$ is the inclusion. Since flabby sheaves are $\Gamma(X, \_\_)$-acyclic, the result follows. $\square$

l-support-long-exact-seq

Proposition 7.4.2. *For every $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ and every closed subspace $Z \subset X$, there exists a long exact sequence*

$$0 \to \Gamma_Z(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X\setminus Z, \mathscr{F}) \to H^1_Z(X, \mathscr{F}) \to H^1(X, \mathscr{F}) \to H^1(X\setminus Z, \mathscr{F}|_{X\setminus Z}) \to \dots$$

Proof. Follows from the lemma and (2.2.8). $\square$

f-local-support-excision

Proposition 7.4.3 (Excision). *Let $Z \subset X$ be a closed subspace and let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be a sheaf. Suppose $V \subset X$ is an open subspace such that $Z \subset V$. Then, for all $n \geq 0$, we have natural isomorphisms*

$$H^n_Z(X, \mathscr{F}) \cong H^n_Z(V, \mathscr{F}|_V).$$

Proof. Consider the composition

$$G : \mathrm{Shf}(X, \mathrm{Ab}) \xrightarrow{|_V} \mathrm{Shf}(V, \mathrm{Ab}) \xrightarrow{\Gamma_Z(V, \_\_)} \mathrm{Ab}\,.$$

Since restriction takes flabby sheaves to flabby sheaves and is an exact functor, we can apply (3.6.5) to conclude that we have

$$H^n_Z(V, \mathscr{F}|_V) \cong R^n G(\mathscr{F}).$$

We'll be done now if we show that $G(\mathscr{F}) \cong \Gamma_Z(X, \mathscr{F})$. For this consider the natural map $\Gamma_Z(X, \mathscr{F}) \to \Gamma_Z(V, \mathscr{F})$. Since $Z \subset V$, this is clearly injective. Suppose $s \in \Gamma_Z(V, \mathscr{F})$; then we can extend it to a section over $X$ simply by gluing it together with the zero section over $X \setminus Z$. This shows surjectivity and finishes the proof. $\square$

The next result will be useful for some Mayer-Vietoris type results for sheaf cohomology.

-support-union-intersect

Proposition 7.4.4. *Let $Z_1, Z_2 \subset X$ be two closed subsets, let $U_i = X \setminus Z_i$, for $i = 1, 2$. Then, for a flabby sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$, we have exact sequences*

$$0 \longrightarrow \Gamma(U_1 \cup U_2, \mathscr{F}) \xrightarrow{\alpha : s \mapsto (s,s)} \Gamma(U_1, \mathscr{F}) \oplus \Gamma(U_2, \mathscr{F}) \xrightarrow{\beta : (s_1, s_2) \mapsto (s_1 - s_2)} \Gamma(U_1 \cap U_2, \mathscr{F}) \longrightarrow 0$$

*and*

$$0 \longrightarrow \Gamma_{Z_1 \cap Z_2}(X, \mathscr{F}) \xrightarrow{s \mapsto (s,s)} \Gamma_{Z_1}(X, \mathscr{F}) \oplus \Gamma_{Z_2}(X, \mathscr{F}) \xrightarrow{(s_1, s_2) \mapsto (s_1 - s_2)} \Gamma_{Z_1 \cup Z_2}(X, \mathscr{F}) \longrightarrow 0$$

Proof. We have the following diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_{Z_1 \cap Z_2}(X, \mathscr{F}) \arrow[r] \arrow[d, "\gamma"'] & \Gamma(X, \mathscr{F}) \arrow[r] \arrow[d] & \Gamma(U_1 \cup U_2, \mathscr{F}) \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & \Gamma_{Z_1}(U, \mathscr{F}) \oplus \Gamma_{Z_2}(U, \mathscr{F}) \arrow[r] \arrow[d, "\delta"'] & \Gamma(X, \mathscr{F}) \oplus \Gamma(X, \mathscr{F}) \arrow[r] \arrow[d] & \Gamma(U_1, \mathscr{F}) \oplus \Gamma(U_2, \mathscr{F}) \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & \Gamma_{Z_1 \cup Z_2}(X, \mathscr{F}) \arrow[r] \arrow[d] & \Gamma(X, \mathscr{F}) \arrow[r] \arrow[d] & \Gamma(U_1 \cap U_2, \mathscr{F}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The column in the middle is trivially exact, and the column on the right is exact by the sheaf axiom and the flabbiness of $\mathscr{F}$ Therefore, the column on the left must also be exact. □

l-support-mayer-vietoris

Proposition 7.4.5. *Let $Z_1, Z_2 \subset X$ be two closed subsets, let $U_i = X \setminus Z_i$, for $i = 1, 2$, and let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be any sheaf. Then we have a long exact sequences*

$$\ldots \to H^n_{Z_1 \cap Z_2}(X, \mathscr{F}) \to H^n_{Z_1}(X, \mathscr{F}) \oplus H^n_{Z_2}(X, \mathscr{F}) \to H^n_{Z_1 \cup Z_2}(X, \mathscr{F}) \to H^{n+1}_{Z_1 \cap Z_2}(X, \mathscr{F}) \to \ldots$$

*and*

$$\ldots \to H^n(U_1 \cup U_2, \mathscr{F}) \to H^n(U_1, \mathscr{F}) \oplus H^n(U_2, \mathscr{F}) \to H^n(U_1 \cap U_2, \mathscr{F}) \to H^{n+1}(U_1 \cup U_2, \mathscr{F}) \to \ldots$$

Proof. Follows from the Proposition above and (2.2.8). □

## 5. Sheaves on Noetherian Spaces

n-filtered-colim-acyclic

Lemma 7.5.1. *Let $X$ be a Noetherian topological space, and let $I$ be a filtered category. Then for any functor $\mathscr{F} : I \to \mathrm{Shf}(X, \mathrm{Ab})$, the presheaf*

$$U \mapsto \operatorname{colim}(\mathscr{F}_i(U))$$

*is already a sheaf, where for an object $i \in I$, we denote the sheaf $\mathscr{F}(i)$ by $\mathscr{F}_i$, is already a sheaf. In other words, we have*

$$\Gamma(U, \operatorname{colim} \mathscr{F}_i) = \operatorname{colim}(\Gamma((, \mathscr{F}_i)U)),$$

