Algebraic Geometry

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Contents

CHAPTER 1

Schemes

chap:scheme

1. Spec of a Ring

The basic construction in the theory of schemes is the spectrum of a commutative ring. Given such a ring R, consider the set $\mathrm{spc}(\operatorname{Spec} R)$ whose elements are the prime ideals of R. For every ideal $I \subset R$, we define the subset $V(I) \subset \mathrm{spc}(\operatorname{Spec} R)$ to be the set of primes containing I.

scheme-vofi-prps

PROPOSITION 1.1.1. *The assignment $I \mapsto V(I)$ satisfies the following properties:*

(1) $V(I) = \emptyset$, *if and only if* $I = R$.
(2) $V(I) = \mathrm{spc}(\operatorname{Spec} R)$, *if and only if* $I \subset \mathrm{Nil}(R)$.
(3) $V(I) = V(\mathrm{rad}(I))$.
(4) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$, *for two ideals* I, J.
(5) $V(\sum_k I_k) = \cap_k V(I_k)$, *for an arbitrary collection of ideals* $\{I_k\}$.
(6) $V(I) \subset V(J)$ *if and only if* $J \subset \mathrm{rad}(I)$.
(7) $V(I)$ *is homeomorphic to* $\mathrm{spc}(\operatorname{Spec}(R/I))$.

PROOF. The only property that needs a proof is the last one. Observe first that we have a natural bijection from $V(I)$ to $\mathrm{spc}(\operatorname{Spec}(R/I)) =: Y$ that just takes a prime $P \supset I$ to the prime $P/I \subset R/I$. It's easy to see that this bijection pulls back a closed set $V(J/I) \subset Y$ to the closed set $V(J) \subset X$. So it's continuous; moreover the homeomorphism maps a closed set $V(J) \supset V(I)$ to $V(\mathrm{rad}\, J/I) \subset Y$. Hence it's a closed map and is thus a homeomorphism. □

Given these properties, we can define a topology on $X := \mathrm{spc}(\operatorname{Spec} R)$ whose collection of closed sets is $\{V(I) : I \subset R \text{ an ideal}\}$.

Now, if $f \in R$ is any element, we have an open set $X_f = X \setminus V((f))$. This is called a *principal open set*, and consists of all primes that don't contain f.

heme-principal-open-prps

PROPOSITION 1.1.2. *We can say the following things about principal open sets:*

(1) $X_f = X$, *if and only if f is a unit.*
(2) $X_f = \emptyset$ *if and only if* $f \in \mathrm{Nil}(R)$.
(3) $X_{fg} = X_f \cap X_g$.
(4) *Given any open set $U \subset X$, and a prime $P \in U$, there is $f \in R$ such that $P \in X_f \subset U$.*
(5) *The principal open sets form an open base for the topology on X.*
(6) $X_f \subset X_g$ *if and only if* $ag = f^k$, *for some* $a \in R$, $k \in \mathbb{N}$.
(7) X_f *is homeomorphic to* $\mathrm{spc}(\operatorname{Spec} R_f)$.

PROOF. (1) Obvious.
(2) Follows from part (2) of Proposition before this.

(3) Follows from the fact that a prime doesn't contain the product fg if and only if it doesn't contain both f and g.
(4) Suppose $U = X \setminus V(I)$, for some ideal $I \subset R$. Then, any $f \in I$ will work.
(5) Immediate from the previous parts.
(6) Observe that

$$\begin{aligned} X_f \subset X_g &\Leftrightarrow V((f)) \supset V((g)) \\ &\Leftrightarrow (f) \subset \mathrm{rad}((g)) \\ &\Leftrightarrow ag = f^k, \text{ for some } a \in R, k \in \mathbb{Z}. \end{aligned}$$

(7) There is a natural bijection

$$\begin{aligned} X_f &\to \mathrm{spc}(\operatorname{Spec} R_f) \\ P &\mapsto P_f. \end{aligned}$$

The pull-back of a closed set $V(I_f)$ is of course just $V(I) \subset X$. So the map is continuous. Suppose $X_g \subset X_f$ is a principal open subset; then by the previous part $g^k = af$, for some $a \in R$, $k \in \mathbb{Z}$. But $X_g = X_{g^k} = X_{af} = X_a \cap X_f$. So X_g consists of all the primes that don't contain both a and f. This maps onto the principal open subset of $\mathrm{spc}(\operatorname{Spec} R_f)$ that consists of all primes not containing a. Since the principal open sets form a basis for the topology on X, this shows that the natural bijection is an open map, and thus a homeomorphism.

□

cheme-spec-quasi-compact

PROPOSITION 1.1.3. *Any principal open subset X_f of X is quasi-compact: every open cover of X_f has a finite subcover.*

PROOF. Since X_f is homeomorphic to $\mathfrak{sp}(\operatorname{Spec} R_f)$, it suffices to prove the statement for $X = \mathfrak{sp}(\operatorname{Spec} R)$. First, suppose we have an open cover of X by principal open sets. That is, $X = \bigcup_i X_{f_i}$ for some $f_i \in R$. Now, consider the ideal I generated by the f_i. We find that $V(I) = \emptyset$; so we must have $I = R$. This means that there is some finite linear combination of the f_i that equals 1. Let $\{f_1, \ldots, f_n\}$ be the finite subset of the f_i involved in this linear combination (that is, they're the ones that have non-zero coefficients). Then, we see that $(1) = (f_1, \ldots, f_n)$. Hence $\bigcap_{i=1}^n V((f_i)) = \emptyset$, and so $\bigcup_{i=1}^n X_{f_i} = X$.

If $\mathcal{V}$ is any open cover of X, then we can find a refinement consisting of principal open sets. Since we can find a finite subcover of the refinement, we can find one for $\mathcal{V}$. □

This allows us to give X a natural structure of a locally ringed space. Call a collection of elements $\Xi \subset R$ *complete* if the correspondence $f \mapsto X_f$ from Ξ to the collection of principal open sets is bijective. That is, each principal open set corresponds to a unique element in Ξ. Thus, we can think of Ξ as representing an open base on X.

Now, for $f \in \Xi$, set $\mathscr{F}_\Xi(X_f) = R_f$. If $X_f \subset X_g$, then we see from the Proposition that $ag = f^k$ for some a and k. This gives us an assignment

$$\begin{aligned} R_g &\to R_f \\ \frac{r}{g^n} &\mapsto \frac{ra^n}{f^{nk}}. \end{aligned}$$

Recall now the definition of a presheaf on an open base from [NOS, 11.2].

scheme-affine-presheaf

PROPOSITION 1.1.4. *The assignment $X_f \mapsto R_f$ gives us a presheaf $\mathscr{F}_\Xi(X_f)$ on the base of principal open sets with the restriction maps as described above. Moreover, if Υ is another complete collection, we have an isomorphism $\mathscr{F}_\Xi \cong \mathscr{F}_\Upsilon$ of presheaves on an open base.*

PROOF. The first thing we need to check is that the purported restriction map is well-defined. Suppose $\frac{r}{g^n} = \frac{s}{g^m} \in R_g$. Then, we have $g^{p+m}r = g^{p+n}s$, for some $p \in \mathbb{N}$. Now, we multiply both sides by $a^{(n+p+m)}$ to get

$$a^n f^{k(p+m)} r = a^m f^{k(p+n)} s,$$

which gives us

$$\frac{ra^n}{f^{nk}} = \frac{sa^m}{f^{mk}}.$$

Also, if there is another pair b, l such that $f^l = bg$, then we should see that we get the same map by using the assignment

$$\frac{r}{g^n} \mapsto \frac{rb^n}{f^{nl}},$$

instead. But this follows from the equalities

$$f^{nl}a^n = b^n g^n a^n = b^n f^{nk}.$$

It's time now to check that these restriction maps compose as they should. Suppose we have $X_f \subset X_g \subset X_h$; then we can find $a, b \in R$ and integers $k, l \in \mathbb{Z}$ such that $f^k = ag$ and $g^l = bh$. This implies that $f^{kl} = a^l bh$. If we look at the composition of maps $R_h \to R_g \to R_f$, we get

$$\frac{r}{h^n} \mapsto \frac{rb^n}{g^{nl}} \mapsto \frac{ra^{nl}b^n}{f^{nkl}}$$

If we take the restriction $R_h \to R_f$ straightaway, we get

$$\frac{r}{h^n} \mapsto \frac{ra^{nl}b^n}{f^{nkl}}.$$

So we see that they're both the same map. We've shown now that $\mathscr{F}_\Xi$ is a presheaf on the open base of principal open sets.

Suppose Υ is another complete collection and let $\mathscr{F}_\Upsilon$ be the presheaf on an open base obtained from it. For every principal open set $U \subset X$, we can find $f \in \Xi$ and $g \in \Upsilon$ such that $X_f = X_g$. This implies that we can find $a, b \in R$ and $k, l \in \mathbb{N}$ such that $f^k = ag$ and $g^l = bf$. This gives us maps in both direction between R_f and R_g exactly in the fashion described above in gory detail. In fact these maps are inverses to each other, since the composite of the two is just the map from R_f to itself induced by the equality $f^{kl} = a^l bf$. As we showed above, this map is the same as the map induced by the tautology $f = f$, which is the identity map. The same is of course true for R_g.

Now, all we need to do is to show that this collection of maps between corresponding localizations defines a morphism of presheaves on a base between $\mathscr{F}_\Xi$ and $\mathscr{F}_\Upsilon$. For this, suppose we have $f, h \in \Xi$ and $e, g \in \Upsilon$ such that $X_f = X_e$, $X_h = X_g$

and $X_f \subset X_h$. Then, we have to show that the diagram

$$\begin{array}{ccc} R_h & \longrightarrow & R_g \\ \downarrow & & \downarrow \\ R_f & \longrightarrow & R_e \end{array}$$

commutes. But the argument that we used to show that the restriction maps composed properly also shows that the compositions $R_h \to R_g \to R_e$ and $R_h \to R_f \to R_e$ are the same: the restriction map from R_h to R_e. □

At last we're in a position to describe what the ringed space structure on X is.

scheme-spec-ringed-space

THEOREM 1.1.5 (Definition). *The presheaf $\mathscr{O}_{\mathrm{Spec}(R)}$ associated to the presheaf on an open base $\mathscr{F}_\Xi$ is in fact a sheaf. Upto isomorphism, it's independent of the choice of Ξ. X equipped with this sheaf of rings is referred to as* Spec R.

PROOF. That $\mathscr{O}_X$ is independent of the choice of Ξ follows from the last proposition. We need to show that it's a sheaf. Choose any complete collection Ξ and set $\mathscr{F} = \mathscr{F}_\Xi$.

By the criterion in [NOS, 11.1], we need to show that for any principal open set $X_f \subset X$, and any weak covering sieve $\mathcal{V} = \{X_{f_i}\}$ of X_f consisting entirely of principal open sets, the natural map

$$R_f = \mathscr{F}(X_f) \to \mathcal{V}(\mathscr{F})$$

is an isomorphism.

Now, we have $X_f = \bigcup_i X_{f_i}$; since X_f is quasi-compact, we see that there are finitely many indices $i = 1, \dots, n$ such that $X_f = \bigcup_{i=1}^n X_{f_i}$. If $Y = \mathrm{spc}(\mathrm{Spec}\, R_f)$ and $Y_i = Y_{\frac{f_i}{1}}$, then we see that under the homeomorphism from Y to X, Y_i gets mapped onto X_{f_i}. This means that in R_f, we have $(1) = (\frac{f_1}{1}, \dots, \frac{f_n}{1})$. So, by [CA, 7.1.6], we see that an element in R_f goes to 0 in each of the localizations at the f_i if and only if it is already 0 in R_f. This tells us that the natural map above is injective.

We wish to show that it is also surjective. So suppose we have a coherent sequence on the right. So we have a collection of elements $b_i \frac{a_i}{f_i^r} \in (R_f)_{f_i}$, such that b_i and b_j restrict to the same element in $(R_f)_{f_i f_j}$. We will restrict our attention now to the finite subcover by the X_{f_i} for $i = 1, \dots n$.

This means that we can find $k \in \mathbb{N}$ such that

$$f_j^k f_i^{r+k} a_j = f_i^k f_j^{r+k} a_i.$$

Observe that we still have $R_f = (f_1^k, \dots, f_n^k)$, where, when we say f_i, we actually mean $\frac{f_i}{1}$. In any case, we see that we can find $c_i \in R_f$ such that $\sum_{i=1}^n c_i f_i^k = 1$.

Let $b = \sum_{i=1}^n c_i f_i^{r+k} a_i$; we see that we have

$$\begin{aligned} f_j^k b &= \sum_{i=1}^n c_i f_j^k f_i^{r+k} a_i \\ &= \sum_{i=1}^n c_i f_j^{r+k} f_i^k a_j \\ &= f_j^{r+k} (\sum_{i=1}^n c_i f_i^k) a_j = f_j^{r+k} a_j. \end{aligned}$$

This shows that b restricts to a_j for each open set in the finite subcover. Now, if $U = X_{f_k}$ is any other element in the open cover, we have $U = \bigcup_{i=1}^n (U \cap X_{f_i})$. Now, both the restriction of b to $\mathscr{F}(U)$, and $a_k \in \mathscr{F}(U)$ restrict to the same element in each $\mathscr{F}(U \cap X_{f_i})$, for $i = 1, \ldots, n$. So by the injectivity of the natural map, we see that b must in fact restrict to a_k, which shows surjectivity and finishes our proof. □

heme-spec-locally-ringed

LEMMA 1.1.6. *The ringed space* Spec R *is in fact a locally ringed space.*

PROOF. Since the principal open sets form a basis for the topology, it suffices to show that, for any prime $P \subset R$, the natural map

$$\varinjlim_{f \notin P} R_f \to R_P$$

is an isomorphism.

First we show that it's surjective. For this, just observe that every element in R_P can be written in the form $\frac{a}{f}$, where $a \in R$ and $f \notin P$. It remains to show injectivity. If $\frac{a}{f}$ goes to zero in R_P, then there is some $g \notin P$ such that $ga = 0$. This means that $\frac{a}{f}$ restricts to 0 in R_{fg}, which of course means that it represents the zero element in the direct limit. □

scheme-spec-fullfaithful

THEOREM 1.1.7. *The assignment* $R \mapsto \operatorname{Spec} R$ *gives rise to a contravariant, full and faithful functor from the category of commutative rings to the category of locally ringed spaces.*

PROOF. First, let's show that it's in fact a functor. Suppose $\phi : R \to S$ is a ring homomorphism. Then, it induces a morphism of ringed spaces $(f, f^\sharp) : \operatorname{Spec} S \to \operatorname{Spec} R$ in the following fashion.

We set $f(P) = \phi^{-1}(P)$; this is a continuous map, since $f^{-1}(V(I)) = V(\phi(I))$. To see this, observe that a prime contains $\phi(I)$ if and only if it contracts under ϕ to a prime containing I.

Now, if $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, and X_a is a principal open subset of X, we see that

$$f^{-1}(X_a) = f^{-1}(X \setminus V((a))) = Y \setminus V((\phi(a))) = Y_{\phi(a)}.$$

So we can define the map

$$f^\sharp : \mathscr{O}_{\operatorname{Spec} R} \to f_*(\mathscr{O}_{\operatorname{Spec} S})$$

by setting

$$f^\sharp_{X_a} : \mathscr{O}_{\operatorname{Spec} R}(X_a) \to \mathscr{O}_{\operatorname{Spec} S}(Y_{\phi(a)})$$

to be the localization $R_a \to S_{\phi(a)}$ of the map ϕ.

Now, suppose $P \subset S$ is a prime; then for any principal open set X_a containing $\phi^{-1}(P)$, we see that $Y_{\phi(a)}$ is a principal open set containing P. This gives us a map $f^{\sharp}_{f(P)} : \mathscr{O}_{\phi^{-1}(P)} \to \mathscr{O}_P$, which is simply the localization $\phi_P : R_{\phi^{-1}(P)} \to S_P$. This is evidently a local homomorphism of rings.