*for every open set $U \subset X$*

Proof. Let the presheaf in question be denoted $\mathscr{G}$. We'll denote the maps in the directed system by $\phi_{k,l} : \mathscr{F}_k \to \mathscr{F}_l$. Suppose $U$ is an open set in $X$ and $\mathcal{V} = \{V_i\}$ is a weak covering sieve of $U$. We want to show that the natural map $\mathscr{G}(U) \to \mathcal{V}(\mathscr{G})$

is an isomorphism. Since $X$ is Noetherian, $U$ is quasicompact [NS, 3.3 ], and we can find a finite subcover $\{V_1, \dots, V_n\}$ of $\mathcal{V}$ for $U$.

Let's show injectivity first: suppose $s \in \mathscr{G}(U)$ is such that $\mathrm{res}_{U,V_i}(s) = 0$, for $i = 1, \dots, n$. Let $s$ be represented by $t \in \mathscr{F}_k(U)$, for some $k$. Then, there is some $l \geq k$ (by which we mean there is a morphism $k \to l$) such that $\phi_{k,l_{V_i}}(\mathrm{res}_{U,V_i}(t)) = 0$, for all $i$. This then means that $\phi_{k,l_U}(t) = 0$ in $\mathscr{F}_l(U)$, and so $s = 0$ in $\mathscr{G}(U)$. This shows that the presheaf $\mathscr{G}$ is separated.

Now, on to surjectivity: suppose we have $s_i \in \mathscr{G}(V_i)$ such that $\mathrm{res}_{V_i,V_i \cap V_j}(s_i) = \mathrm{res}_{V_j,V_i \cap V_j}(s_j) \in \mathscr{G}(V_i \cap V_j)$, for all $i, j$. Since the presheaf is separated, it suffices to piece together an $s \in \mathscr{G}(U)$ from $s_i$ for $i = 1, \dots, n$. So we can find $k$ such that for all $i = 1, \dots, n$, we have $t_i \in \mathscr{F}_k(V_i)$ representing $s_i$, such that they form a coherent sequence for $\mathscr{F}_k$ over the weak covering sieve generated by the open cover $\{V_1, \dots, V_n\}$. Since $\mathscr{F}_k$ is a sheaf, we can piece the $t_i$ together to get a section $t$ of $\mathscr{F}_k$ over $U$. It's easy to check now that the image of $t$ in $\mathscr{G}(U)$ restricts to each of the $s_i$. $\square$

an-filtered-colim-flabby

LEMMA 7.5.2. *Let $X$ be a Noetherian space and suppose $\mathscr{F} : I \to \mathrm{Shf}(X, \mathrm{Ab})$ is a functor from a filtered category $I$, with $\mathscr{F}_i$ flabby, for all objects $i$ in $I$. Then* $\operatorname{colim} \mathscr{F}_i$ *is also flabby, and in particular is $\Gamma_Z(X, \_\_)$-acyclic, for all closed subsets $Z \subset X$.*

PROOF. Using the lemma above, this reduces to the fact that colimits of abelian groups preserve surjections, which is of course true. $\square$

cohomology-colim-commute

PROPOSITION 7.5.3. *Let $X$ be a Noetherian topological space, let $Z \subset X$ be a closed subspace, and let $\mathscr{F} : I \to \mathrm{Shf}(X, \mathrm{Ab})$ be a functor from a filtered category $I$. Then, for every $n \geq 0$, we have natural isomorphisms*

$$H^n_Z(X, \operatorname{colim} \mathscr{F}_i) \cong \operatorname{colim} H^n_Z(X, \mathscr{F}_i).$$

PROOF. Since $\mathrm{Shf}(X, \mathrm{Ab})$ and Ab are both Grothendieck categories, we see that $\mathrm{Funct}(I, \mathrm{Shf}(X, \mathrm{Ab}))$ is also a Grothendieck category. In particular, it has enough injectives, and the colimit functor $\operatorname{colim} : \mathrm{Funct}(I, \mathrm{Shf}(X, \mathrm{Ab})) \to \mathrm{Shf}(X, \mathrm{Ab})$ is exact. Therefore, the Proposition will follow from (3.6.7) and the lemma above, if we show that there is a natural isomorphism:

$$\Gamma_Z(X, \operatorname{colim} \mathscr{F}_i) \cong \operatorname{colim} \Gamma_Z(X, \mathscr{F}_i).$$

First we will show that the functor $\underline{H}^0_Z(\_\_)$ commutes with colimits. For this observe that we have a natural morphism

$$\operatorname{colim}(\underline{H}^0_Z(\mathscr{F}_i)) \to \underline{H}^0_Z(\operatorname{colim} \mathscr{F}_i).$$

We now consider the action of this morphism induced on stalks. When $x \notin Z$, then the stalk at $x$ of both sheaves involved is zero; so we can assume that $x$ is in $Z$. In this case, since stalks commute with colimits, we have

$$\begin{aligned} \operatorname{colim}(\underline{H}^0_Z(\mathscr{F}_i))_x &= \operatorname{colim}(\underline{H}^0_Z(\mathscr{F}_i)_x) \\ &= \operatorname{colim} \end{aligned}$$

*TOBEDONE*

$\square$

xtzero-closed-cohomology

LEMMA 7.5.4. *Let $X$ be a topological space, and suppose $Y, Z \subset X$ are closed subsets. If $\mathscr{F} \in \operatorname{Shf}(Y, \mathrm{Ab})$ is a sheaf of abelian groups over $Y$, then, for all $n$, we have a natural isomorphism*

$$H^n_{Z\cap Y}(Y, \mathscr{F}) \cong H^n_Z(X, i_*\mathscr{F}),$$

*where $i : Y \to X$ is the inclusion map.*

PROOF. Follows from (3.6.5), since $i_*$ is exact and takes injective sheaves to flabby sheaves by (7.1.6) and (7.1.5). □

We now come to the main result of this section: the Vanishing Theorem of Grothendieck. It's quite a strong result with a strikingly simple proof. Before that, a definition.