This gives us a map between $\mathrm{Hom}_{\mathrm{Ring}}(R, S)$ and $\mathrm{Hom}(\operatorname{Spec} S, \operatorname{Spec} R)$. Let's construct a map in the other direction. Suppose we have a morphism of locally ringed spaces $(f, f^{\sharp}) : \operatorname{Spec} S \to \operatorname{Spec} R$. Then $f^{\sharp}$ induces a map $\phi : R \to S$ on the rings of global sections. For any prime $P \subset S$, we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\ \phi\ } & S \\ \downarrow & & \downarrow \\ R_{f(P)} & \xrightarrow[f^{\sharp}_{f(P)}]{} & S_P, \end{array}$$

where the vertical maps are localizations. It follows that every element outside $f(P)$ is taken to an invertible element in S_P by ϕ. Hence $\phi(R \setminus f(P)) \subset S \setminus P$. Moreover, since $f^{\sharp}$ is a local homomorphism, we see that $\phi(f(P)) \subset P$. Therefore, $\phi^{-1}(P) = f(P)$. But now, as morphisms of sheaves of abelian groups, $f^{\sharp}$, and the morphism induced by the localizations of ϕ agree on stalks, and so $(f, f^{\sharp})$ is in fact the morphism of locally ringed spaces induced by ϕ.

It's clear that these two assignments are inverses of each other. □

Now, we'll relate properties of ring homomorphisms to the properties of the morphisms they induce between the Specs.

`scheme-ringmap-specmap`

PROPOSITION 1.1.8. *Let $\phi : R \to S$ be a ring homomorphism, and let $(f, f^{\sharp}) : \operatorname{Spec} S \to \operatorname{Spec} R$ be the induced morphism of locally ringed spaces. Then the following statements hold:*

(1) *If $I \subset S$ is an ideal, then $\overline{f(V(I))} = V(\phi^{-1}(I))$.*
(2) *ϕ is injective if and only if $f^{\sharp}$ is injective, and in this case f is dominant.*
(3) *ϕ is surjective if and only if f is a homeomorphism onto a closed subset of X, and $f^{\sharp}$ is surjective.*

PROOF. (1) First assume that $I = \mathrm{rad}(I)$. Let $I = \cap P$, for primes $P \in V(I)$. Then, $\phi^{-1}(I) = \cap \phi^{-1}(P)$. Now, a closed subset $V(J) \subset X$ contains $f(V(I))$ if and only if $J \subset \phi^{-1}(P)$, for every prime $P \subset S$. This is equivalent to saying that J is a subset of $\phi^{-1}(I)$. So we see that

$$\overline{f(V(I))} = \bigcap_{J \subset \phi^{-1}(I)} V(J) = V(\phi^{-1}(I)).$$

Now, for I arbitrary, it suffices to show that

$$\mathrm{rad}(\phi^{-1}(I)) = \mathrm{rad}(\phi^{-1}(\mathrm{rad}(I))).$$

Quotienting S by I, it suffices to show that

$$\mathrm{rad}(\ker \psi) = \mathrm{rad}(\psi^{-1}(\mathrm{Nil}(T)),$$

for any homomorphism of rings $\psi : R \to T$. One inclusion is easy; for the other, suppose a is on the right hand side; then $\phi(a)$ is nilpotent, and so $\psi(a^n) = 0$, for some n, implying $a^n \in \ker \psi$.

(2) We see that $f^\sharp$ is injective if and only if the map

$$R_a \to S_{\phi(a)}$$

is injective for all elements $a \in R$. But this is also a sufficient condition for ϕ to be injective as a map of R-modules, and hence as a map of rings (see [CA, 7.1.6].

Now, observe from part (1) that

$$\overline{f(X)} = V(\phi^{-1}(0)) = V(\ker\phi),$$

where $X = \operatorname{Spec} S$. Now, since ϕ is injective, $\ker\phi = 0$, and so

$$\overline{f(X)} = V(\operatorname{Nil}(R)) = Y,$$

where $Y = \operatorname{Spec} R$.

(3) First suppose that ϕ is surjective; then we get an isomorphism $\widetilde{\phi} : R/\ker\phi \to S$ that gives rise to a homeomorphism

$$\widetilde{f} : \operatorname{Spec} S \to \operatorname{Spec} R/\ker\phi.$$

But the natural projection $R \to R/\ker\phi$ gives us an inclusion $\operatorname{Spec} R/\ker\phi \hookrightarrow \operatorname{Spec} R$, and composing this with $\widetilde{f}$ gives us f, showing that f is a homeomorphism onto a closed subset of $\operatorname{Spec} R$. It remains to show that $f^\sharp$ is surjective, but this follows trivially, since the localization of surjective maps is surjective.

In the other direction, observe that the map $\phi : R \to S$ factors as

$$R \to R/\ker\phi \hookrightarrow S.$$

So the induced map $(f, f^\sharp)$ is the composition

$$Y \xrightarrow{(h,h^\sharp)} X' \xrightarrow{(g,g^\sharp)} X,$$

where $X' = \operatorname{Spec} R/\ker\phi$. Now, since the map $h : Y \to X'$ is induced by an injective map, we see that it's dominant, by the first part. Also, since the map $g : X' \to X$ is induced by a surjective map, we see that it is a homeomorphism onto a closed subset of X, by the paragraph above (in fact, it's the closed subset $V(\ker\phi)$). We claim that $(h, h^\sharp)$ is in fact an isomorphism. First, we show that h is a homeomorphism. Since both g and f are homeomorphisms onto their images, we see that h must also be a homeomorphism onto its image. But that means h is closed, and since it's dominant, its image must be the whole of X'.

Moreover, by what went before, $h^\sharp : \mathscr{O}_{X'} \to h_*\mathscr{O}_Y$ is injective and $g^\sharp : \mathscr{O}_X \to g_*\mathscr{O}_{X'}$ is surjective. Since we know that $f^\sharp = h^\sharp g^\sharp$ is surjective, we see that $h^\sharp$ must also be surjective, and is thus an isomorphism of sheaves of rings. This shows that $(h, h^\sharp) : Y \to X'$ is an isomorphism of affine schemes, and so the map $R/\ker\phi \to S$ must also be an isomorphism, by Proposition 1.1.7, which shows that ϕ is surjective.

□

2. Schemes

In this section, we define the basic objects of study. Just as Euclidean space is the building block of the theory of manifolds, the Specs that we defined above (or affine schemes-see below) are the building blocks of scheme theory.

Definition 1.2.1. An *affine scheme* is just the locally ringed space $\operatorname{Spec} R$ for some commutative ring R.

A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ such that for every point $x \in X$, there is an open neighborhood U of x with the locally ringed space $(U, \mathscr{O}_X|_U)$ isomorphic to an affine scheme. We call $\mathscr{O}_X$ the *structure sheaf* on X.

A *morphism* of schemes $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is just a morphism of locally ringed spaces.

This gives us a *category* Sch of schemes.

Note on Notation 1. From now on, we'll use $\mathscr{O}_{X,x}$ to denote the local ring at a point $x \in X$. Also, we'll usually abuse notation and use just X to refer to the scheme $(X, \mathscr{O}_X)$. In a similar vein, we'll talk about an open set $U \subset X$ being an affine open: what we mean is that $(U, \mathscr{O}_X|_U)$ is isomorphic to an affine scheme. Also we'll refer to the 'morphism' $f : X \to Y$, when we in fact mean a morphism $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$.

Note that for any open subset $U \subset X$, the locally ringed space $(U, \mathscr{O}_X|_U)$ is also a scheme: this follows from the fact that, for any affine scheme $\operatorname{Spec} R$, any principal open subset of $\operatorname{Spec} R$ is again an affine scheme (see [HPII, 2.1]). This is called an *open subscheme* of X, and we'll usually refer to its structure sheaf as $\mathscr{O}_U$ instead of $\mathscr{O}_X|_U$.

We can think of the Spec functor as being right adjoint to the global sections functor. The only problem with this is that Spec is contravariant, and there's no canonical choice between rightness or leftness, it would seem to me. In any case, formally stated, the proposition is thus.

eme-spec-globsec-adjoint Proposition 1.2.2. *The functor* $\operatorname{Spec} : \mathrm{Ring}^{op} \to \mathrm{Sch}$ *is right adjoint to the global sections functor* $\Gamma(\_\_, \mathscr{O}_\_) : \mathrm{Sch} \to \mathrm{Ring}^{op}$.

Proof. There is a natural transformation in one direction that takes every morphism $X \to \operatorname{Spec} A$ to the ring homomorphism $A \to \Gamma(X, \mathscr{O}_X)$ induced on the global sections. We will show that this is bijective. Note first, by the Gluing Lemma [RS, 1.17] that giving a morphism $X \to \operatorname{Spec} A$ is equivalent to choosing an affine open cover $\mathcal{V} = \{V_i\}$ of X, and giving morphisms $f_i : V_i \to \operatorname{Spec} A$ which agree on the intersections of their domains of definition. This means exactly that we have an equalizer diagram

$$\mathrm{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} A) \to \prod_i \mathrm{Hom}_{\mathrm{Sch}}(V_i, \operatorname{Spec} A) \rightrightarrows \prod_{i,j,k} \mathrm{Hom}_{\mathrm{Sch}}(V_{ijk}, \operatorname{Spec} A),$$

where the V_{ijk}, for fixed i, j, give an affine open cover for $V_i \cap V_j$, where the two maps on the right are given by restricting each section over V_i to $V_i \cap V_j$, for varying j, and then to $V_j \cap V_i$, for varying j.

Now, observe that we have an equalizer diagram given by the sheaf axiom

$$\Gamma(X, \mathscr{O}_X) \to \prod_i \Gamma(V_i, \mathscr{O}_{V_i}) \rightrightarrows \prod_i \Gamma(V_{ijk}, \mathscr{O}_{V_{ijk}}).$$

Applying the $\mathrm{Hom}_{\mathrm{Ring}}(A, \_\_)$ functor to this, we get a map between equalizer diagrams

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} A) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathrm{Sch}}(V_i, \operatorname{Spec} A) & \rightrightarrows & \prod_{i,j,k} \mathrm{Hom}_{\mathrm{Sch}}(V_{ijk}, \operatorname{Spec} A) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathscr{O}_X)) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(V_i, \mathscr{O}_{V_i})) & \rightrightarrows & \prod_{i,j,k} \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(V_{ijk}, \mathscr{O}_{V_{ijk}})) \end{array}$$

The maps in the middle and on the right are isomorphisms, by 1.1.7; therefore, so is the map on the left, as is easily checked. □

3. The Affine Communication Lemma

The next two lemmas are fundamental in the reduction of local properties of schemes to properties of affine schemes.

nt-open-cover-propensets

Lemma 1.3.1. *If $(X, \mathscr{O}_X)$ is a scheme and $U, V \subset X$ are two affine opens, then the intersection $U \cap V$ has an open cover by sets that are principal open sets inside both U and V.*

Proof. Suppose $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$; let $U_f \subset U \cap V$ be a principal open subset of U, and let $V_g \subset U_f$ be a principal open subset of V. We claim that V_g is also a principal open subset of U_f, and thus of U. Let g' be the image of g in A_f; so $g' = \frac{a}{f^n}$, for some $a \in A$. I claim that $B_g = (A_f)_{g'} = A_{fa}$. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & A_f \\ \downarrow & \swarrow & \downarrow \\ B_g & \leftrightarrows & A_{fa}, \end{array}$$

where we get the maps in the bottom row by the universal property of localization (a is invertible in B_g, and g is invertible in A_{fa}). These maps are inverses to each other, and so

$$V_g = \operatorname{Spec} B_g = \operatorname{Spec} A_{fa} = U_{fa}.$$

Hence, open subsets of $U \cap V$ that are principal in both U and V in fact form a basis for $U \cap V$. □

The next lemma is something that I picked up from Ravi Vakil's notes. It formalizes an argument that lets you translate statements proved for a given affine open cover to a statement about every affine open.

eme-affine-communication

Lemma 1.3.2 (Affine Communication Lemma). *Let $(X, \mathscr{O}_X)$ be a scheme, and let Π be a property of affine open subsets of X that satisfies the following two conditions:*

(1) *If Π is true for $\operatorname{Spec} R$, then it's true for $\operatorname{Spec} R_f$, for all $f \in R$.*
(2) *If $(1) = (f_1, \dots, f_n) \subset R$, and Π is true for each $\operatorname{Spec} R_{f_i}$, then Π is true for $\operatorname{Spec} R$.*

Now, suppose $X = \bigcup_i \operatorname{Spec} R_i$*, with* Π *true for each* $\operatorname{Spec} R_i$*. Then* Π *is true for every affine open subset of* X*.*

PROOF. Let $\operatorname{Spec} R$ be an affine open subset of X. Then, by the previous Lemma, $\operatorname{Spec} R \cap \operatorname{Spec} R_i$ is covered by open subsets that are principal in both R and R_i. Since $\operatorname{Spec} R$ is quasi-compact, we can take finitely many such subsets. Then, by property (1), Π is true for every such subset, and by property (2), it's true for $\operatorname{Spec} R$. □

Observe that the property Π is really a property of rings. The following Proposition lists some properties that satisfy the conditions of the lemma.

prps-satisfying-aff-comm

PROPOSITION 1.3.3. *The following properties satisfy the conditions of the Lemma. That is, if they hold for a ring* R*, then they hold for all localizations* R_f*, and conversely, if they hold for localizations* R_{f_i}*, where* $\{f_i\}$ *is a finite generating set for* R*, then they hold for* R*.*

(1) *Noetherianness.*
(2) *Reducedness.*
(3) *Normality, if* R *is reduced and Noetherian.*
(4) *Being a finitely generated* S*-algebra over some ring* S*.*
(5) *Being flat over a ring* S*.*

PROOF. (1) It's clear that if R is Noetherian, then every localization of R is also Noetherian. For the other direction, suppose $I \subset R$ is an ideal, and $\phi_i : R \to R_{f_i}$ is the natural map; we claim that $I = \cap_i \phi_i^{-1}(\phi_i(I)R_{f_i})$. One inclusion is clear. For the other, suppose $a \in R$ lies in the intersection. Then, for all i, we can find $b_i \in I$ and $n, k \in \mathbb{N}$ such that

$$f_i^{n+k} a = f_i^k b_i,$$

for all i. Now, since $(f_1, \ldots, f_n) = R$, we can find $c_i \in R$ such that $\sum_i c_i f_i^{n+k} = 1$. Then we see that

$$a = \sum_i c_i f_i^{n+k} a = \sum_i c_i f_i^k b_i \in I.$$

Now, suppose $I_1 \subset I_2 \subset \ldots$ is a chain of ideals in R. Since R_{f_i} is Noetherian for each i, we can find a $k > 0$ such that

$$\phi_i(I_k)R_{f_i} = \phi_i(I_{k+1})R_{f_i}, \text{ for all } i.$$

So we see that

$$I_k = \cap_i \phi_i^{-1}(\phi_i(I_k)R_{f_i}) = \cap_i \phi_i^{-1}(\phi_i(I_{k+1})R_{f_i}) = I_{k+1}.$$

This shows that every chain of ideals in R stabilizes, and so R is also Noetherian.

(2) Consider $\operatorname{Nil} R$: this is 0 if and only if its localizations at each of the f_i is 0 (see [CA, 7.1.6]), and of course R is reduced if and only if $\operatorname{Nil} R = 0$.

(3) Let S be the integral closure of R in $K(R)$. By (2), we know that every localization of R is reduced. Moreover, the integral closure of R_{f_i} in $K(R_{f_i})$ is just S_{f_i}. Now, R is integrally closed if and only if the inclusion $R \hookrightarrow S$ is surjective. By [CA, 7.1.6], this can happen if and only if each localization $R_{f_i} \hookrightarrow S_{f_i}$ is surjective if and only if each R_{f_i} is also integrally closed.

(4) It is clear that if R is a finitely generated S-algebra, then so is every localization of R. For the converse, suppose R_{f_i} is a finitely generated S-algebra, for every i. Then, we can find elements $s_{ij} \in R$ and $n_{ij} \in \mathbb{N}$ such that R_{f_i} is generated over S by the $\frac{s_{ij}}{f_i^{n_{ij}}}$ for varying j. Let R' be the S-subalgebra of R generated by the f_i and the s_{ij} together. We claim that $R' = R$. Indeed, choose an element $a \in R$; in every localization R_{f_i}, we can write $\frac{a}{1} = \frac{p_i(s_{ij}, f_i)}{f_i^N}$, for some polynomial p_i over S, and some $N \in \mathbb{N}$. So we can find some bigger $M \in \mathbb{N}$ and some other polynomials q_i over S such that

$$f_i^M a = q_i(s_{ij}, f_i).$$

As usual, if $c_i \in R$ are such that $\sum_i c_i f_i^M = 1$, then we see that

$$a = \sum_i c_i q_i(s_{ij}, f_i) \in R'.$$

(5) Consider, for any ideal $J \subset S$, the R-module $\operatorname{Tor}_1^S(S/J, R)$. R is S-flat if and only if this is 0 for all such ideals J if and only if its localizations $\operatorname{Tor}_1^S(S/J, R_{f_i})$ are 0, for every i if and only if R_{f_i} is S-flat, for all i. Note that we used the fact that the rings R_{f_i} were R-flat in the penultimate equivalence.

□

4. A Criterion for Affineness

Definition 1.4.1. Let X be a scheme, and let $A = \Gamma(X, \mathscr{O}_X)$ be its ring of global sections. For $a \in A$, we set

$$X_a = \{x \in X : a_x \notin \mathfrak{m}_x\}$$

e-principal-open-schemes

Proposition 1.4.2. *Let X and A be as in the definition above.*

(1) *For every $a \in A$, and every affine open $U = \operatorname{Spec} R \subset X$, $X_a \cap U = U_{a'}$, where a' is the image of a in $\Gamma(U, \mathscr{O}_X)$. In particular, X_a is open.*
(2) *Suppose X is quasi-compact, and thus has a finite affine cover $\{U_i = \operatorname{Spec} R_i\}$; if $s \in A$, and $\operatorname{res}_{X,X_a}(s) = 0$, then there exists $n \in \mathbb{N}$ such that $a^n s = 0 \in A$. In particular, when X is quasi-compact, the map*

$$A_a \to \Gamma(X_a, \mathscr{O}_X)$$

is injective.
(3) *Suppose further that for the finite affine cover above, $U_i \cap U_j$ is quasi-compact for every pair of indices (i, j). Then, for every section $b \in \Gamma(X_a, \mathscr{O}_X)$, there exists $n \in \mathbb{N}$ such that $a^n b$ extends to a global section of $\mathscr{O}_X$. In particular, the map*

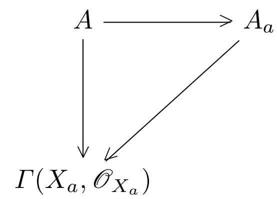
$$A_a \to \Gamma(X_a, \mathscr{O}_X)$$

is also surjective, and is thus an isomorphism.
(4) *Let $f : Y \to X$ be a morphism of schemes, and let $\widetilde{a}$ be the image of a in $\Gamma(Y, \mathscr{O}_Y)$ under $f^\sharp$. Then*

$$Y_{\widetilde{a}} = f^{-1}(X_a).$$

PROOF. (1) Clearly, the stalk a_x is not contained in $\mathfrak{m}_x$, for some point $x \in U$ if and only if the stalk a'_x is not contained in $\mathfrak{m}_x$. But this is precisely the same as saying that the prime in R corresponding to the point x does not contain a'. Thus, $U \cap X_a$ consists exactly of those primes in R that don't contain a', and so equals $U_{a'}$. The conclusion that X_a is an open subset follows immediately, because its intersection with every open affine is open.

(2) Let $a_i, s_i \in \Gamma(U_i, \mathscr{O}_{U_i})$ be the restrictions of a and s over the U_i. Then, we see that there is an $n > 0$ such that $a_i^n s_i = 0$, for all i. This means that $a^n s$ restricts to 0 on all sets in an open cover, which in turn means that it was zero to begin with. For the second statement, consider the following diagram:



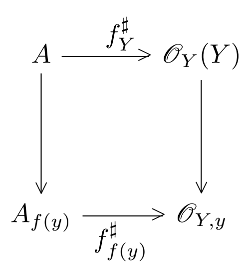
where the diagonal map is obtained from the universal property of localization, since a is invertible in $\Gamma(X_a, \mathscr{O}_{X_a})$. An element $\frac{s}{a^r} \in A_a$ goes to zero under the diagonal map if and only if s goes to zero under the vertical map if and only if $a^n s = 0 \in A$, for some $n \geq 0$. But this means that $\frac{s}{a^r}$ is already 0 in A_a! So the diagonal map is injective.

(3) Again, let $a_i \in \Gamma(U_i, \mathscr{O}_{U_i})$ and $s_i \in \Gamma((U_i)_{a_i}, \mathscr{O}_{U_i})$ be the restrictions of a and s to each of the U_i. Since U_i is affine, we see that

$$\Gamma((U_i)_{a_i}, \mathscr{O}_{U_i}) = (R_i)_{a_i}.$$

So we can find $t > 0$ such that $a_i^t s_i$ is the restriction of a section b_i over U_i. Consider now the section $c_{ij} = b_i - b_j$ over $U_i \cap U_j$ (strictly speaking, we're taking the difference between the restrictions of the b_i). This restricts to 0 on $(U_i \cap U_j)_a$. So by part (2) we can find $n_{ij} > 0$ such that $a_{ij}^{n_{ij}} c_{ij} = 0$, where a_{ij} is the restriction of a to $U_i \cap U_j$. Taking k to be the maximum of these n_{ij} for varying i, j, we see that the sequence $(a_i^k b_i)$ is coherent. Hence it glues together to give a global section that restricts to $a^{k+t} s$ on X_a. For the second statement, consider again the diagram in the previous part. We've shown precisely that it's surjective!

(4) Indeed, let $y \in Y$ be a point; then we have a commutative diagram



So we see that $f^\sharp_{f(y)}(a_{f(y)}) = \widetilde{a}_y$. Since $f^\sharp_{f(y)}$ is a local homomorphism, this means that

$$y \in Y_{\widetilde{a}} \Leftrightarrow \widetilde{a}_y \notin \mathfrak{m}_y \Leftrightarrow a_{f(y)} \notin \mathfrak{m}_{f(y)} \Leftrightarrow f(y) \in X_a,$$

□

The first thing we'd like is a decent criterion that'll tell us when a scheme is affine. We'll develop a much more powerful cohomological criterion later. For now, we must be satisfied with this.

eme-affineness-criterion

PROPOSITION 1.4.3. *A scheme X is affine if and only if there are finitely many elements $f_1, \ldots, f_n \in A := \Gamma(X, \mathscr{O}_X)$ such that $A = (f_1, \ldots, f_n)$, and X_{f_i} is affine for each i.*

PROOF. Let $X_{f_i} = \operatorname{Spec} B_i$, for each i. By (1.2.2), the identity map from A to A gives rise to a morphism of schemes $f : X \to \operatorname{Spec} A$, with its restriction $X_{f_i} \to \operatorname{Spec} A$ to each X_{f_i} being induced by the map $A \to B_i$ given by restriction.

Now, consider $X_{f_i} \cap X_{f_j}$: this is just the principal open subset $(X_{f_i})_{f'_j}$, where f'_j is the image of f_j in B_i. In particular, it's quasi-compact; so we can use the result from (1.4.2) to conclude that $B_i = A_{f_i}$, and the restriction maps are just the natural maps $A \to A_{f_i}$, for all i. Let $Y = \operatorname{Spec} A$; then we see that the maps $X_{f_i} \to Y$ are induced by these natural restrictions, and are thus isomorphisms onto Y_{f_i} in Y. So we see that f induces isomorphisms $f^{-1}(Y_{f_i}) \to Y_{f_i}$ for all i. Since the (f_i) generate A, the sets Y_{f_i} form an open cover for Y. We claim that this means f is itself an isomorphism, which will of course mean that $X \cong \operatorname{Spec} A$ is affine. This is easy: the map of sheaves $f^\sharp$ induces isomorphisms on stalks, since each $x \in X$ is contained in some $f^{-1}(Y_{f_i})$. Hence, $f^\sharp$ is an isomorphism. It remains to show that f is a homeomorphism of topological spaces. It's surjective, since the Y_{f_i} cover Y, and it's injective, since, if $y \in Y_{f_i}$, then there's only one point in $f^{-1}(Y_{f_i})$, and hence only one point in X mapping to Y. The map is also closed, since the intersection of the image of a closed set with every Y_{f_i} will be closed. All this combines to show that f is homeomorphism. □

5. Irreducibility and Connectedness

5.1. Irreducibility.

DEFINITION 1.5.1. A scheme X is *irreducible* if its underlying topological space is irreducible [NS, 1].

An *irreducible component* of X is an irreducible component of its underlying topological space.

A scheme $(X, \mathscr{O}_X)$ is *locally Noetherian*, if there is an open affine cover $\mathcal{V} = \{V_i\}$ of X such that $\mathscr{O}_{V_i}$ is a Noetherian ring for every i. It is *Noetherian*, if it is also quasi-compact.

REMARK 1.5.2. By the Affine Communication Lemma (1.3.2), and the Proposition following it, this is equivalent to requiring that $\mathscr{O}_V$ be Noetherian for *every* affine open subscheme of X.

cheme-irreducible-affine

PROPOSITION 1.5.3. *Let $X = \operatorname{Spec} R$ be an affine scheme.*

(1) *The irreducible components of X are precisely the closed subsets of the form $V(P)$, where $P \subset R$ is a minimal prime.*

(2) *X is irreducible if and only if R has a unique minimal prime, if and only if* $\operatorname{Nil} R$ *is prime.*
(3) *If $I \subset R$ is an ideal, then $V(I)$ is an irreducible subset if and only if* $\operatorname{rad}(I)$ *is prime.*

Proof. It's enough to prove (1), since everything else follows immediately from it: (2) is obvious, and (3) follows from (2), because $V(I)$ is homeomorphic to $\operatorname{Spec} R/I$. Let's prove (1). Let $I \subset R$ be an ideal; then $V(I)$ is irreducible if and only if $\operatorname{rad}(I)$ is prime. For this, we can replace I with $\operatorname{rad}(I)$, and assume that I is radical. Now, to say that $V(I)$ is irreducible is equivalent to saying that whenever $I = J \cap J'$, with $J \not\subset I$, then $J' \subset I$. This is precisely equivalent to saying that I is prime. Let P now be any prime; it's now easy to see that $V(P)$ is a maximal irreducible subset if and only if P is a minimal prime of R, since $V(Q) \subset V(P)$ if and only if $Q \supset P$. □

-noetherian-qc-fin-irred

Proposition 1.5.4. *Let X be a Noetherian scheme. Then the space underlying X is Noetherian, and hence X has only finitely many irreducible components.*

Proof. X has a finite affine open cover $\{U_1, \ldots, U_n\}$, where each U_i is a Noetherian affine scheme. Since the finite union of Noetherian spaces is Noetherian, it suffices to show that $X = \operatorname{Spec} R$, where R is a Noetherian ring, is a Noetherian space. For this, it's enough to show that every open subset $U \subset X$ is quasi-compact [NS, 3.3]. But if $\{X_{f_i}\}$ is a principal open cover for U, then $Z = U^c = V(\{f_i\})$. Since R is Noetherian, we can find finitely many elements $f_1, \ldots, f_n$ such that $Z = V(f_1, \ldots, f_n)$. Then $\{X_{f_1}, \ldots, X_{f_n}\}$ is a finite open subcover, which finishes our proof. The second part follows from [NS, 3.4]. □

Recall the definition of a generic point from [NS, 2].

scheme-generic-points

Proposition 1.5.5. *Let X be a scheme.*
(1) *If $X = \operatorname{Spec} R$ is affine, then the generic points of X are precisely the minimal primes of R.*
(2) *Every irreducible closed subset of X has a unique generic point. In particular, any scheme is quasi-Zariski.*
(3) *If $x \in X$, then the irreducible components of* $\operatorname{Spec} \mathscr{O}_{X,x}$ *are in bijective correspondence with the irreducible components of X containing x. In particular, if X is irreducible, then $\mathscr{O}_{X,x}$ has a unique minimal prime, for all $x \in X$.*

Proof. (1) Follows from part (1) of (1.5.3). To show that no other prime can be a generic point, just note that if $P \subsetneq Q$, then $V(P) \supsetneq V(Q)$.
(2) Let $Z \subset X$ be an irreducible closed subset; since every open subset of Z is dense in Z, and hence contains any generic points, it suffices to show that any irreducible affine scheme has a unique generic point. But this follows from part (1).
(3) By [NS, 1.3], the irreducible components of any open subset of X are in bijective correspondence with the irreducible components of X meeting that open subset. So we can assume that $X = \operatorname{Spec} R$ is affine, and that x corresponds to some prime $P \subset R$. In this case $\mathscr{O}_{X,x} = R_P$, and the minimal primes of R_P are in bijective correspondence with the minimal primes of R contained in P, which are in turn in bijective correspondence with the irreducible components of X containing x.

□

5.2. Connectedness.

Definition 1.5.6. A scheme X is *connected* if its underlying topological space is connected.

A *connected component* of a scheme X is simply a connected component of the underlying topological space of X.

heme-connected-equivprps

Proposition 1.5.7. *Let X be a scheme, and let $A = \Gamma(X, \mathscr{O}_X)$. Then the following are equivalent.*

(1) *X is connected.*
(2) $\operatorname{Spec} A$ *is connected.*
(3) *A has no non-trivial idempotents.*

Proof. The equivalence of (2) and (3) follows in standard fashion [HPII, **??**]. We'll prove (3) $\Leftrightarrow$ (1): First suppose A has a non-trivial idempotent e. Let $f = 1-e$; then f is also an idempotent, and $ef = 0$. So, for any $x \in X$, if $e_x \notin \mathfrak{m}_x$, then $f_x = 0$ and vice versa. Therefore, $X_e \cap X_f = \emptyset$; moreover $X_e \cup X_f = X$. This follows, because $e + f = 1$, and so both e and f cannot be in $\mathfrak{m}_x$, for any $x \in X$. This shows that X is disconnected. Conversely, suppose X is disconnected, and let $U, V \subset X$ be disjoint non-empty open subsets such that $U \cup V = X$. Then, by the sheaf axiom

$$A = \Gamma(U, \mathscr{O}_X) \times \Gamma(V, \mathscr{O}_X)$$

is a product of rings, and thus has non-trivial idempotents. □

6. Reduced and Integral Schemes: The Fourfold Way

Definition 1.6.1. A scheme $(X, \mathscr{O}_X)$ is *reduced* if, for every open set $U \subset X$, the ring $\mathscr{O}_X(U)$ is reduced.

duced-iff-stalks-reduced

Proposition 1.6.2. *A scheme $(X, \mathscr{O}_X)$ is reduced if and only if for every $x \in X$ the local ring $\mathscr{O}_{X,x}$ is reduced.*

Proof. We'll show that X is reduced if and only if every affine open subscheme of X is also reduced. Since a ring is reduced if and only if its localizations at every prime are reduced, this will prove the statement. One direction of this equivalence is trivial. For the other, suppose $U \subset X$ is any open subscheme, and let $\mathcal{V} = \{V_i\}$ be an affine open cover of U. If $s \in \mathscr{O}_X(U)$ is a nilpotent element, then, since $\mathscr{O}_X(V_i)$ is reduced by hypothesis, s must restrict to 0 over each V_i. But then s was 0 to begin with! □

Now it's time to present the Fourfold Way to Universal Arrows! Here, we want to construct, for every scheme X, an arrow $PX \to X$ satisfying some universal property. There's a very general philosophy behind such constructions that I'll outline now.

scheme-threefold-way

Step 1: Construct the arrow $PX \to X$ for all affine schemes $X = \operatorname{Spec} A$ using some explicit construction. More specifically, dualize the property and show that a universal arrow in the other direction exists in the category of ring maps going out of A, or, to redualize the statement, in the category of all maps of affine schemes *to* X. Then use the fact that any morphism between schemes is determined by a collection of compatible morphisms between affine open subschemes to extend the universality to the category of all scheme morphisms to X.