DEFINITION 7.5.5. The *homological dimension* of a topological space $X$ is defined to be the quantity

$$\sup\{\Gamma(X, \_\_) - \dim(\mathscr{F}) :\ \mathscr{F} \in \operatorname{Shf}(X, \mathrm{Ab})\}.$$

It is either a non-negative integer or $\infty$.

f-grothendieck-vanishing

THEOREM 7.5.6 (Grothendieck's Vanishing Theorem). *Let $X$ be a finite dimensional Noetherian topological space, with $\dim X = n$, and let $Z \subset X$ be a closed subspace; then, for every sheaf $\mathscr{F} \in \operatorname{Shf}(X, \mathrm{Ab})$, we have $H^r_Z(X, \mathscr{F}) = 0$, for $r > n$. In particular, the homological dimension of $n$ is at most $n$.*

PROOF. The proof will be Bourbakiesque; that is, in several steps, each of which cuts away at the complexities of the problem, till, at the end, the statement that we have to actually prove becomes an obvious fact. We'll do a double induction on the number of irreducible components and on $\dim X$.

**Reduction to the case $X$ irreducible:** Suppose the Theorem is true for an irreducible space. Let $Y \subset X$ be an irreducible component of $X$ and let $U = X \setminus Y$. Then, for every sheaf $\mathscr{F} \in \operatorname{Shf}(X, \mathrm{Ab})$, we have an exact sequence

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Y) \to 0$$

Using the long exact sequence associated to this short exact sequence, and observing that $j_!(\mathscr{F}|_U)$ is a sheaf supported on $W = \overline{U}$, where $Z$ has one fewer irreducible component than $X$, we use induction on the number of irreducible components and (7.5.4) to reduce to the case where $X$ is irreducible.

**The Base Case:** So now we can suppose that $X$ is irreducible. Suppose $dimX = 0$; then the only closed subsets of $X$ are $X$ and $\emptyset$, and so the only open subsets of $X$ are $X$ and $\emptyset$. We see then that either $Z = \emptyset$ and $\Gamma_Z(X, \_\_)$ is the 0 functor, or $Z = X$ and $\operatorname{Shf}(X, \mathrm{Ab})$ is isomorphic to Ab via the functor $\Gamma(X, \_\_)$. In particular, $\Gamma(X, \_\_)$ is exact, and so the homological dimension of $X$ is 0.

**Reduction to the Finitely Generated case:** Now suppose $\dim X = n > 0$ (assuming still that $X$ is irreducible), and let $\mathscr{F} \in \operatorname{Shf}(X, \mathrm{Ab})$ be a sheaf. Set $\mathscr{S} = \bigcup_{U \subset X} \Gamma(U, \mathscr{F})$, and let $I$ be the poset of finite subsets of $\mathscr{S}$. Considering this as a filtered category, take the functor $\mathscr{G} : I \to \operatorname{Shf}(X, \mathrm{Ab})$ that takes $i \in I$ to the subsheaf $\mathscr{G}_i$ of $\mathscr{F}$ generated by all the sections inside $i$. Then we see that $\mathscr{F} = \operatorname{colim} \mathscr{G}_i$, and so by (7.5.3) we have

$$H^r_Z(X, \operatorname{colim} \mathscr{G}_i) \cong \operatorname{colim} H^r_Z(X, \mathscr{G}_i).$$

In particular, it suffices to show that $H^r_Z(X, \mathscr{F}) = 0$, for all sheaves of finite type over $X$, and for all $r \geq n$.

**Reduction to the Case of One Generator:** Now suppose $\mathscr{F}$ is a sheaf generated by $r$ sections, and let $\mathscr{F}'$ be a subsheaf of $\mathscr{F}$ generated by $r-1$ sections; then we have an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is generated by one section over some open set $U \subset X$. Looking at the long exact sequence of cohomology associated to this sequence, and using induction on the number of generators, we find that it is enough to do the case where $\mathscr{F}$ is generated by one generator.

**Reduction to the Case of Ideal Sheaves:** In this case, there is some open subset $U \subset X$ such that $\mathscr{F}|_U$ is a quotient of $\underline{\mathbb{Z}}_U$. Moreover, we also have $\Gamma(V, \mathscr{F}) = 0$, for $V \not\subseteq U$. Therefore, $\mathscr{F}$ is in fact a quotient of $j_!(\underline{\mathbb{Z}}_U)$, where $j : U \to X$ is the inclusion map. From the long exact sequence associated to the short exact sequence

$$0 \to \mathscr{I} \to j_!(\underline{\mathbb{Z}}_U) \to \mathscr{F} \to 0,$$

we reduce to the case where $\mathscr{F}$ is a subsheaf of $\mathscr{G} = j_!(\underline{\mathbb{Z}}_U)$, for some open subset $U \subset X$.

**Reduction to the case $\mathscr{F} = j_!(\underline{\mathbb{Z}}_U)$:** Let $\mathscr{I} \subset \mathscr{G}$ be a subsheaf. If $\mathscr{I} = 0$, then we're done; otherwise, let $d \in \mathbb{N}$ be the smallest positive integer such that $\mathscr{I}_x = d\mathbb{Z}$ (where we consider $\mathbb{Z}$ to be the local ring of $\underline{\mathbb{Z}}_X$ at $x$). Now, $\mathscr{G}|_U = \underline{\mathbb{Z}}_U$ is a constant sheaf on $U$, since $U$ is irreducible. Therefore, we can find a neighborhood $V \ni x$ contained in $U$ such that $\mathscr{I}|_V = d\mathbb{Z}_V$. That means that we have the following short exact sequence:

$$0 \to j'_!(\underline{\mathbb{Z}}_V) \to^d \mathscr{I} \to \mathscr{I}'' \to 0,$$

where $j' : V \to X$ is the inclusion map, and where $\mathscr{I}''$ is supported on $U \setminus V$, which is contained in the proper closed subset $Y = X \setminus V$ of $X$. Since $\dim Y < n$, using the long exact sequence of cohomology obtained from this short exact sequence, and induction on the dimension, we reduce to proving the vanishing theorem for the case where $\mathscr{F} = j_!(\underline{\mathbb{Z}}_U)$ for the inclusion $j : U \to X$ of some open subset into $X$.