Step 2: Suppose we've constructed the universal arrow $f_X : PX \to X$ for some scheme X. Check that for any open subscheme $V \subset X$ the restriction $f_X^{-1}(V) \to V$ is the universal arrow $f_V : PV \to V$.

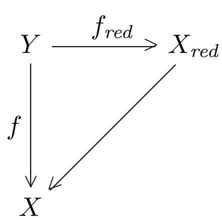
Step 3: Now, suppose we can cover every scheme X by an open cover $\mathcal{V} = \{V_i\}$, such that we have a universal arrow $f_i : PV_i \to V_i$ for each i. Now, by the previous step the restrictions of f_i and f_j to the open subschemes of PV_i and PV_j lying over $V_i \cap V_j$ both satisfy the universal property needed of the arrow $P(V_i \cap V_j) \to V_i \cap V_j$, and so there is a unique isomorphism $\phi_{ij} : f_i^{-1}(V_i \cap V_j) \to f_j^{-1}(V_i \cap V_j)$ such that $f_i = f_j \circ \phi_{ij}$. This means that we can glue together the PV_i along the ϕ_{ij} to get a scheme PX, and we can glue together the morphisms f_i to get an arrow $f : PX \to X$. We might be able to show that this satisfies the universal property by using the fact that its restrictions to an open cover do.

Step 4: We just cover X by an affine cover and use the last step. Sometimes, this step might be subsumed in step (3).

We'll use reducedness to provide a baby example of this procedure.

scheme-reduced-scheme

Proposition 1.6.3 (Definition). *For every scheme X, there is a universal arrow $X_{red} \to X$ from reduced schemes to X that's a homeomorphism on the underlying topological spaces. That is, X_{red} is reduced, and for every other morphism $f : Y \to X$ there is a unique morphism $f_{red} : Y \to X_{red}$ such that the following diagram commutes.*



The scheme X_{red} is known as the reduced scheme *associated to X.*

Proof. We'll work through the steps of the Fourfold Way.

Step 1: For a ring R, let $R_{\text{red}} = R/\operatorname{Nil} R$. Then, if $X = \operatorname{Spec} R$, we set $X_{\text{red}} = \operatorname{Spec} R_{\text{red}}$, and we let $X_{\text{red}} \to X$ be the morphism induced by the natural map $R \to R_{\text{red}}$. If S is another reduced ring, and $R \to S$ is a map of rings, then every nilpotent element goes to 0 in S. This shows that any such map factors uniquely through R_{red}. So any map from a reduced affine scheme to X factors uniquely through X_{red}. If Y is any reduced scheme; then, as we saw earlier, to give a morphism from Y to X is equivalent to taking an open affine cover $\mathcal{V} = \{V_i\}$ of Y and giving morphisms $V_i \to X$ that agree on the intersections of their domains of definition. Now, all these restrictions factor uniquely through X_{red}. If we consider any affine cover of the intersection $V_i \cap V_j$, then we see that the restrictions of these factorings to each of the affine opens in this cover is unique; so the restriction of the factorings $V_i \to X_{\text{red}}$ and $V_j \to X_{\text{red}}$ must agree on $V_i \cap V_j$. This means that we can glue them together to factor f through a map $f_{\text{red}} : Y \to X_{\text{red}}$. The uniqueness of such a factoring is forced by the construction. Observe that in this case the map $X_{\text{red}} \to X$ is a homeomorphism of topological spaces.

Step 2: Suppose we've constructed the universal arrow $X_{\text{red}} \to X$ for some scheme X, not necessarily affine. Suppose $U \subset X$ is an open subscheme; let $\widetilde{U}$ be the open subscheme of X_{red} that's the pullback of U under the map $X_{\text{red}} \to X$. Now, if $f : Y \to U$ is a morphism to U from a reduced scheme Y; then since we can also consider it a map into X, we see that it factors uniquely through a map $f_{\text{red}} : Y \to X_{\text{red}}$. But in fact it factors through $\widetilde{U}$ since the composition of f_{red} with the map $X_{\text{red}} \to X$ goes into U. Such a factoring must be unique, since it's determined by its composition with the inclusion $\widetilde{U} \to X_{\text{red}}$, which is by hypothesis unique. Since $\widetilde{U}$ is a subscheme of a reduced scheme, we see that $\widetilde{U} \to U$ is indeed the universal arrow that we wanted. Moreover, if the map $X_{\text{red}} \to X$ is a homeomorphism of the underlying topological spaces, then so is $\widetilde{U} \to U$.

Step 3: So suppose we've taken an open affine cover V_i for X and patched together an arrow $\widetilde{X} \to X$ from the universal arrows $V_{i\text{red}} \to V_i$. We can do this by [NOS, 11.1]. The maps on the triple intersections agree by the uniqueness of the isomorphisms between the restrictions of the universal arrows to the double intersections. Then $\widetilde{X}$ is clearly reduced. Also, if $f : Y \to X$ is a morphism from a reduced scheme Y to X, then by taking a fine enough affine open cover $\mathcal{W} = \{W_j\}$ of Y, we can ensure that f is determined by compatible morphisms $f_j : W_j \to V_{\sigma(j)}$ for some correspondence σ between the indexing sets of $\mathcal{W}$ and $\mathcal{V}$. Each of these maps then factors uniquely through $f_{j\text{red}} : W_j \to V_{\sigma(j)\text{red}}$. We can glue together these morphisms to build a map $\widetilde{f} : Y \to \widetilde{X}$ such that $\widetilde{f}$ composed with the map $\widetilde{X} \to X$ is the original map f. Uniqueness of such a factoring is forced by the uniqueness of the factoring on each set of the affine open cover. Note that the map $\widetilde{X} \to X$, having been glued together from homeomorphisms, also gives rise to a homeomorphism of topological spaces.

□

This proof had a lot of boring and obvious details. We'll not be presenting them in all their inane glory in the future.

DEFINITION 1.6.4. A term that I do not like: A scheme $(X, \mathscr{O}_X)$ is *integral* if $\mathscr{O}_X(U)$ is an integral domain for every open set $U \subset X$.

-iff-reduced-irreducible PROPOSITION 1.6.5. *The following statements are equivalent for a scheme X:*

(1) *X is integral.*
(2) *X is reduced and irreducible.*

If X has only finitely many irreducible components, then the two statements are also equivalent to this one: X is connected, and $\mathscr{O}_{X,x}$ is a domain for all $x \in X$.

PROOF. (1) ⇒ (2)**:** If X is integral, then it's clearly reduced. Also, if $U_1, U_2 \subset X$ are two disjoint open sets, then

$$\mathscr{O}_X(U_1 \coprod U_2) = \mathscr{O}_X(U_1) \times \mathscr{O}_X(U_2)$$

is not a domain. Hence X must be irreducible.

(2) ⇒ (1)**:** Suppose we have sections $f, g \in \mathscr{O}_X(U)$ such that $fg = 0$. Let $Y = U \setminus U_f$ and $Z = U \setminus U_g$; then these are both closed subsets of U. Since $fg = 0$, at least one of f_x or g_x must be 0 in $k(x)$ for every $x \in U$. So we see that $Y \cup Z = U$. But U is irreducible, and so we see that one of Y or Z must be equal to U. Assume it's Y; then for every $x \in U$, we have $f_x \in \mathfrak{m}_x$. This of course implies that over every affine open subset of U, the restriction of f is nilpotent (it will be in every prime of the ring corresponding to the affine scheme) and thus 0. But then f is 0 over U, thus showing that X is integral.

Now, suppose X has only finitely many irreducible components. One direction is clear. For the converse, suppose X is connected with $\mathscr{O}_{X,x}$ a domain, for all $x \in X$. Since reducedness is a local criterion (1.6.2), it follows that X is reduced. Moreover, for every $x \in X$, $\operatorname{Spec} \mathscr{O}_{X,x}$ is irreducible, and so each $x \in X$ is contained only in one irreducible component, by (1.5.5). Suppose X has more than one irreducible component. Let $Y \subset X$ be such a component, and let Z be the union of the rest of the irreducible components; since X has only finitely many irreducible components, Z is also closed. It follows that $Y \cup Z = X$ and $Y \cap Z = \emptyset$, implying that X is disconnected, which is a contradiction. □

-integral-rational-field

PROPOSITION 1.6.6. *Let X be an integral scheme with generic point ξ.*

(1) *For every tower of open subsets $U \subset V$, the restriction*

$$\Gamma(V, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$$

is an injection.

(2) *For every open subset $V \subset X$, the natural injection*

$$\Gamma(V, \mathscr{O}_X) \to \mathscr{O}_{X,\xi}$$

induces an isomorphism

$$K(\Gamma(V, \mathscr{O}_X)) \xrightarrow{\cong} \mathscr{O}_{X,\xi}.$$

(3) *For every $x \in X$, there is a natural injection*

$$\mathscr{O}_{X,x} \to \mathscr{O}_{X,\xi}.$$

(4) *Identifying $\Gamma(U, \mathscr{O}_X)$ and $\mathscr{O}_{X,x}$ with subrings of $\mathscr{O}_{X,\xi}$, we have*

$$\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x}.$$

PROOF. (1) For this, it suffices to show that the natural map $\Gamma(V, \mathscr{O}_X) \to \mathscr{O}_{X,\xi}$ is injective, for all open subsets $V \subset X$. Since $\Gamma(V, \mathscr{O}_X)$ is the inverse limit of the rings of sections of the affine open subsets of V, and since inverse limits preserve monomorphisms, we can assume that $V = \operatorname{Spec} R$ is affine, with R a domain. In this case, ξ corresponds to the (0) ideal in R, and so the natural map under consideration is nothing but the injection

$$R \to K(R).$$

(2) From the proof of the first part, this holds for any affine open subset $U \subset X$. For the general statement, just use the argument using inverse limits as in the last part.

(3) Let $U = \operatorname{Spec} R$ be any affine neighborhood of x. Then U also contains ξ (corresponding to the (0) ideal in R), and so the local ring $\mathscr{O}_{X,\xi}$ at ξ is just a localization of the domain $\mathscr{O}_{X,x}$.
(4) Using the same inverse limit argument, we can reduce this to the case where U is affine. In this case, the statement is equivalent to the fact that for any domain R, we have

$$R = \bigcap_{P \subset R \text{ prime}} R_P,$$

where the intersection is taken in the quotient field $K(R)$. Indeed, given an element a in the intersection on the right hand side, set $I = (R : a) \subset R$. This is an ideal of R, and, for every prime $P \subset R$, we have $I_P = R_P$, which shows that $I = R$, and thus $a \in R$.

□

Definition 1.6.7. For an integral scheme X with generic point ξ, we denote the field $\mathscr{O}_{X,\xi}$ by $K(X)$ and call it the *function field* or *field of rational functions* of X.

Remark 1.6.8. For a more general treatment of this notion, see (2).

7. The Fiber Product and Base Change

7.1. Fiber products. In this section, we'll construct fiber products in the category of schemes. There is a more conceptual way of doing this that involves a detour into representable functors, but we'll build them by hand for now.

Definition 1.7.1. Given a scheme $(X, \mathscr{O}_X)$, a *scheme over* X is a morphism of schemes $f : Y \to X$. If we have two schemes over X, then a morphism between them is a map between the domains that makes the obvious diagram commute. We'll also sometimes call a scheme over X an X-scheme.

This gives us a *category* of schemes over X, which we will denote Sch_X. If $X = \operatorname{Spec} A$, then we will also call a scheme over X an A-scheme, and denote Sch_X by Sch_A instead.

An *affine* scheme over X is just a scheme over X, whose domain is affine.

The *fiber product* between two schemes over X is their product in the category Sch_X. Given two schemes over X, $f : Y \to X$, $g : Z \to X$, the fiber product is given by the data of a scheme $f \times g : Y \times_X Z \to X$ over X and morphisms $p_1 : Y \times_X Z \to Y$, $p_2 : Y \times_X Z \to Z$ of schemes over X. For reasons of economy, we'll just use the domain $Y \times_X Z$ to denote this fiber product.

Remark 1.7.2. Note that the category of affine A-schemes, for some ring A, is equivalent to the opposite category of A-algebras. Also note that any scheme gives rise canonically to a $\mathbb{Z}$-scheme.

The main theorem is this.

scheme-fiber-product

Theorem 1.7.3. *For any scheme $(X, \mathscr{O}_X)$, fiber products exist in the category* Sch_X.

Before we prove the theorem, we'll need a couple of definitions and lemmas.

Definition 1.7.4. A morphism $f : X \to Y$ of schemes is a *monomorphism* if it's monic in the category of schemes and scheme morphisms.

An *open immersion* is a morphism of schemes $U \to Y$, which is an isomorphism onto an open subscheme of Y.

eme-open-immersion-monic

Lemma 1.7.5. *Any open immersion is a monomorphism.*

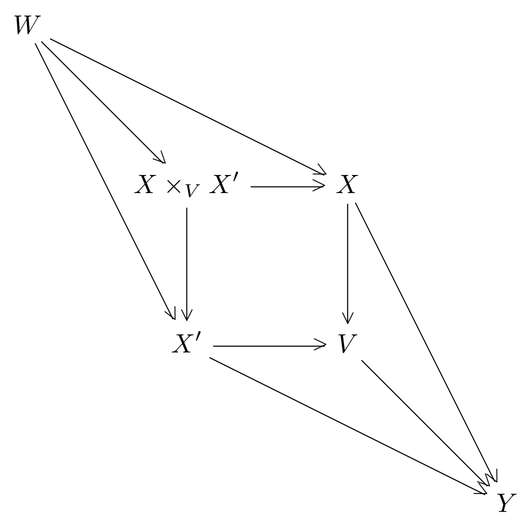
Proof. It suffices to show that, for any open subscheme U of Y, the inclusion $U \hookrightarrow Y$ is a monomorphism. We reduce immediately to the case where $Y = \operatorname{Spec} R$ is affine and $U = \operatorname{Spec} R_f$ is a principal open. In this case, we just have to show that $R \to R_f$ is an epimorphism. This follows from the universal property of localizations. □

nomorphism-fiber-product

Lemma 1.7.6. *Let $i : V \to Y$ be a monomorphism, and let X and X' be two V-schemes. Then we have a natural isomorphism of Y-schemes*

$$X \times_V X' \cong X \times_Y X'$$

Proof. It suffices to show that $X \times_V X'$ satisfies the same universal property as $X \times_Y X'$. Suppose we had two morphisms $W \to X$ and $W \to X'$ of Y-schemes. Then, since both $X \to Y$ and $X' \to Y$ factor through i, and i is a monomorphism, we see that $W \to X$ and $W \to X'$ are in fact morphisms of V-schemes. This gives us a morphism $W \to X \times_V X'$ of V-schemes through which they both factor. But in fact they are morphisms of Y-schemes, since the structure morphisms factor through V. The picture for this is as below:



□

Proof. The proof is essentially a somewhat layered application of the Fourfold Way. Remember that we have three different objects to deal with! Here, all morphisms will be morphisms of schemes over X.

Step 1: We'll construct the fiber product in the category of affine A-schemes, for some ring A. This is easy, since we just have to build the coproduct in the category of A-algebras. But given two A-algebras R, S, it's easy to see that $R \otimes_A S$ is the coproduct in the category of A-algebras. Thus

$$\operatorname{Spec} R \times_{\operatorname{Spec} A} \operatorname{Spec} S \cong \operatorname{Spec}(R \otimes_A S).$$

Now, using the usual trick of taking an affine open cover and gluing together the unique maps obtained on the sets in the cover, we see that this is in fact a fiber product in the category of A-schemes, in general.

Step 2: Now, suppose that we've constructed the fiber product $Y \times_X Z$ of two schemes $Y \to X$ and $Z \to X$ over X. We claim that, for any open subscheme $U \subset Y$, the preimage $p_1^{-1}(U)$ is the fiber product of $U \to X$ and $Z \to X$. Indeed, let $g_1 : W \to U$ and $g_2 : W \to Z$ be morphisms of schemes over X. Then, we can consider g_1 as a map into Y, and thus get a unique morphism $g : W \to Y \times_X Z$ through which both g_1 and g_2 factor. But now, the image of $p_1 \circ g = g_1$ lies in U, and so g in fact maps into $p_1^{-1}(U)$. Uniqueness is clear.

Step 3: Suppose now that we have an open cover $\mathcal{V} = \{V_i\}$ of Y, and we've constructed the fiber product $V_i \times_X Z$, for all i. Let $p_{i,1}$ be the projection of this product onto V_i. Observe that by the last part, both $p_{i,1}^{-1}(V_i \cap V_j)$ and $p_{j,1}^{-1}(V_i \cap V_j)$ are fiber products of $V_i \cap V_j \to X$ and $Z \to X$. So we have a unique isomorphism from one fiber product to the other. Using these isomorphisms, we can glue together $V_i \times_X Z$ to form the purported fiber product $Y \times_X Z$, with $p_1^{-1}(V_i) \cong V_i \times_X Z$. It remains to check that this does have the universal property. So let $g_1 : W \to Y$ and $g_2 : W \to Z$ be two morphisms over X. Each restriction $g_1^{-1}(V_i) \to V_i$ factors uniquely through the fiber product $p_1^{-1}(V_i)$. By uniqueness, these factorings restrict to the same map on the intersections $g_1^{-1}(V_i \cap V_j)$, and so can be glued together to get a factoring of g_1 through $Y \times_X Z$.

Step 4: Now, let Y, Z be A-schemes; then, if $\{V_i\}$ and $\{W_j\}$ are affine open coverings of Y and Z respectively, then we know from part (1) that $V_i \times_{\operatorname{Spec} A} W_j$ exists for all i, j. From part (3), we see that this implies that $Y \times_{\operatorname{Spec} A} W_j$ exists for all j, and so $Y \times_{\operatorname{Spec} A} Z$ exists, again by part (3). So we've constructed all fiber products in the category of A-schemes.

Step 5: Let X be any scheme, and let $\mathcal{V} = \{V_i\}$ be an affine open cover of X. Suppose we have two schemes over X, $f : Y \to X$ and $g : Z \to X$. By the above, the fiber products $f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$ exist for all i. By the two Lemmas above, these are isomorphic to the fiber products $f^{-1}(V_i) \times_X g^{-1}(V_i)$. So, again, by two applications of part (3), we see that the fiber product $Y \times_X Z$ exists.

□

EXAMPLE 1.7.7. We'll see now that the fiber product is not very intuitive at first sight. Also look at Example (7.4.6). Let t and s be two transcendental elements over a field k. Consider the tensor product $A = k(t) \otimes_k k(s)$: one can look at this as a ring of fractions of $k[t, s]$, where we invert everything in the multiplicative set $T = \{p(t)q(s) : \ p(t), q(s) \neq 0\}$. Now, suppose $\mathfrak{m} \subset A$ is a maximal ideal; then it's of the form $T^{-1}\mathfrak{n}$, for some prime ideal $\mathfrak{n} \subset k[t, s]$, maximal among those that don't intersect T. But in that case $\mathfrak{n} \cap k[t] = 0$, for otherwise there will be a non-zero $p(t) \subset \mathfrak{n} \cap T$. Similarly $\mathfrak{n} \cap k[s] = 0$, which implies that $\operatorname{ht} \mathfrak{n} \leq 1$. Hence, either $\mathfrak{n} = (0)$, or $\mathfrak{n} = (g)$, for some irreducible $g \notin k[u] \cup k[t]$. In sum, we see that $\dim A = 1$ (in fact, A is a Dedekind domain), and A has infinitely many maximal ideals. Hence $\operatorname{Spec} A$ is an infinite set, even though it is the fiber product of two schemes with one-point sets.

7.2. Preimages and Fibers. One of the useful things about the fiber product is that it lets us define the preimage of a subscheme under a morphism of schemes. So suppose we have a morphism of schemes $f : X \to Y$, and a subscheme $Y' \hookrightarrow Y$ of Y. Then, the *preimage* of Y' under f is just the fiber-product $X \times_Y Y'$. We'll see later that in good cases, this is also a nice subscheme of X.

In particular, we can treat a point $y \in Y$ as the subscheme $\operatorname{Spec} k(y)$. In this case, we say that the *fiber* of f over y is the fiber product

$$X_y := X \times_Y \operatorname{Spec} k(y).$$

The next proposition shows that this is in fact a good generalization of the notion of a fiber.

r-set-fiber-homeomorphic

PROPOSITION 1.7.8. *The fiber over a point X_y is homeomorphic as a topological space to the subspace $f^{-1}(y) \subset X$.*

PROOF. Let $\{V_i\}$ be an affine open cover for Y. From the construction of the fiber product, we know that

$$X \times_Y \operatorname{Spec} k(y) = \bigcup_{y \in V_i} X_i \times_{V_i} \operatorname{Spec} k(y) = \bigcup_{y \in V_i} (X_i)_y$$

where $X_i = f^{-1}(V_i)$. So it suffices to show that $f^{-1}(y) \cap X_i$ is homeomorphic to $(X_i)_y$. But if $\{W_{ij}\}$ is an affine open cover for X_i, then, again from the construction of the fiber product, we know that $(X_i)_y = \bigcup_j (W_{ij})_y$. Essentially, we've reduced the problem to where both $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$ are affine. In this case, $y = P$, for some prime $P \subset R$, and

$$X_y = \operatorname{Spec}(S \otimes k(P)) = \operatorname{Spec}(S_P/PS_P).$$

and if f is induced by a map $\phi : R \to S$, then

$$f^{-1}(y) = \{Q \subset R : \phi^{-1}(Q) = P\} \subset \operatorname{spc}(\operatorname{Spec} S).$$

Now, we have the natural topological embeddings

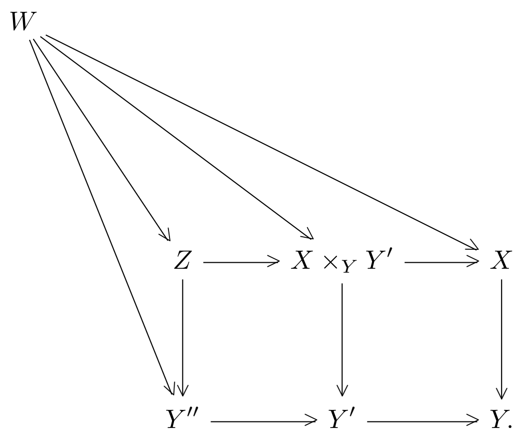
$$\operatorname{spc}(\operatorname{Spec}(S_P/PS_P)) \hookrightarrow \operatorname{spc}(\operatorname{Spec} S_P) \hookrightarrow \operatorname{spc}(\operatorname{Spec} S).$$

Under these embeddings, the space on the left maps exactly onto $f^{-1}(y)$, as is easily checked. □

7.3. Base Changes.

DEFINITION 1.7.9. If $f : X \to Y$ is a scheme over Y, and $Y' \to Y$ is a morphism, then the *base change* of f is the scheme over Y' $X \times_Y Y' \to Y'$.

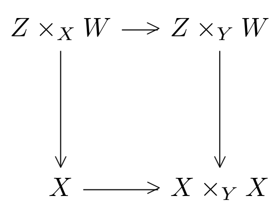
Base change is transitive. More specifically, suppose we have a sequence $Y'' \to Y' \to Y$ of morphisms. Then we can take the iterated fiber product $Z = (X \times_Y Y') \times_{Y'} Y''$, with maps $q_1 : Z \to X \times_Y Y'$ and $q_2 : Z \to Y''$. Now, if $W \to X$ and $W \to Y''$ are morphisms over Y, then we see that $W \to Y'' \to Y'$ is also a morphism over Y; hence we get a unique morphism $W \to X \times_Y Y'$ through which $W \to Y'$ and $W \to X$ factor, and then, by iterating the process, we get a unique morphism $W \to Z$ through which $W \to X \times_Y Y'$ and $W \to Y''$ factor. This in fact gives us a unique morphism through which $W \to X$ and $W \to Y''$ factor. Thus, we see that $Z \cong X \times_Y Y''$ as schemes over Y. Pictorially, we have



As one sees, this was a very arrow-theoretic argument, and holds for base changes in any category.

eme-useful-fiber-diagram

7.4. A Very Useful Fiber Diagram. Suppose we have two morphisms of Y-schemes $Z \to X$ and $W \to X$. Then I claim that the following diagram is a fiber diagram:

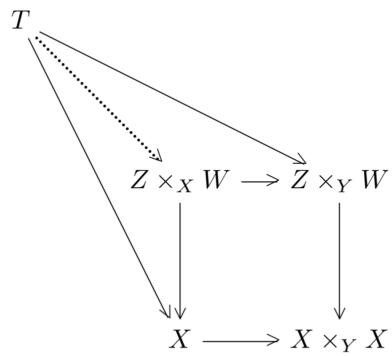


First, the arrows in the diagram need to be explained. The one on the left is the obvious arrow. The arrow at the bottom is the one whose projections onto each copy of X are the identity morphisms (this'll be very important when we study separated morphisms further down the line), and the arrow at the top is obtained from the universal property of the fiber product. The arrow on the right is the one whose projections onto each copy of X are the compositions

$$Z \times_Y W \to Z \to X; Z \times_Y W \to W \to X.$$

The diagram commutes since, whichever way we go, we get the map from $Z \times_X W$ to $X \times_Y X$ whose projections are the canonical map $Z \times_X W \to X$.

Now, suppose we have morphisms $T \to Z \times_Y W$ and $T \to X$ making the following picture commute:



We get the unique dotted map in the following fashion. The arrow $T \to Z \times_Y W$ gives us morphisms of Y-schemes $T \to Z$ and $T \to W$. But we see immediately from the diagram that these are in fact morphisms of X-schemes, and so we get the unique morphism $T \to Z \times_X W$, showing that we indeed do have a fiber diagram.

CHAPTER 2

Morphisms of Schemes

chap:mos

1. Open and Closed Immersions

Let's now see what base changes actually look like for certain simple morphisms. The simplest case is when the morphism $Y' \to Y$ is an isomorphism. In this case, we see immediately that $X \times_Y Y' \cong X$. In particular, $X \times_Y Y \cong X$.

What is the base change of an open immersion over $g : Y' \to Y$? Note that we can assume that U is in fact an open subscheme of Y. Consider the open subscheme $g^{-1}(U)$ of Y'. If we have morphisms $Z \to U$ and $Z \to Y'$ over Y, then, treating the first map as a morphism $Z \to Y$, we see that these two morphisms factor through a unique morphism $Z \to Y \times_Y Y' \cong Y'$. Now, since the composition of this factoring with f maps into U, we see that we in fact have a unique factoring through $Z \to g^{-1}(U)$. So we have $g^{-1}(U) \cong U \times_Y Y'$.

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

Definition 2.1.1. Let Ξ be a property of morphisms of schemes. Then, a morphism $f : X \to Y$ is *universally* Ξ if any base change $X \times_Y Y' \to Y'$ has the property Ξ.

We showed above that any open immersion is universally an open immersion. In language we will define below, we can say that open immersions are *stable under base change*. A more interesting case arises from closed immersions, which we define now.

Definition 2.1.2. A *closed immersion* is a morphism of schemes $f : Z \to Y$, which is closed as a map of topological spaces, and which is such that the map of sheaves $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_Z$ is surjective.

Observe now that we can rephrase part (2) of (1.1.8) in the following fashion: a morphism of affine schemes $f : \operatorname{Spec} R' \to \operatorname{Spec} R$ is a closed immersion if and only if it is induced by a surjective map of rings $R \to R'$. But, in fact, any closed immersion into $\operatorname{Spec} R$ is of this form.

rsion-into-affine-affine

Proposition 2.1.3. *Suppose $Z \to Y = \operatorname{Spec} R$ is a closed immersion. Then Z is affine, and the immersion is induced by some surjective map of rings $R \to R'$.*

Proof. It will suffice to show that Z is affine; the second part of the statement will then follow from the remark just above. For this, we'll use the criterion from

(1.4.3). We'll show that Z has a covering by finitely many affine open sets of the form Z_{a_i} such that the a_i generate the ring of global sections on Z.

First, let $a \in R$ be any element, and let $\widetilde{a} = f_Y^\sharp(a) \in \mathscr{O}_Z(Z)$ be its image in $\mathscr{O}_Z(Z)$. From (1.4.2), we know that $f^{-1}(X_a) = Z_{\widetilde{a}}$.

Now, suppose $V \subset Z$ is an affine open. Then, for every $z \in V$, there is some $a \in R$ such that $f(z) \in f(Z) \cap Y_a \subset f(V)$. This implies that $z \in Z_{\widetilde{a}} = f^{-1}(Y_a) \subset V$. But we know that, for any $b \in \mathscr{O}_Z(Z)$, $Z_b \cap V$ is affine (1.4.2). So we see that every point in Z has an affine neighborhood of the form $Z_{\widetilde{a}}$, for some $a \in R$. So we can choose $a_1, \ldots, a_k \in R$ such that the affine opens $Z_{\widetilde{a}_i}$, form an open cover for Z. In that case, the sets $f(Z) \cap Y_{a_i}$ form an open cover for $f(Z)$. Let $I \subset R$ be an ideal such that $f(Z)$ is homeomorphic to $\operatorname{spc}(\operatorname{Spec} R/I)$. Then, we see that the images of a_i in R/I must generate R/I. So we can find elements $a_{k+1}, \ldots, a_n \in I$ such that $1 = \sum_{i=1}^k c_i a_i + \sum_{j=k+1}^n a_j \in R$, for some $c_i \in R$. Observe that now we'll have $f(Z) \cap Y_{a_i} = \emptyset$, for $i > k$, which means that $Z_{\widetilde{a}_i} = \emptyset$, for $i > k$.

So we've managed to find finitely many affine opens of the form $Z_{\widetilde{a}_i}$, which cover Z, such that the a_i generate R. But then their images in $S = \mathscr{O}_Z(Z)$ will generate S. This finishes our proof. □

REMARK 2.1.4. The above proof actually shows that for any morphism $f : Z \to Y$, with Y affine and mapping Z homeomorphically onto a closed subset of Y, Z must also be affine.

The above Proposition can be generalized to all schemes, using quasi-coherent sheaves of ideals, but that'll come up after we've gotten to sheaves of modules over a scheme.

Now, suppose $Y = \operatorname{Spec} R$ is affine, $Z = \operatorname{Spec} R/I$, for some ideal I, and $Z \to Y$ is the natural closed immersion induced by the map $R \to R/I$. If $g : Y' \to Y$ is an affine R-scheme, with $Y' = \operatorname{Spec} S$, then we see that the base change $Y' \times_Y Z \to Y'$ is just the morphism induced by the map of rings $S \to S/IS$. This is because we have the following picture for the maps of rings

$$\begin{array}{ccc} S/IS & \longleftarrow & R/I \\ \uparrow & & \uparrow \\ S & \longleftarrow & R \end{array}$$

In particular, we see that $\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} R/I = \operatorname{Spec} S/IS$, and the base change $Y' \times_Y Z \to Y'$ is just the natural morphism $\operatorname{Spec} S/IS \to \operatorname{Spec} S$, which is again a closed immersion.

mos-affine-universal

DEFINITION 2.1.5. We say that a property Ξ of morphisms of schemes is *affine-universal* if the following conditions is true for a morphism $f : X \to Y$ of schemes: Whenever f has property Ξ, and Y is affine, then any base change $X \times_Y Y' \to Y'$, for Y' affine, also has property Ξ.

losed-immersion-aff-univ

REMARK 2.1.6. It follows from the preceding discussion and Proposition (2.1.3) that a closed immersion is affine-universal.

Observe now that being a closed immersion is a very local property, in the sense that, if we had an open cover $\{V_i\}$ of Y such that the restrictions $f^{-1}(V_i) \to V_i$ are all closed immersions, then f is again a closed immersion. Clearly, in this case, the map of sheaves $f^\sharp$ is surjective on stalks, and so is surjective. It's definitely a homeomorphism onto its image, since it's glued together from homeomorphisms onto their images (the main problem is to show injectivity: for this just note that every point lies in some V_i, and so there's precisely one point in $f^{-1}(V_i)$, and hence just one point in the domain that maps to it). So we can think of a closed immersion as being glued together from lots of natural maps of the form $\operatorname{Spec} R/I \to \operatorname{Spec} R$.