**The case $\mathscr{F} = j_!(\underline{\mathbb{Z}}_U)$:** Let $Y = X \setminus U$, and let $i : Y \to X$ be the inclusion map. Then we have an exact sequence

$$0 \to j_!(\underline{\mathbb{Z}}_U) \to \underline{\mathbb{Z}}_X \to i_*(\underline{\mathbb{Z}}_Y) \to 0$$

Since $Y \subsetneq X$ (we can of course assume that $U \neq \emptyset$, for otherwise we are trivially done), $\dim Y < \dim X$, and so by the induction on dimension we see that the homological dimension of $Y$ is at most $n-1$. Since $X$ is irreducible $\underline{\mathbb{Z}}_X$ is the constant sheaf, and is thus flabby, and hence $\Gamma_Z(X, \_\_)$-acyclic. Putting these two facts together with the long exact sequence of cohomology arising from this short exact sequence, we see that $H^r_Z(X, j_!(\underline{\mathbb{Z}}_U)) = 0$, for $r > n$.

□

REMARK 7.5.7. Note the analogy of this proof with the characterization of the global dimension of a ring $R$ (in increasing order of strength) in terms of finitely generated $R$-modules and then in terms of the ideals of $R$.

## 6. Čech Cohomology

The quickest way to get a computable cohomology of sheaves is via Čech cohomology. Unfortunately, this does not always agree with the cohomology theory obtained above from more abstract homological concerns. But in nice cases (say, for a separated scheme, or for a topological manifold) it does give the same answers.

Let $\mathcal{V} = \{V_i : i \in I$ be an open cover of a topological space $X$, and suppose $I$ has a total ordering. For each $p \geq 0$, set

$$\sigma^p(I) = \{(i_0, \ldots, i_p) \in I^{p+1} : i_0 < i_1 < \ldots < i_p\},$$

and for all $J = (i_0, \ldots, i_p) \in \sigma^p(I)$ set

$$V_J = \bigcap_{r=0}^{p} V_{i_r}.$$

For every $0 \leq k \leq p+1$, we have a map

$$\begin{aligned} \tau_k^p &: \sigma^{p+1}(\mathcal{V}) \to \sigma^p(I) \\ (i_0, \ldots, i_{p+1}) &\mapsto (i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}). \end{aligned}$$

For $0 \leq l < k \leq p+1$, these maps satisfy the formula

$$\tau_{k-1}^{p-1}\tau_l^p = \tau_l^{p-1}\tau_k^p.$$

Now, for $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$, set

$$C^p(\mathcal{V}, \mathscr{F}) = \prod_{J \in \sigma^p(I)} \Gamma(V_J, \mathscr{F}).$$

An element $s \in C^p(\mathcal{V}, \mathscr{F})$ is given by a collection of sections $s_J \in \Gamma(V_J, \mathscr{F})$, for all $J \in \sigma^p(I)$. We now define an ostensible differential

$$\begin{aligned} d^p &: C^p(\mathcal{V}, \mathscr{F}) \to C^{p+1}(\mathcal{V}, \mathscr{F}) \\ (d^p s)_J &= \sum_{0 \leq k \leq p+1} (-1)^k s_{\tau_k^p(J)} \qquad \text{for } J \in \sigma^{p+1}(\mathcal{V}). \end{aligned}$$

By $s_{\tau_k^p(J)}$, we of course mean $s_{\tau_k^p(J)}$ restricted to $V_J$, but we omit this additional information for convenience. To check that this is in fact a differential we compute, for $J \in \sigma^{p+2}(\mathcal{V})$,

$$\begin{aligned} (d^{p+1}d^p s)_J &= \sum_{0 \leq k \leq p+2} (-1)^k (d^p s)_{\tau_k^{p+1}(J)} \\ &= \sum_{0 \leq k \leq p+2} (-1)^k \left( \sum_{0 \leq l \leq p+1} (-1)^l s_{\tau_l^p(\tau_k^{p+1}(J))} \right) \\ &= \sum_{0 \leq l < k \leq p+2} (-1)^{k+l} s_{\tau_l^p(\tau_k^{p+1}(J))} + \sum_{0 \leq l \leq k \leq p+1} (-1)^{k+l} s_{\tau_k^p(\tau_l^{p+1}(J))} \\ &= 0, \end{aligned}$$

since every summand in the sum on the right hand side has a counterpart in the sum on the left hand side that differs from it only in sign. This follows from the formulas for $\tau^p_k$ that we found above.

Remark 7.6.1. For convenience, take the following convention: for any ordered $(p+1)$-tuple $(i_0, \dots, i_p) \in \sigma^p(I)$, any map $\alpha : [p] \to [p]$, and any element $s \in C^p(\mathcal{V}, \mathscr{F})$, we set

$$s_{(\alpha(i_0),\dots,\alpha(i_p))} = \begin{cases} (-1)^{\mathrm{sgn}(\alpha)} s_{(i_0,\dots,i_p)} & \text{if } \alpha \text{ is a bijection} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 7.6.2. Given an open covering $\mathcal{V} = \{V_i : i \in I\}$ of a topological space $X$, and a sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$, the *Čech complex* of $\mathscr{F}$ over $\mathcal{V}$ is the complex $C^\bullet(\mathcal{V}, \mathscr{F})$.

The cohomology $H^\bullet(C(\mathcal{V}, \mathscr{F}))$ of this complex is called the *Čech cohomology* of $\mathscr{F}$ over $\mathcal{V}$, and is denoted $\check{H}^\bullet(\mathcal{V}, \mathscr{F})$.