This leads to a definition.

mos-affine-local

DEFINITION 2.1.7. We say that a property Ξ of morphisms of schemes is *local on the base* if the following is true:

(1) If $f : X \to Y$ is a morphism with property Ξ, then for every open set $V \subset Y$, the restriction $f^{-1}(V) \to V$ also has property Ξ.
(2) With f as above, if there is an open cover $\{V_i\}$ of Y such that all the restrictions $f^{-1}(V_i) \to V_i$ have property Ξ, then f also has the property.

REMARK 2.1.8. In fact, we don't lose anything by restricting ourselves to affine opens in the second condition. For if $\{V_i\}$ is any open cover which satisfies the hypotheses of condition (2), then so will any affine refinement, by condition (1). For this reason, being local on the base is also sometimes referred to as *affine-localness*.

REMARK 2.1.9. Usually, any property of schemes is defined locally, so one implication (about the restrictions satisfying the same property) will be tautological. It's the other one that's more significant.

We showed above that the property of being a closed immersion is local on the base. Note that it follows immediately from its definition that an open immersion is local on the base. It will now follow from the next general proposition that any closed immersion is in fact universally a closed immersion.

DEFINITION 2.1.10. We say that a property Ξ is *stable under base change* if whenever $X \to Y$ has property Ξ, then so does the base change $X \times_Y Y' \to Y'$ over any scheme $Y' \to Y$ over Y.

fine-universal-universal

PROPOSITION 2.1.11. *Suppose Ξ is a property of morphism of schemes that is local on the base and affine-universal. Then Ξ is stable under base change.*

PROOF. The proof will be in the same vein as the Fourfold Way, although a little simpler in some ways. Before that, recall from (2.1.5) the condition for a property to be affine-universal.

Step 1: Suppose $f : X \to Y$ and $g : Y' \to Y$ are morphisms over Y, so that f has property Ξ, and suppose there is an open cover $\{V_i\}$ of Y such that the base changes

$$f^{-1}(V_i) \times_{V_i} g^{-1}(V_i) \to g^{-1}(V_i)$$

all have property Ξ. But if $p_1 : X \times_Y Y' \to Y'$ is the base change, then these base changes are just the maps

$$p_1^{-1}(g^{-1}(V_i)) \to g^{-1}(V_i).$$

Since Ξ is local on the base, we see that the base change $p_1 : X \times_Y Y' \to Y'$ also has property Ξ.

Step 2: Now, with the same notation as above, let $f : X \to Y$ be a morphism of schemes, with Y affine. Let $\{W_i\}$ be an affine open cover for Y'. Then, for each i, the base change

$$X \times_Y g^{-1}(W_i) \cong p_1^{-1}(g^{-1}(W_i)) \to g^{-1}(W_i)$$

has property Ξ by affine-universality of Ξ. Since Ξ is local on the base, we see that the base change $X \times_Y Y' \to Y'$ also has property Ξ.

Step 3: Let $f : X \to Y$ be any morphism of schemes with property Ξ. Then, if $\{V_i\}$ is an affine open cover for Y, we see by the previous step that, for each i, the base change

$$f^{-1}(V_i) \times_{V_i} g^{-1}(V_i) \to g^{-1}(V_i)$$

has property Ξ. By Step (1), this means that the base change $X \times_Y Y' \to Y'$ also has property Ξ.

□

-closed-immersion-proper

Corollary 2.1.12. *Closed immersions are stable under base change.*

2. The Reduced Induced Subscheme

Definition 2.2.1. A *closed subscheme* of a scheme X is an equivalence class of closed immersions into X, under the equivalence relation of 'isomorphic as schemes over X'.

A closed subscheme *associated* to a closed subset Y of X is the equivalence class of a closed immersion whose topological image is Y. Note that by the definition of the equivalence relation, any other immersion in the same equivalence class will also need to have topological image Y.

A morphism between two closed subschemes associated to Y is just a morphism between representatives of each equivalence class as schemes over X.

Remark 2.2.2. Usually, when we talk about closed subschemes we'll be talking about a specific closed immersion in the equivalence class that it represents, and we'll often conflate the two. This should not be an issue.

Now, if $V(I) \subset \operatorname{spc}(\operatorname{Spec} R)$ is a closed subset, then there are many closed subschemes associated to it: one for each ideal $J \subset R$ with $\operatorname{rad}(J) = \operatorname{rad}(I)$. But the subscheme associated to $\operatorname{rad}(I)$ is in some ways the most canonical choice. We will formalize and generalize this notion to arbitrary schemes in the next construction.

Definition 2.2.3. The *reduced induced subscheme* associated to a closed subset Y of X is a closed subscheme $Y' \to X$ associated to Y such that for every other closed subscheme $Y'' \to X$ associated to Y factors uniquely through $Y' \to X$. In other words, the reduced induced subscheme is terminal in the category of closed subschemes associated to Y.

mos-reduced-induced

Proposition 2.2.4. *For every closed subset Y of a scheme X, there exists an associated reduced induced structure.*

Proof. This is ripe for some Fourfold Way action, but with a little twist.

Step 1: Suppose $X = \operatorname{Spec} R$ is affine, and $Y = V(I)$, for some ideal $I \subset R$. Then, consider the closed subscheme

$$\operatorname{Spec} R/\operatorname{rad}(I) \to \operatorname{Spec} R.$$

Then, every other closed subscheme associated to $V(I)$ corresponds to an ideal $J \subset R$ with $\operatorname{rad}(J) = \operatorname{rad}(I)$, and so factors through $Y' = \operatorname{Spec} R/\operatorname{rad}(I)$.

Step 2: Now, suppose $X_a \subset X = \operatorname{Spec} R$ is a principal open subscheme; then since $(R/\operatorname{rad}(I))_a = R_a/\operatorname{rad}(I_a)$, we see that if $f : Y' \to X$ is the reduced induced subscheme associated to Y, then $f^{-1}(X_a) \to X_a$ is the reduced induced subscheme associated to $Y \cap X_a$.

Step 3: Now, let $U \subset X = \operatorname{Spec} R$ be any open subscheme. We claim that $f^{-1}(U) \to U$ is the reduced induced subscheme associated to $Y \cap U$. To see this, cover U by principal opens $V_i = \operatorname{Spec} R_{a_i}$. Then $f^{-1}(V_i) \to V_i$ is the reduced induced subscheme associated to $Y \cap V_i$ by the previous step. Now, suppose $g : Z \to U$ is any closed subscheme associated to $Y \cap U$. Then, for each i, $g^{-1}(V_i) \to V_i$ factors uniquely through $f^{-1}(V_i) \to V_i$, say via maps $h_i : g^{-1}(V_i) \to f^{-1}(V_i)$. Since $V_i \cap V_j$ is also a principal open (1.1.2), $g^{-1}(V_i \cap V_j) \to V_i \cap V_j$ also factors uniquely through $f^{-1}(V_i \cap V_j) \to V_i \cap V_j$ via a map $h_{ij} : g^{-1}(V_i \cap V_j) \to f^{-1}(V_i \cap V_j)$. From this, we see that the h_i agree on their domains of intersection, and so can be glued together to get a factoring of g through $f^{-1}(V_i) \to V_i$.

Step 4: Let X be an arbitrary scheme now, and cover X with affine opens V_i. Then, for each i, we have a reduced induced subscheme $f_i : V_i \to X$ associated to $Y \cap V_i$. By the previous steps, we can glue these together to get a scheme $f : Y' \to X$ over X such that $f^{-1}(V_i) \to V_i$ is the reduced induced subscheme associated to $Y \cap V_i$. Now, we use the usual argument to conclude that f satisfies the universal property for the reduced induced subscheme associated to Y.

□

REMARK 2.2.5. It's the *reduced* induced subscheme, because it's clearly a reduced scheme (look at it locally). If we had taken $Y = X$, then we would have gotten back X_{red}.

3. Surjections and Dominant Maps

DEFINITION 2.3.1. A morphism of schemes $f : X \to Y$ is *surjective* if it's surjective as a map on the underlying topological spaces. Using 1.7.8), we see that this is the same as saying that for every $y \in Y$, the underlying topological space of the fiber X_y is non-empty.

mos-surjective-aff-loc

PROPOSITION 2.3.2. *The property of being surjective is local on the base and affine universal. It's stable under base change and composition.*

PROOF. That it's stable under base change will follow from (2.1.11) and the first part of the statement, and it's evident that it's stable under composition. So it remains to prove the first part of the Proposition. Locality on the base is immediate, so it's enough to show affine universality. For this, assume $g : Y' = \operatorname{Spec} S' \to Y = \operatorname{Spec} S$ is a morphism of affine schemes and let $f : X \to Y$ be a surjective morphism. Since $X \times_Y Y'$ is the union of open subschemes $V \times_Y Y'$, for $V \subset X$ an affine open, it's enough to consider the case where $X = \operatorname{Spec} R$ is also affine. In this case, the fiber product is just $\operatorname{Spec} R \otimes_S S'$. Since $\operatorname{Spec} R \to \operatorname{Spec} S$ is surjective, we see that for every prime $P \subset S$, the ring $R \otimes_S k(P)$ is non-zero. We must show that, for every prime $Q \subset S'$, the ring $R \otimes_S S' \otimes'_S k(Q)$ is also non-zero.

Let $P = Q^c \subset S$ be the contraction of Q to S, and let $R' = R \otimes_S k(P)$ be the fiber over P. We know that R' is non-zero; moreover we also see that

$$R \otimes_S S' \otimes'_S k(Q) \cong R \otimes_S k(Q) \cong R' \otimes_{k(P)} k(Q).$$

So we reduce to showing that if K is a field extension of another field k, and if T is a non-zero k-algebra, then $T \otimes_k K \neq 0$ (to translate the earlier equation to this, take $T = R'$, $k = k(P)$ and $K = k(Q)$). But observe that $k(Q)$ is faithfully flat over $k(P)$, and so this follows from part (3) of [CA, 3.6.4]. □

DEFINITION 2.3.3. A morphism $f : X \to Y$ is *dominant* if $f(X)$ is dense in Y. Clearly, surjective morphisms are dominant, and a dominant morphism with closed image is surjective.

mos-dominant-aff-loc

PROPOSITION 2.3.4. *Dominant morphisms are local on the base and are stable under composition.*

PROOF. Trivial. □

ominant-integral-schemes

PROPOSITION 2.3.5. *Let $f : X \to Y$ be a morphism between integral schemes. Let ξ_X and ξ_Y be the generic points of X and Y, respectively. Then the following are equivalent:*

(1) *f is dominant.*
(2) *$f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is injective.*
(3) *For every open subset $V \subset Y$ and every open subset $U \subset f^{-1}(V)$, the map*
$$\Gamma(V, \mathscr{O}_Y) \to \Gamma(U, \mathscr{O}_X)$$
is injective.
(4) *$f(\xi_X) = \xi_Y$.*
(5) *$\xi_Y \in f(X)$.*

PROOF. **(1) ⇒ (2):** Suppose f is dominant; then, for every affine open $U = \operatorname{Spec} R \subset Y$, the morphism $f^{-1}(U) \to U$ is dominant. From (1.1.8), we see that this is induced by map $\varphi : R \to \Gamma(f^{-1}(U), \mathscr{O}_X)$ such that $\ker\varphi \subset \operatorname{Nil} R$. Since R is integral, this shows that φ is injective. So we see that $f^\sharp_U$ is injective for all affine open subsets $U \subset Y$. Now, to finish the proof we utilize the fact that, for every open subset $V \subset Y$, $\Gamma(V, \mathscr{O}_Y)$ is the inverse limit of the rings $\Gamma(U, \mathscr{O}_Y)$, as U ranges over all affine open subsets of V, and $\Gamma(f^{-1}(V), \mathscr{O}_X)$ is the inverse limit of the rings $\Gamma(f^{-1}(U), \mathscr{O}_X)$. Since inverse limits preserve injections, this implication is proved.

(2) ⇒ (3): Observe that for open subset $U \subset f^{-1}(V)$, the natural inclusion map is dominant, and hence the restriction map
$$\Gamma(f^{-1}(V), \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$$
is injective by the proof of the first implication. Now, (3) follows easily from (2).

(3) ⇒ (4): If (3) is true, then we can assume that both X and Y are affine. In this case, it reduces to showing that the (0) ideal in $\Gamma(X, \mathscr{O}_X)$ contracts to the (0) ideal in $\Gamma(Y, \mathscr{O}_Y)$, which it of course does.

(4) ⇒ (5): This is trivial.

(5) ⇒ (1): So is this.

□

mos-dominant-local-rings

COROLLARY 2.3.6. *Let $f : X \to Y$ be a dominant morphism between integral schemes. For every $x \in X$, the induced map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is an injection.*

PROOF. We reduce at once to the case where $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$ are affine, and $f : X \to Y$ is induced by an injective map of rings $\varphi : R \to S$. Then it reduces to showing that if $Q \subset S$ is a prime contracting to a prime $P \subset R$ under φ, then the natural map $R_P \to S_Q$ is injective. But $\varphi_P : R_P \to S_P$ is injective, and the localization map $S_P \to S_Q$ is always injective, since S is integral, and so the composition $R_P \to S_Q$ must also be injective. □

4. Affine and Quasi-compact Morphisms

Still more properties of morphisms in this section.

DEFINITION 2.4.1. A morphism $f : X \to Y$ of schemes is *quasi-compact* if, for every affine open $V \subset X$, $f^{-1}(V)$ is quasi-compact.

A morphism $f : X \to Y$ is *affine* if, for every affine open $V \subset X$, $f^{-1}(V)$ is affine.

REMARK 2.4.2. It's clear that affine morphisms are quasi-compact. We also showed in (2.1.3) that closed immersions are affine.

compact-specmap-dominant

PROPOSITION 2.4.3. *Let $f : X \to \operatorname{Spec} R$ be a quasi-compact morphism. Then, the natural morphism $X \to \operatorname{Spec} \Gamma(X, \mathscr{O}_X)$ is dominant.*

PROOF. The fact that f is quasi-compact means that the underlying topological space of X is quasi-compact. We will be done, if we can show that, for every non-nilpotent $a \in \Gamma(X, \mathscr{O}_X)$, the open subset $X_a \subset X$ is non-empty. This is equivalent to saying that $\Gamma(X_a, \mathscr{O}_X)$ non-zero. From (1.4.2), we see that $\Gamma(X, \mathscr{O}_X)_a$ injects into $\Gamma(X_a, \mathscr{O}_X)$. Since a is not nilpotent, $\Gamma(X, \mathscr{O}_X)_a \neq 0$: our proof is finished. □

mos-aff-qc-aff-loc

PROPOSITION 2.4.4. *Both quasi-compactness and affineness are local on the base and affine-universal. In particular, quasi-compact and affine morphisms are stable under base change.*

PROOF. The second assertion follows from the first by Proposition (2.1.11). Recall the definitions of the concepts of local on the base and affine-universal from (2.1.7) and (2.1.5). Let $f : X \to Y$ be a morphism of schemes. We'll prove localness on the base first, using the Affine Communication Lemma (ACL, for short) (1.3.2).

Quasi-compact: We'll go through the two conditions for localness on the base.

(1) If $V \subset Y$ is an open subscheme, then perforce the preimage of every affine open in V is quasi-compact.

(2) In the notation of ACL, let property P for an affine open $V \subset Y$ be true if $f^{-1}(V)$ is quasi-compact. We'll show that this property satisfies the two conditions of this lemma. Suppose $V = \operatorname{Spec} R$, and let $a \in R$ be any element. Then, we saw earlier in the proof of (1.4.2), that $f^{-1}(\operatorname{Spec} R_a) = f^{-1}(V)_a$. Now, if $f^{-1}(V)$ is quasi-compact, then it has a covering by finitely many affine opens, and the intersection of $f^{-1}(V)_a$ with each such affine is quasi-compact. This implies that $f^{-1}(V)_a$ is also quasi-compact. Now, suppose $\{a_1, \ldots, a_n\}$ is a finite generating set for R, and $f^{-1}(V)_{a_i}$ is quasi-compact for each i.