We also define a sheaf theoretic version of the Čech complex. For $p \geq 0$, and a sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$, set

$$\underline{C}^p(\mathcal{V}, \mathscr{F}) = \prod_{J \in \sigma^p(I)} f^U_*(\mathscr{F}|_{V_J}),$$

where, for every open subset $U \subset X$, $f^U : U \to X$ is the inclusion map. Observe that we have:

$$\Gamma(U, \underline{C}^p(\mathcal{V}, \mathscr{F})) = C^p(\mathcal{V} \cap U, \mathscr{F}|_U).$$

Therefore, the graded sheaf $\underline{C}^\bullet(\mathcal{V}, \mathscr{F})$ has a structure of a complex of sheaves.

xactness-sheafified-cech

Lemma 7.6.3. *Let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be a sheaf of abelian groups over a topological space $X$, and let $\mathcal{V} = \{V_i : i \in I\}$ be an open cover for $X$. Then the complex*

$$0 \to \mathscr{F} \xrightarrow{\epsilon} \underline{C}^0(\mathcal{V}, \mathscr{F}) \to \underline{C}^1(\mathcal{V}, \mathscr{F}) \to \cdots$$

*is exact in $\mathrm{Shf}(X, \mathrm{Ab})$, where $\epsilon : \mathscr{F} \to \underline{C}^0(\mathcal{V}, \mathscr{F})$ is the product of the natural restrictions.*

Proof. It's enough to show exactness locally; more precisely, we'll show that for every $k \in I$ and every open set $U \subset V_k$, the complex

$$0 \to \Gamma(U, \mathscr{F}) \to C^0(\mathcal{V} \cap U, \mathscr{F}) \to C^1(\mathcal{V} \cap U, \mathscr{F}) \to \cdots$$

is exact. For this, consider the maps

$$h^p : C^p(\mathcal{V} \cap U, \mathscr{F}) \to C^{p-1}(\mathcal{V} \cap U, \mathscr{F})$$
$$(h^p s)_J = s_{J \cup \{k\}} \qquad \text{for } J \in \sigma^{p-1}(I).$$

Here we take $\sigma^{-1}(I) = \emptyset$ and $C^{-1}(\mathcal{V} \cap U) = \Gamma(U, \mathscr{F})$. We have, for $s \in C^p(\mathcal{V} \cap U, \mathscr{F})$, and $J \in \sigma^p(I)$,

$$\begin{aligned}\big((d^{p-1}h^p + h^{p+1}d^p)s\big)_J &= \sum_{0 \leq r \leq p} (-1)^r (h^p s)_{\tau_r^{p-1}(J)} + (d^p s)_{J \cup \{k\}} \\ &= \sum_{0 \leq r \leq p} (-1)^r s_{\tau_r^{p-1}(J) \cup \{k\}} + \sum_{0 \leq t \leq p+1} (-1)^t s_{\tau_t^p(J \cup \{k\})}\end{aligned}$$

□

Now we come to the main theorem.

sheaf-cech-derived-agree

THEOREM 7.6.4. *Let $X$ be a topological space, and let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be a sheaf of abelian groups over it. Suppose $\mathcal{V} = \{V_i : i \in I\}$ is an open cover such that $\mathscr{F}$ is $\Gamma(V_J, \_\_)$-acyclic, for all $J \in \sigma^p(I)$, and for $p \geq 0$. Then there is a natural isomorphism*

$$\check{H}^\bullet(\mathcal{V}, \mathscr{F}) \cong H^\bullet(X, \mathscr{F}).$$

PROOF. We will use the Grothendieck spectral sequence. Consider the sequence of functors:

$$\mathrm{Shf}(X, \mathrm{Ab}) \xrightarrow{C^\bullet(\mathcal{V}, \_\_)} \mathrm{Ch}^{\geq 0} \mathrm{Ab} \xrightarrow{H^0} \mathrm{Ab}\,.$$

We claim that $C^\bullet(\mathcal{V}, \_\_)$ takes flabby sheaves to acyclic complexes in $\mathrm{Ch}^{\geq 0} \mathrm{Ab}$. Indeed, if $\mathscr{I}$ is flabby, then $C^\bullet(\mathcal{V}, \mathscr{I})$ is just the complex obtained from applying the global sections functor to $\underline{C}^\bullet(\mathcal{V}, \mathscr{I})$, which according to the lemma above is an acyclic complex of flabby sheaves. Since exact sequences of flabby sheaves are preserved by the global sections functor, we have our claim.

Moreover, observe that we have $H^0(C^\bullet(\mathcal{V}, \_\_)) \cong \Gamma(X, \_\_)$. This is just the sheaf axiom.

Given this, we're in a position to apply Grothendieck's spectral sequence, to conclude that we have a spectral sequence $\{E_r^\bullet\}$ such that

$$H^p(R^q(C^\bullet(\mathcal{V}, \_\_))(\mathscr{F})) = E_2^{p,q} \Rightarrow H^{p+q}(X, \mathscr{F}).$$

In particular, we have a natural surjection

$$\check{H}^n(\mathcal{V}, \mathscr{F}) = H^n(R^0(C^\bullet(\mathcal{V}, \_\_))(\mathscr{F})) \to H^n(X, \mathscr{F}),$$

whose kernel lies in $E_2^{n-2,1} = H^{n-2}(R^1(C^\bullet(\mathcal{V}, \_\_))(\mathscr{F}))$.