Then, it follows immediately that $f^{-1}(V)$ is also quasi-compact. So, using ACL, we conclude that quasi-compactness is local on the base.

Affine: Again, the two conditions, one by one.

(1) The first condition follows trivially from the definition.

(2) Again, we use ACL: let property P for an affine open $V \subset Y$ be true if $f^{-1}(V)$ is affine. If $V = \operatorname{Spec} R$, and $a \in R$ is any element, then $f^{-1}(\operatorname{Spec} R_a) = f^{-1}(V)_a$ is clearly affine, whenever $f^{-1}(V)$ is affine. Now, suppose $\{a_1, \ldots, a_n\}$ is a generating set for R, with U_{a_i} affine for every i, where $U = f^{-1}(V)$. Then, since the a_i generate R, their images in S will generate S. So we can apply (1.4.3) to conclude that U is affine.

Now, we move on to showing affine-universality. For affine morphisms, it follows trivially. For quasi-compact morphisms, we have to work a little harder. So suppose $g : Y' \to Y$ is a morphism of affine schemes, and $f : X \to Y$ is a quasi-compact morphism. Since quasi-compactness is local on the base, it suffices to show that $X \times_Y Y'$ is quasi-compact. But, if $\{V_i\}$ is a finite affine cover for X, then $V_i \times_Y Y'$ is a finite affine cover for the base change, and so we see that $X \times_Y Y'$ is quasi-compact. $\square$

Here's a nice application of this proposition that I filched from Ravi Vakil.

Example 2.4.5. Let $X = \operatorname{Spec} R$ be an affine scheme, and let $Y = \operatorname{Spec} R/I$ be a closed subscheme associated to an invertible ideal $I \subset R$. See [CA, 7.2.1] for definitions. So there is an affine open cover $\{X_i = \operatorname{Spec} R_{a_i}\}$ of X such that $Y \cap X_i$ is associated to a principal ideal (b_i) of R_{a_i} [CA, 7.1.5]. Consider the open immersion $X - Y \hookrightarrow X$. For every i, the restriction is the map $(X_i)_{b_i} \hookrightarrow X_i$, and is hence affine. So we see from the localness on the base of affineness that $X - Y$ is also affine.

5. The Scheme Theoretic Image

Now we turn to the issue of what the image of a morphism $f : X \to Y$ is. We want it to be a subscheme of Y, preferably a closed subscheme, but which one? The best way to go about this is by defining the image through some universal property. What universal property does the image of a morphism satisfy? In general, if $\operatorname{im} f : Z \to Y$ is the image morphism, then f should factor through it. Moreover, if f also factors through any other inclusion $Z' \to Y$, then $\operatorname{im} f$ should also factor through it. Basically, the image of f should be the smallest subobject of Y containing the set-theoretic image of f in this sense. Let's formalize this now.

Definition 2.5.1. The *scheme-theoretic image* of a morphism $f : X \to Y$ of schemes is a closed subscheme $Z \to Y$ through which f factors. Moreover, if $Z' \to Y$ is any other closed subscheme through which f factors, then $Z \to Y$ also factors through $Z' \to Y$. Clearly, this is initial in a suitably defined category, and is thus determined up to unique isomorphism, if it exists.

s-scheme-theoretic-image

Proposition 2.5.2. *The scheme-theoretic image of every quasi-compact morphism $f : X \to Y$ of schemes exists. Moreover, it's a closed subscheme associated to the closure of the set-theoretic image of f.*

Remark 2.5.3. We'll return to this after we've investigated quasi-coherent sheaves over a scheme.

PROOF. A simple(r) application of the Fourfold Way

Step 1: Suppose first that $Y = \operatorname{Spec} R$ is affine. Then $f : X \to Y$ is determined by a map $\phi : R \to \mathscr{O}_X(X)$. Consider the closed immersion $g : \operatorname{Spec} R/\ker\phi \to \operatorname{Spec} R$. Clearly f factors through this; also, if f factored through any other closed immersion $\operatorname{Spec} R/I \to \operatorname{Spec} R$, then it's easy to see that $I \subset \ker\phi$, and so g also factors through $\operatorname{Spec} R/I \to \operatorname{Spec} R$. Hence g is the scheme-theoretic image of f. Observe moreover that the set-theoretic image of g is $V(\ker\phi)$, which, according to (1.1.8), is locally the closure of the set-theoretic image of f. Since the image of f is a finite union of the set-theoretic images of affine opens, its closure is also the set-theoretic image of g.

Step 2: Now, consider a principal open $Y_a \subset Y$. The map $f^{-1}(Y_a) \to Y_a$ is determined by a map $R_a \to \mathscr{O}_{X_a}(X_a)$ that factors through the localization ϕ_a. But recall from (1.4.2) that, since X is quasi-compact, the map $\mathscr{O}_X(X)_a \to \mathscr{O}_{X_a}(X_a)$ is an injection. Hence, the kernel of the map $R_a \to \mathscr{O}_{X_a}(X_a)$ is just $\ker\phi_a$, and the scheme theoretic image of the map $X_a \to Y_a$ is therefore $g_a : \operatorname{Spec}(R/I)_a \to \operatorname{Spec} R_a$, which is just the map $g^{-1}(Y_a) \to Y_a$. Now, given any open set $U \subset Y$, we can cover it with principal opens, and show that the map $g^{-1}(U) \to U$ is the scheme-theoretic image of $f^{-1}(U) \to U$.

Step 3: Let Y be arbitrary now, and let $\{V_i\}$ be an affine open cover for Y. Let $g_i : Z_i \to V_i$ be the scheme-theoretic images of the maps $f^{-1}(V_i) \to V_i$ given to us by Step (1). Then, by Step (2), we know that both $g_i^{-1}(V_i \cap V_j) \to V_i \cap V_j$ and $g_j^{-1}(V_i \cap V_j) \to V_i \cap V_j$ are scheme-theoretic images of $f^{-1}(V_i \cap V_j) \to V_i \cap V_j$. So there's a unique isomorphism over $V_i \cap V_j$ between them. We can use these isomorphisms to glue together the Z_i and the g_i, and get a morphism $g : Z \to Y$, that we can show easily to be the scheme-theoretic image of f. That its set-theoretic image is the closure of the set-theoretic image of f follows from Step (1).

□

EXAMPLE 2.5.4. The scheme theoretic image can be quite horrible, if we remove the quasi-compactness condition. Consider the natural morphism $\coprod_n \operatorname{Spec} k[x]/(x^n) \to \operatorname{Spec} k[x]$: its scheme theoretic image is the identity map $\operatorname{Spec} k[x] \to \operatorname{Spec} k[x]$, but its set theoretic image is just the closed point (x).

red-scheme-theoretic-img

PROPOSITION 2.5.5. *If, in the proposition above, X is reduced, then the scheme-theoretic image of f is just the reduced induced subscheme associated to the closure of the set-theoretic image of f.*

PROOF. Observe that for every affine open $V = \operatorname{Spec} R \subset Y$, the map $f^{-1}(V) \to V$ is induced by a map of rings $\phi : R \to \mathscr{O}_X(f^{-1}(V))$, and the restriction of the scheme-theoretic image $g^{-1}(V) \to V$ is just the closed immersion $\operatorname{Spec} R/\ker\phi \to \operatorname{Spec} R$. Since $f^{-1}(V)$ is quasi-compact, we can cover it with finitely many affines $W_k = \operatorname{Spec} S_k$, and, if $J_k = \ker\phi_k$, where $\phi_k : R \to S_k$ induces the morphism $W_k \to V$, then we see that $\ker\phi = \cap J_k$. Now, the closure of the image of $W_k \to V$ is just $V(J_k)$, by (1.1.8).

Since S_k is reduced and R/J_k is a subring of S_k, we see that the scheme-theoretic image $\operatorname{Spec} R/J_k \to \operatorname{Spec} R$ of $W_k \to V$ is the reduced induced subscheme associated to $V(J_k)$. This implies that $J_k \subset R$ is radical, and so also is the finite intersection $\ker\phi \subset R$. So the scheme theoretic image $\operatorname{Spec} R/\ker\phi \to \operatorname{Spec} R$ of $f^{-1}(V) \to V$ is the reduced induced subscheme associated to the set-theoretic closure $V(\ker\phi)$. This shows that the scheme theoretic image of $X \to Y$ is the reduced induced subscheme associated to the set-theoretic closure of the image of f. □

6. Locally Closed Immersions

DEFINITION 2.6.1. A morphism $f : Z \to Y$ is a *locally closed immersion* or just an *immersion* if it factors into a closed immersion $Z \to U$ followed by an open immersion $U \to Y$.

mos-loc-closed-aff-loc

PROPOSITION 2.6.2. *The property of being a locally closed immersion is local on the base and are stable under base change and compositions.*

PROOF. First we show that locally closed immersions are local on the base

(1) Suppose $f : X \to Y$ is a locally closed immersion factoring as $X \xrightarrow{g} U \xrightarrow{h} Y$, with g a closed immersion and h an open immersion. Let $V \subset Y$ be any open set, then $f^{-1}(V) \to V$ factors as $g^{-1}(h^{-1}(V)) \to h^{-1}(V)$ followed by $h^{-1}(V) \to V$.
(2) Let $f : X \to Y$ be a morphism, and let the property P be true of an open set $V \subset Y$ if $f^{-1}(V) \to V$ is a locally closed immersion. Let $\{V_i\}$ be an open cover for V. If P is true for V, then it's true for all V_i by (1) above. Conversely, suppose P is true for each V_i; then $f^{-1}(V_i) \to V_i$ factors via a closed immersion $g_i : f^{-1}(V_i) \to U_i$ and an open immersion $f_i : U_i \to V_i$. Identifying U_i with its open image in V, let $U = \bigcup_i U_i$ equipped with the inclusion $U \hookrightarrow V$. Then, the g_i glue together to give a closed immersion $g : f^{-1}(V) \to U$ (since closed immersions are local on the base), followed by the open immersion $U \to V$.

Let $f : X \to Y$ be a locally closed immersion with the usual factoring $X \to U \to Y$, and let $g : Y' \to Y$ be any Y-scheme. We have the following diagram:

$$\begin{array}{ccc} X \times_Y Y' & \longrightarrow & X \\ \downarrow & & \downarrow \\ U \times_Y Y' & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

where we've used the isomorphism

$$X \times_Y Y' \cong X \times_U (U \times_Y Y')$$

as Y-schemes. Since open and closed immersions are stable under base change, we see from the diagram that so are locally closed immersions.

To show that immersions are stable under composition, it suffices to show that if $f = i \circ u$, where i is a closed immersion and u is an open immersion, then f can be expressed as a composition $v \circ j$, where v is an open immersion and j is a closed immersion. Let $u : X \to Z$, and $i : Z \to Y$, and let $U \subset Y$ be an open subset such that $u(X) = i^{-1}(U)$, and let $v : U \to Y$ be the natural open immersion. We're done now, by observing that $i^{-1}(U) \to U$ is a closed immersion, and that $X \cong i^{-1}(U)$. □

One might legitimately wonder, given the last paragraph of the proof, why we choose to define immersions the way we do: why not define them to be compositions of closed immersions with open immersions, instead of the other way round? The next Proposition tells us that, if we're willing to accept certain weak finiteness hypotheses, we can in fact do this.

s-loc-closed-open-closed

PROPOSITION 2.6.3. *Let $f : X \to Y$ be a quasi-compact immersion. Then, we can express f as a composition $i \circ u$, where i is a closed immersion and u is an open immersion.*

PROOF. Let $i : Z \to Y$ be the scheme-theoretic image of f. Hence $f = i \circ v$, where $v : X \to Z$. We claim that $v = k \circ u$, where k is a closed immersion and u is open. For this, we can assume that Y, and hence also Z, is affine. Now, since $f : X \to Y = \operatorname{Spec} R$ is an immersion, we can find an open subset $U \subset Y$ such that f factors through a closed immersion $j : X \to U$. We can cover U by principal affine opens Y_{a_i}, and consider the restriction $j_i : X_{a_i} \to Y_{a_i}$. Now, the restriction of v, $v_i : X_{a_i} \to Z_{a_i}$ is the scheme theoretic image of $f_i : X_{a_i} \to Y$, as we saw in the construction of the scheme theoretic image. Observe also that it is now sufficient to prove that v_i is the composition of a closed immersion with an open immersion, for all i. So we've reduced the situation to where $Y = \operatorname{Spec} R$, and $X = \operatorname{Spec}(R_a/J_a)$, for some element $a \in R$ and some ideal $J \subset R$ that doesn't contain a. Let $I = \ker(R \to R_a/J_a)$; then $Z = \operatorname{Spec} R/I$, and $u : X \to Z$ is induced by the map $R/I \to (R_a/J_a)$, which factors as

$$R/I \to R/J \to (R_a/J_a).$$

This gives us our result. □

7. Morphisms of Finite Type and of Finite Presentation

7.1. Morphisms of Finite Type. The regular maps that one encounters in classical affine algebraic geometry are all induced by maps of finite type between k-algebras. Such maps still play a big role in scheme theoretic geometry. So we're going to define them and look at some of their properties in this section.

DEFINITION 2.7.1. A morphism $f : X \to Y$ of schemes is *locally of finite type* if, for every affine open $V = \operatorname{Spec} R \subset Y$, we can cover $f^{-1}(V)$ with affine opens $W_i = \operatorname{Spec} S_i$ such that the map of rings $R \to S_i$ induced by the restriction $W_i \to V$ is of finite-type.

The morphism f is of *finite type* if, in the notation above, the collection of W_i can be taken to be finite for every V.

mos-fin-type-equiv-prps

PROPOSITION 2.7.2. *The following are equivalent for a morphism $f : X \to Y$.*

(1) *f is locally of finite type.*
(2) *There is an open cover $\{U_i : i \in I\}$ of Y such that the restrictions $f^{-1}(U_i) \to U_i$ are locally of finite type.*
(3) *For every affine open $V = \operatorname{Spec} R \subset Y$ and every affine open $U = \operatorname{Spec} S \subset f^{-1}(V)$, S is a finitely generated R-algebra.*

PROOF. (1) ⇒ (2) and (3) ⇒ (1) are trivial, so we'll be done if we prove (2) ⇒ (3). By replacing Y with V and U_i with $U_i \cap V$ and taking a further refinement of $\{U_i\}$, we can assume that $Y = \operatorname{Spec} R$ is affine, and that there is a finite open cover $\{U_i = Y_{f_i} : 1 \leq i \leq n\}$ for some set of generators $\{f_1, \dots, f_n\}$ of the unit ideal of R such that the restriction $X_{f_i} \to Y_{f_i}$ is locally of finite type. Now, for every affine open subscheme $\operatorname{Spec} S \subset X$, the images of the f_i in S also generate the unit ideal in S, and, moreover, for each i, S_{f_i} is of finite type over R_{f_i} and thus over R. Now, it follows from (1.3.3) that S is also of finite type over R. □

REMARK 2.7.3. Condition (3) above can be rephrased as saying that morphisms that are locally of finite type are *local on the domain.*

mos-fin-iff-loc-fin-qc

LEMMA 2.7.4. *A morphism $f : X \to Y$ of schemes is of finite type if and only if it is of locally finite type and quasi-compact.*

PROOF. Trivial. □

-loc-fin-type-properties

COROLLARY 2.7.5.
(1) *Morphisms that are locally of finite type are local on the base and on the domain.*
(2) *The class of morphisms that are locally of finite type is stable under base change.*
(3) *The class of morphisms that are locally of finite type is stable under composition.*

All the above assertions are also true with 'locally of finite type' replaced by 'of finite type'.

PROOF. The last assertion follows from the Proposition, the lemma above and (2.4.4).