It remains to compute the derived functors of the Čech complex functor. For this, take an injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}$, and consider the double complex $K^{\bullet,\bullet} = C^\bullet(\mathcal{V}, \mathscr{I}^\bullet)$. We have

$$\begin{aligned} \left(R^p(C^\bullet(\mathcal{V}, \_\_))(\mathscr{F})\right)^q &= H_{II}^{q,p}(K) \\ &= \prod_{J \in \sigma^p(I)} H^q(V_J, \mathscr{F}) \\ &= 0, \quad \text{if } q > 0. \end{aligned}$$

We applied our hypothesis on the local acyclicity of $\mathscr{F}$ in the last equality. From this we find that the kernel of the natural surjection is in fact zero, and so we have the isomorphism promised to us in the Theorem. □

EXAMPLE 7.6.5 (Cohomology of $\mathbb{A}^2 - \{(0,0)\}$). Let $U = \mathbb{A}^2 - \{(0,0)\}$ be the complement of the origin in the affine plane. Then $U$ has a covering by the principal affine opens $U_1 = \operatorname{Spec} k[x,y]_x$ and $U_2 = \operatorname{Spec} k[x,y]_y$. Let's compute the Čech complex of $\mathscr{O}_U$ corresponding to the open cover $\mathcal{V} = \{U_1, U_2\}$ of $U$. We have

$$\begin{aligned} C^0(\mathcal{V}, \mathscr{O}_U) &= k[x,y,x^{-1}] \times k[x,y,y^{-1}] \\ C^1(\mathcal{V}, \mathscr{O}_U) &= k[x,y,x^{-1},y^{-1}] \\ d^1 : C^0(\mathcal{V}, \mathscr{O}_U) &\to C^1(\mathcal{V}, \mathscr{O}_U) \\ (f(x,y,x^{-1}), g(x,y,y^{-1})) &= (f(x,y,x^{-1}) - g(x,y,x^{-1}))C^n(\mathcal{V}, \mathscr{O}_U) \quad = 0, \quad \text{for } n > 1, \end{aligned}$$

The image of $d^1$ consists of all linear combinations of monomials $x^i y^j$, where at least one of $i$ or $j$ is non-negative. Therefore, we find that

$$H^1(U, \mathscr{O}_U) \cong \check{H}^1(\mathcal{V}, \mathscr{O}_U)$$

is an infinite dimensional vector space over $k$ spanned by the images of the monomials $x^i y^j$, where $i, j < 0$.

Observe that we didn't use the acyclicity condition on $\mathscr{F}$ right till the end of the proof of the Theorem. In particular, we did not use it to construct the spectral sequence. We extract this spectral sequence from the proof.

mology-spectral-sequence

Proposition 7.6.6 (The Čech Cohomology Spectral Sequence). *Let $X$ be a topological space, let $\mathcal{V} = \{V_i : i \in I\}$ be an open cover of $X$, and let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be a sheaf of abelian groups over $X$. For $n \geq 0$, let $\underline{H}^n(\mathscr{F})$ be the presheaf on $X$ given by $U \mapsto H^n(U, \mathscr{F}|_U)$. Then we have a first quadrant spectral sequence $\{E_\bullet^r : r \geq 0\}$ such that*

$$H^p(C^\bullet(\mathcal{V}, \underline{H}^q(\mathscr{F}))) \cong E_2^{p,q} \Rightarrow H^{p+q}(X, \mathscr{F}).$$

Proof. We have already shown that there is a spectral sequence boundedly converging to $H^{p+q}(X, \mathscr{F})$, on whose second page we have

$$E_2^{p,q} \cong H^p(R^q(C^\bullet(\mathcal{V}, \text{--}))(\mathscr{F})).$$

It remains to compute the derived functors of the Čech complex functor. We can consider the Čech complex functor to be the composition:

$$\mathrm{Shf}(X, \mathrm{Ab}) \xrightarrow{i} \mathrm{Pre}(X, \mathrm{Ab}) \xrightarrow{C^\bullet(\mathcal{V}, \text{--})} \mathrm{Ch}^{\geq 0} \mathrm{Ab}\,.$$

Since the Čech complex functor is clearly exact on $\mathrm{Pre}(X, \mathrm{Ab})$, we have by (3.6.5):

$$R^q(C^\bullet(\mathcal{V}, \text{--}))(\mathscr{F}) = C^\bullet(\mathcal{V}, R^q i(\mathscr{F})).$$

So what we really need to do is compute the right derived functors of the forgetful functor $i$. But, by taking a flasque resolution $\mathscr{I}^\bullet$ of $\mathscr{F}$, it's easy to see that we have

$$\begin{aligned} \Gamma(U, R^q i(\mathscr{F})) &= H^q(\Gamma(U, \mathscr{I}^\bullet)) \\ &= H^q(U, \mathscr{F}|_U). \end{aligned}$$

□

## 7. Ext Groups

There are two other fundamental left exact functors on the category $\mathscr{O}_X$-mod, for a ringed space $(X, \mathscr{O}_X)$. These are the functors $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \text{--})$ and $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{F}, \text{--})$, for some fixed $\mathscr{O}_X$-module $\mathscr{F}$. We will investigate their derived functors in this section and construct a spectral sequence relating the derived functors of the pair.

Definition 7.7.1. Let $(X, \mathscr{O}_X)$ be a ringed space. For $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, we define, for $n \geq 0$,

$$\begin{aligned} \mathrm{Ext}^n_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) &= \mathrm{Ext}^n_{\mathscr{O}_X\text{-mod}}(\mathscr{F}, \mathscr{G}). \\ \underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) &= R^n(\underline{\mathrm{Hom}}_{\mathscr{O}_X\text{-mod}}(\mathscr{F}, \text{--}))(\mathscr{G}). \end{aligned}$$

Here are some preliminary properties of and relations between the two derived functors.

heaf-ext-sext-properties

PROPOSITION 7.7.2. *Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules.*

(1) *For every open subset $U \subset X$, we have natural isomorphisms*
$$\underline{\operatorname{Ext}}^{\bullet}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})|_U \cong \underline{\operatorname{Ext}}^{\bullet}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U).$$
(2) *If $\mathscr{F} = \mathscr{O}_X$, then we have*
$$\operatorname{Ext}^n_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong H^n(X,\mathscr{G}).$$
$$\underline{\operatorname{Ext}}^n_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong \begin{cases} \mathscr{G} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. In (1), both $\delta$ functors are clearly effaceable for $n > 0$, since $\mathscr{I}|_U$ is injective for any injective sheaf $\mathscr{I}$ (7.1.8) and agree in degree 0; hence they are isomorphic. For (2), just note that we have
$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong \Gamma(X,\mathscr{G}).$$
$$\underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{G}) \cong \mathscr{G}.$$
□

Here's a useful spectral sequence that relates $\underline{\operatorname{Ext}}$ and Ext.

al-ext-spectral-sequence

PROPOSITION 7.7.3 (Local Ext Spectral Sequence). *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules; then we have a first quadrant spectral sequence $\{E_r^{\bullet} : r \geq 0\}$ such that*
$$H^p(X, \underline{\operatorname{Ext}}^q_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})) \cong E_2^{p,q} \Rightarrow \operatorname{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}).$$

PROOF. Observe that we have
$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \text{--}) \cong \Gamma((, \underline{\operatorname{Hom}}_{\mathscr{O}_X})\mathscr{F}, \text{--}).$$
Moreover, for every injective sheaf $\mathscr{I}$, $\underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{F},\mathscr{I})$ is flabby by (7.1.6). Now the proposition follows from Grothendieck's spectral sequence (3.6.4). □

Now we look at how these functors behave when tensored with locally free sheaves of finite rank. This will prove helpful when we consider the cohomology of quasi-coherent sheaves over projective space.

sext-locally-free-tensor

PROPOSITION 7.7.4. *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules, and let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank, and let $\check{\mathscr{E}}$ be its dual.*

(1) *For every injective sheaf $\mathscr{I}$, $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{I}$ is also injective.*
(2) *We have natural isomorphisms:*
$$\operatorname{Ext}^{\bullet}_{\mathscr{O}_X}(\mathscr{F}\otimes\mathscr{E},\mathscr{G}) \cong \operatorname{Ext}^{\bullet}_{\mathscr{O}_X}(\mathscr{F},\check{\mathscr{E}}\otimes\mathscr{G})$$
$$\begin{aligned} \underline{\operatorname{Ext}}^{\bullet}_{\mathscr{O}_X}(\mathscr{F}\otimes\mathscr{E},\mathscr{G}) &\cong \operatorname{Ext}^{\bullet}_{\mathscr{O}_X}(\mathscr{F},\check{\mathscr{E}}\otimes\mathscr{G}) \\ &\cong \underline{\operatorname{Ext}}^{\bullet}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})\otimes\check{\mathscr{E}}. \end{aligned}$$

PROOF. For (1), just note that we have natural isomorphisms
$$\begin{aligned} \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{E}\otimes_{\mathscr{O}_X}\mathscr{I}) &\cong \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F},\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{L},\mathscr{I})) \\ &\cong \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}\otimes\mathscr{L},\mathscr{I}), \end{aligned}$$
and the last functor is clearly exact in $\mathscr{F}$.

For the first or second isomorphisms in (2), observe that on either side of the isomorphism we have $\delta$-functors agreeing in degree 0, both of which are effaceable, the first, quite trivially, and the second by part (1). For the last isomorphism, we only need to check that we have

$$\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \otimes \check{E} \cong \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{F} \otimes \mathscr{E},\mathscr{G}).$$

There is a natural morphism in one direction, and it's easy to check locally on open sets where $\mathscr{E}$ is free that this is indeed an isomorphism. □

-locally-free-resolution

PROPOSITION 7.7.5. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and let $\mathscr{E}^\bullet$ be a left resolution of $\mathscr{F}$ by locally free $\mathscr{O}_X$-modules of finite rank. Then, for every $\mathscr{O}_X$-module $\mathscr{G}$, we have natural isomorphisms*

$$\underline{\mathrm{Ext}}^\bullet_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \cong H^{-n}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E}^\bullet,\mathscr{G})).$$

PROOF. Follows from (3.4.3), since $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E}, \_\_)$ is an exact functor, for every locally free $\mathscr{O}_X$-module $\mathscr{E}$ (locally it is isomorphic to the functor that takes $\mathscr{G}$ to $\mathscr{G}^n$, for some $n \geq 0$). □

The next Corollary shows that $\underline{\mathrm{Ext}}$ preserves coherence and quasicoherence under some conditions that are common in geometric situations.

sheaf-sext-coherent

COROLLARY 7.7.6. *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules; suppose that $X$ is Noetherian, and that $\mathscr{O}_X$ and $\mathscr{F}$ are coherent $\mathscr{O}_X$-modules.*

1. *If $\mathscr{G}$ is also coherent, then $\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is also coherent.*
2. *If $\mathscr{O}_X$-qcoh is a Serre subcategory of $\mathscr{O}_X$-mod (for example if $X$ is a scheme), then $\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is a quasi-coherent sheaf.*

PROOF. The main point is that when $\mathscr{O}_X$ is coherent and $X$ is Noetherian, every coherent $\mathscr{O}_X$-module has a local resolution by locally free $\mathscr{O}_X$-modules of finite rank. This is quite easy to see from the definition of coherence: for any point $x \in X$, we can find a neighborhood $U$ of $x$ and some free sheaf $\mathscr{E}^0$ surjecting onto $\mathscr{F}|_U$. Now the kernel of this is again coherent, and hence has some other locally free sheaf surjecting onto it on a smaller neighborhood of $x$. Proceeding in this fashion, since $X$ is Noetherian, we can find some smallest neighborhood $W$ of $x$ on which we have a free resolution of $\mathscr{F}|_W$. Replacing $X$ with $W$, we can assume that $\mathscr{F}$ has a free resolution over $X$.

Now, let $\mathscr{E}^\bullet$ be a locally free resolution of $\mathscr{F}$; then by the Proposition we have

$$\underline{\mathrm{Ext}}^\bullet_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \cong H^{-n}(\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E}^\bullet,\mathscr{G})).$$

If $\mathscr{G}$ is coherent, then $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E}^\bullet,\mathscr{G})$ is a complex of coherent sheaves, and hence its cohomology is also coherent. If $\mathscr{O}_X$-qcoh is a Serre subcategory, then it contains its kernels and cokernels, and moreover $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E}^\bullet,\mathscr{G})$ is a complex of quasi-coherent sheaves. Hence its cohomology sheaves are also quasi-coherent. □

sheaf-sext-stalks

COROLLARY 7.7.7. *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules; suppose that $X$ is Noetherian and also that $\mathscr{O}_X$ and $\mathscr{F}$ are coherent $\mathscr{O}_X$-modules; then, for every $x \in X$, we have natural isomorphisms*

$$\underline{\mathrm{Ext}}^\bullet_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x \cong \mathrm{Ext}^\bullet_{\mathscr{O}_x}(\mathscr{F}_x,\mathscr{G}_x).$$

PROOF. Since $\mathscr{F}$ and $\mathscr{O}_X$ are coherent, there exists a neighborhood $U$ around every $x \in X$ and a resolution $\mathscr{E}^\bullet$ of $\mathscr{F}|_U$ by free sheaves of finite rank. Replacing $U$ by $X$, we have by the Proposition above:

$$\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) = H^{-n}(\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}^\bullet,\mathscr{G}))$$

Taking stalks at $x$ on both sides and observing that this is an application of an exact functor, we have

$$\begin{aligned}\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x &= H^{-n}(\mathrm{Hom}_{\mathscr{O}_x}(\mathscr{E}^\bullet_x,\mathscr{G}_x))\\ &\cong \mathrm{Ext}^n_{\mathscr{O}_x}(\mathscr{F}_x,\mathscr{G}_x)\end{aligned}$$

where we have used the isomorphism

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}^\bullet,\mathscr{G})_x \cong \mathrm{Hom}_{\mathscr{O}_x}(\mathscr{E}^\bullet_x,\mathscr{G}_x)$$

that is obtained from the fact that $\mathscr{E}^n$ is free of finite rank and hence finitely presented, for all $n \leq 0$. The second isomorphism follows from the fact that $\mathscr{E}^\bullet_x$ is a projective resolution for $\mathscr{F}_x$ in $\mathscr{O}_X$-mod. □

## 8. Higher Direct Images

DEFINITION 7.8.1. Given a continuous map $f : X \to Y$ of topological spaces, the direct image functor $f_* : \mathrm{Shf}(X, \mathrm{Ab}) \to \mathrm{Shf}(Y, \mathrm{Ab})$ is a right adjoint and is hence left exact. We call its derived functors $R^n f_*$ the *higher direct images* of $f$.

er-direct-image-presheaf

PROPOSITION 7.8.2. *Let $f : X \to Y$ be a continuous map, and let $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ be a sheaf. Then, for $n \geq 0$, $R^n f_*\mathscr{F}$ is the sheafification of the presheaf that assigns to every open subset $U \subset Y$, the group $H^n(f^{-1}(U), \mathscr{F})$.*

PROOF. Let $G^\bullet : \mathrm{Shf}(X, \mathrm{Ab}) \to \mathrm{Pre}(Y, \mathrm{Ab})$ be the $\delta$-functor that assigns to every sheaf $\mathscr{F}$, the presheaf $U \mapsto H^\bullet(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$. Consider now the $\delta$-functor $\mathrm{Shf}\,\mathscr{G}^\bullet : \mathrm{Shf}(X, \mathrm{Ab}) \to \mathrm{Shf}(Y, \mathrm{Ab})$: this is a $\delta$-functor, since Shf is exact. Moreover, we have $\mathrm{Shf}\,\mathscr{G}^0 \cong f_*$; therefore it is enough to show that $\mathrm{Shf}\,\mathscr{G}^n$ is effaceable, for $n \geq 1$. For this, let $\mathscr{I}$ be any injective sheaf over $X$; then $\mathscr{I}|_{f^{-1}(U)}$ is still injective for all open subsets $U \subset Y$ (7.1.8), and so we find that $\mathscr{G}^n\mathscr{I} = 0$, for $n \geq 1$, which shows effaceability. □

mages-flabby-resolutions

COROLLARY 7.8.3. *Let $f : X \to Y$ be a continuous map.*

(1) *Any flabby sheaf over $X$ is $f_*$-acyclic.*
(2) *For any sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$ and any flabby resolution $\mathscr{G}^\bullet$ of $\mathscr{F}$, we have natural isomorphisms*

$$R^\bullet f_*\mathscr{F} \cong H^\bullet((\mathrm{Ch}\, f_*)(\mathscr{G}^\bullet)).$$

(3) *If $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces, then the derived functors of the direct image functor*

$$f_* : \mathscr{O}_X\text{-mod} \to \mathscr{O}_Y\text{-mod}$$

*agree with $R^\bullet f_*$.*

PROOF. (1) follows from the Proposition and (2) follows immediately from (1) and (2.3.8. For (3), just note that any injective resolution in $\mathscr{O}_X$-mod is flasque in $\mathrm{Shf}(X, \mathrm{Ab})$ by (7.1.6). □

We finish this section with a look at the Leray spectral sequence for higher direct images. Like most of the spectral sequences we've seen before, this will be a special case of the Grothendieck spectral sequence.

-leray-spectral-sequence

PROPOSITION 7.8.4 (Leray Spectral Sequence). *Let $f : X \to Y$ be a continuous map, and let $Z \subset Y$ be a closed subset; then, for every sheaf $\mathscr{F} \in \mathrm{Shf}(X, \mathrm{Ab})$, we can naturally associate a first quadrant spectral sequence $\{E_r^\bullet : r \geq a\}$ such that*

$$H_Z^p(Y, R^q f_* \mathscr{F}) \cong E_2^{p,q} \Rightarrow H_{f^{-1}(Z)}^{p+q}(X, \mathscr{F}).$$

PROOF. Observe that we have $\mathit{\Gamma}_{f^{-1}(Z)}(X, \mathscr{F}) \cong \mathit{\Gamma}_Z(Y, f_* \mathscr{F})$. To see this, observe that both sides of the identity are isomorphic to the kernel of the restriction map $\mathit{\Gamma}(X, \mathscr{F}) \to \mathit{\Gamma}(f^{-1}(X \setminus Z), \mathscr{F})$.

Now the statement follows from (3.6.4), and the fact that $f_*$ takes flabby sheaves to flabby sheaves (7.1.5). $\square$

CHAPTER 8

# Derived Categories

chap:derived

CHAPTER 9

# More on Sheaf Cohomology

chap:mosheaf