(1) Follows immediately from the Proposition.
(2) It suffices to show that the property of being locally of finite type is affine universal. Suppose $g : Y' = \operatorname{Spec} S \to Y = \operatorname{Spec} R$ is a morphism of affine schemes, and suppose $f : X \to Y$ is a morphism of locally finite type. By localness on the domain, it's enough to show that $X \times_Y Y'$ can be covered by affine opens each of which corresponds to a finitely generated S-algebra. Now, we can find an affine open cover $\{V_i = \operatorname{Spec} B_i\}$ of X such that B_i is of finite type over S. Then, for each i, $V_i \times_Y Y' = \operatorname{Spec} B_i \otimes_R S$ will be of finite type over S. This gives us an affine cover of $X \times_Y Y'$ that satisfies our requirements.
(3) Let $f : X \to Y$ and $g : Z \to X$ be morphisms that are locally of finite type, and let $V = \operatorname{Spec} R \subset Y$ be an affine open subscheme, and let $W = \operatorname{Spec} T \subset g^{-1}(f^{-1}(V))$ be an affine open subscheme of Z. We have to show that T is finitely generated over R. We can find generators $f_1, \dots, f_r$ of the unit ideal of T such that $g(W_{f_i})$ is contained in some affine subscheme $U_i = \operatorname{Spec} S_i \subset f^{-1}(V)$. Then, for each i, T_{f_i} is finitely generated over S_i, which in turn is finitely generated over R. Hence, for each i, T_{f_i} is finitely generated over R. Now the result follows from (1.3.3).

□

Next follows a very important result on morphisms of finite type.

n-type-img-constructible

Theorem 2.7.6 (Chevalley). *Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes. Then the set-theoretic image under f of any constructible subset $W \subset X$ is a constructible subset of Y.*

Proof. At the outset, we can replace X and Y with X_{red} and Y_{red}, and assume that our schemes are reduced.

Any constructible set is the finite union of locally closed subsets of X. Hence it's enough to show that $f(W)$ is constructible for any locally closed subset $W \subset X$. Now, we can consider the inclusion $W \hookrightarrow X$ as being the composition of a closed immersion with an open immersion. Given this, we see that the morphism $W \to Y$ is also of finite type (since immersions are of finite type). Thus, it suffices to show (replacing W by X), that $f(X)$ is constructible.

X has only finitely many irreducible components (1.5.4), and so we only need to show that $f(X_i)$ is irreducible, for every irreducible component X_i of Y. Since $f(X_i)$ lives inside some irreducible component of Y, we can assume that both X and Y are irreducible.

We'll be done if we show that $f(X) \cap U = f(f^{-1}(U))$ is constructible, for any affine open $U \subset Y$. Since Y is quasi-compact, $f(X)$ can then be expressed as a finite union of constructible sets, and will thus be constructible. In sum, we're now in the situation where $f : X = \operatorname{Spec} S \to Y = \operatorname{Spec} R$, with S an integral domain, finitely generated over R. Now, replacing Y with $Y' = \operatorname{Spec}(R/\ker\varphi)$, where φ is the map of rings inducing f, we can assume that f is in fact dominant, and is induced by an injective map of rings $R' = R/\ker\varphi \hookrightarrow S$, with R' and S both domains, S a finitely generated R'-algebra.

To finish the proof, we use the criterion from [NS, 5.4](this is where we really need the Noetherian hypothesis): we'll show that $f(X)$ contains a non-empty open subset of Y' (note that Y' is irreducible). By [CA, 8.1.3], we can find $0 \neq a \in R'$ such that S_a is finite over $R'_a[x_1, \ldots, x_n]$, and is in particular integral over this polynomial ring. By (2.8.3), the induced map of spectra is surjective; i.e. if $Z = \operatorname{Spec} R_a[x_1, \ldots, x_n]$, then the morphism $X_a \to Y'_a$ factors through a surjective morphism $X_a \to Z$. We'll now show that the morphism $Z \to Y'_a$ is also surjective. Indeed, we find that for any prime $P \subset R_a$, $P[x_1, \ldots, x_n]$ contracts to P (alternatively, one can also observe that the polynomial ring is faithfully flat over R, and apply [CA, 3.6.9]). In sum, we find that the morphism $X_a \to Y'_a$ is the composition of two surjective morphisms, and is thus surjective. But then $Y'_a \subset f(X)$ is a non-empty open subset, and our proof is finished. □

Example 2.7.7. In general, the image of a morphism of finite type need not be open or closed. Consider, for example the morphism $f : \operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x, y]$, where k is algebraically closed, induced by the map of rings

$$\begin{aligned} k[x, y] &\to k[x, y] \\ x &\mapsto xy \\ y &\mapsto y \end{aligned}$$

Now, a maximal ideal $(x - a, y - b) \subset k[x, y]$ contracts to the maximal ideal $(x - ab, y - b) \subset k[x, y]$. Therefore, the subset of closed points of $\operatorname{Spec} k[x, y]$ intersects the image of f in the subset $\{(x - ab, y - b) : (a, b) \in k^2\}$, which contains all closed points except for the subset $\{(x - a, y) : a \neq 0\}$. This subset is the intersection of the set of closed points with $V((y))$ minus the point (x,y). So it's neither closed, nor open, which implies that the image of f is neither open, nor closed.

7.2. Morphisms of Finite Presentation.

DEFINITION 2.7.8. A morphism $f : X \to Y$ is said to be be *locally of finite presentation* if, for every

8. Integral and Finite Morphisms

8.1. Integral morphisms.

DEFINITION 2.8.1. A morphism $f : X \to Y$ of schemes is *integral* if, for every affine open $V = \operatorname{Spec} R \subset Y$, $f^{-1}(V) = \operatorname{Spec} S$ is also affine, and $f^\sharp : R \to S$ is integral.

mos-integral-aff-loc

PROPOSITION 2.8.2. *Let $f : X \to Y$ be a morphism.*

(1) *If there exists an open cover $\{U_i : i \in I\}$ such that $f^{-1}(U_i) \to U_i$ is integral, then f is integral. In other words, integrality is local on the base.*
(2) *If f is integral and $g : Y \to Z$ is another integral morphism, then $g \circ f$ is integral.*
(3) *If f is quasi-compact, X is integral, Y is reduced, and if $\{V_i : i \in I\}$ is an open cover of X such that $f|_{V_i}$ is integral, then there exists an open subscheme $U \subset Y$ such that $f^{-1}(U) \to U$ is integral.*

PROOF. Stability under composition follows from the fact that the composition of two integral maps of rings is still integral. For localness on the base, we'll use ACL as usual: let $f : X \to Y$ be a morphism of schemes, and let an affine open $V \subset Y$ have property P if $f^{-1}(V) \to V$ is an integral morphism.

(1) If $V = \operatorname{Spec} R$ has property P, with $f^{-1}(V) = \operatorname{Spec} S$ such that S is integral over R, then, for any $a \in R$, it follows that S_a is integral over R_a (integrality is preserved under localization [CA, 4.2.7]).
(2) Now, suppose we have $a_1, \dots, a_n \in R$ generating the unit ideal such that V_{a_i} has property P. By localness on the base of affineness, it follows that $f^{-1}(V) = \operatorname{Spec} S$ is affine over V. So, we have a ring S such that S_{a_i} is integral over R_{a_i}, for all i; we want to show that S is then integral over R. We can of course assume that $R \subset S$. Let S' be the integral closure of R in S; then we see that $S'_{a_i} = S_{a_i}$, for all i. From this, it follows that $S' = S$, and so S is integral over R.

It remains to prove (3), which is essentially saying that integrality is generically local on the domain for a quasi-compact morphism. For this, we can assume that $Y = \operatorname{Spec} R$ is affine, and that X has a finite open cover $\{V_i = \operatorname{Spec} S_i : 1 \le i \le n\}$, with S_i integral over R, for each i. We'll find $r \in R$ such that $(V_i)_r \subset V_n$, for all i. Given this, we'll find that $f^{-1}(Y_r) = X_r \subset V_n$, and so $f^{-1}(Y_r) = (V_n)_r$ is affine and integral over Y_r. Now, we can take $U = Y_r$ to finish the proof. We still need to find such an r. For this we do the following: since $V_i \cap V_n \neq \emptyset$, for all i, we can find $a_i \in S_i$ such that $\emptyset \neq (V_i)_{a_i} \subset V_n$. Now, since a_i is integral over R, it's the root of some monic polynomial $p_i \in R[t]$. Since S_i is a domain, the constant term b_i in p_i is non-zero. Let $r = \prod_i b_i$; since R is reduced, $R_r \neq 0$ and so $Y_r \neq \emptyset$. Moreover, a prime in S_i that contains a_i also contains r, and so $(V_i)_r \subset (V_i)_{a_i} \subset V_n$. So we're finally done. ∎

The next Proposition will be useful when we encounter the dimension theory of schemes (6.1.5). For this, recall the definitions of going up and incomparability from [NS, 6]

mos-integral-closed

PROPOSITION 2.8.3. *Let $f : X \to Y$ be an integral morphism of schemes.*

(1) *f is a closed map.*
(2) *f has the going up and incomparability properties.*

PROOF. (1) Replacing Y by the closure of $f(X)$, we may assume that f is dominant. Now, we can assume that Y, and hence X, is affine, and show that f is in fact surjective. In this case, f is induced by an integral extension $R \subset S$, where $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$. That f is surjective now follows from (4.4.5).

(2) Both these properties are local, so we can assume Y, and hence X, is affine. Now, the result follows immediately from [CA, 4.4.5] and [CA, 4.4.7].

□

8.2. Finite and Quasifinite Morphisms.

DEFINITION 2.8.4. A morphism $f : X \to Y$ is *finite* if, for every affine open $V = \operatorname{Spec} A$, $f^{-1}(V) = \operatorname{Spec} B$ is affine, with B a finitely generated module over A. Hence a morphism is finite if and only if it is integral and of finite type.

A morphism $f : X \to Y$ is *quasifinite* if it's of finite type and has finite fibers.

REMARK 2.8.5. Note that this disagrees with the definition given in [HPII, 3.5].

The next Proposition should be predictable.

mos-finite-aff-local

PROPOSITION 2.8.6. *Being a finite or quasi-finite morphism is a local on the base and affine-universal property. In particular, the classes of finite morphisms and quasifinite morphisms are stable under base change.*

PROOF. We'll prove localness on the base first.

Finite: The two conditions, one at a time.

(1) Follows from the definition.
(2) As usual, we'll use ACL. Let $f : X \to Y$ be a morphism, and let P be true of an affine open $V = \operatorname{Spec} R \subset Y$, if $f^{-1}(V) = \operatorname{Spec} S$ is affine with S a finite R-module. Then, for any $a \in R$, S_a will be a finite R_a-module. Now, suppose $\{a_1, \dots, a_n\}$ is a finite generating set for R, and $f^{-1}(V)_{a_i} = \operatorname{Spec} S_i$ is affine, with S_i a finite R_{a_i}-module. Then, since affineness is local on the base (2.4.4), we know that $f^{-1}(V) = \operatorname{Spec} S$ is affine, and so $S_i = S_{a_i}$. Let $r_{ij} \in S$ be such that S_{a_i} is generated as an R_{a_i}-module by r_{ij} for varying j. Then, for every element $s \in S$, we can find $c_{ij} \in R$ and $N \in \mathbb{N}$ such that $a_i^N s = \sum_j c_{ij} r_{ij}$. Now, we use the usual partition of unity argument to conclude that s is in the R-submodule of S generated by the r_{ij}, and so S is finitely generated over R.

Quasifinite: This is easy, since morphisms of finite type are local on the base, and having finite fibers is clearly a local condition.

Now, we'll do affine-universality.

First, we take care of finite morphisms: Let $Y' = \operatorname{Spec} R' \to Y = \operatorname{Spec} R$ be a morphism of affine schemes, and let $X \to Y$ be a finite morphism. Then, by definition, $X = \operatorname{Spec} S$ is affine, and so we're reduced to showing that if S is a finitely generated R-module, then $S \otimes_R R'$ is a finitely generated R'-module. But this is obvious.

Now, for quasifinite morphisms: Let Y' and Y be as above, and let $f : X \to Y$ be a quasifinite morphism. Then, there is a finite affine open cover $\{V_i = \operatorname{Spec} S_i\}$ of X such that S_i is a finitely generated R-algebra with only finitely many primes lying over any given prime $Q \subset R$. That is $S_i \otimes_R k(Q)$ is a finite $k(Q)$-algebra. Then, for any prime $P \subset R'$ with $P^c = Q$, we have

$$S_i \otimes_R R' \otimes_{R'} k(P) = S_i \otimes_R k(Q) \otimes_R R' \otimes_{R'} k(P).$$

We see that $S_i \otimes_R (k(Q) \otimes_R R')$ is finitely generated over $k(Q) \otimes_R R'$, and so $S_i \otimes_R R' \otimes_{R'} k(P)$ is a finite $k(P)$-algebra, and we see that $X \times_Y Y'$ is covered by finitely many affines $\operatorname{Spec} S_i \otimes_R R'$ such that each is quasifinite over Y'. This shows that $X \times_Y Y'$ is itself quasifinite over Y'. □

s-finite-local-on-domain

Proposition 2.8.7. *Suppose $f : X \to Y$ is a quasi-compact morphism, with X integral and Y reduced, and suppose there is a finite open cover $\{V_i : 1 \leq i \leq n\}$ of X such that $f|_{V_i}$ is finite. Then there exists a non-empty open subscheme $U \subset Y$ such that $f^{-1}(U) \to U$ is a finite morphism.*

Proof. Follows from part (3) of (2.8.2) and (2.7.2). □

egral-generically-finite

Corollary 2.8.8. *Let $f : X \to Y$ be a dominant morphism of finite type between integral schemes, and suppose that the fiber X_υ over the generic point υ of Y is finite. Then $K(X)/K(Y)$ is a finite field extension and there exists an open subscheme $U \subset Y$ such that $f^{-1}(U) \to U$ is finite.*

Proof. There is no harm in assuming that everything in sight is affine. In this case, the first assertion reduces to showing that if $R \subset S$ is a tower of domains with S a finitely generated R-algebra, and $S \otimes_R K(R)$ a finite $K(R)$-module, then $K(S)$ is finite over R. But we have, by [CA, 8.5.1]:

$$\operatorname{tr} \deg_{K(R)} K(S) = \dim(S \otimes_R K(R)) = 0.$$

So $K(S)$ is an algebraic extension of $K(R)$, which, since it's a finitely generated $K(R)$-algebra, shows that it's in fact finite.

For the second assertion, by the Proposition above, it suffices to find an open subscheme $V \subset Y$ and an affine open cover $\{V_i : 1 \leq i \leq n\}$ of $f^{-1}(V)$ such that $f|_{V_i}$ is finite. For this, we might as well assume that $Y = \operatorname{Spec} R$ is affine, and that X has a finite affine open cover $\{W_i : 1 \leq i \leq n\}$, where $W_i = \operatorname{Spec} S_i$, for some finitely generated R-algebra S_i. Now, by hypothesis, $\operatorname{tr} \deg_{K(R)} K(S_i) = 0$; and so, by [CA, 8.1.3], we can find $a_i \in R$ such that $(S_i)_a$ is finite over R_{a_i}. Thus, if we take $V = \bigcap_i Y_{a_i}$, and $V_i = (W_i)_{a_i}$, we see that all our prayers are answered. □

s-finite-morphism-closed

Proposition 2.8.9. *If $f : X \to Y$ is a finite morphism, then it is a closed map of topological spaces.*

Proof. Follows from (2.8.3). □

9. Separated and Quasi-separated Morphisms

Separatedness is the algebro-geometrical relativized version of Hausdorffness. Without much ado, then...

Definition 2.9.1. For a Y-scheme $f : X \to Y$, the *diagonal* $\Delta_f : X \to X \times_Y X$ is the morphism whose projections onto each copy of X are just the identity morphisms.

-diagonal-locally-closed

Proposition 2.9.2. *The diagonal Δ_f in the definition above is a locally closed immersion.*

Proof. If $\{W_j\}$ is an affine open cover for Y, then we know from the construction of the fiber product (1.7.3) that $\{f^{-1}(W_j) \times_{W_j} f^{-1}(W_j)\}$ is an open cover of $X \times_Y X$, and the pullback of such an open set in X is just $f^{-1}(W_j)$. So, since local closedness is local on the base, we can assume that $Y = \operatorname{Spec} R$ is affine. Let $\{V_j = \operatorname{Spec} S_j\}$ be an affine open cover for X; then $X \times_Y X = \bigcup_{i,j} V_i \times_Y V_j$. Now, the map $X \to X \times_Y X$ factors through the open subscheme $\bigcup_i V_i \times_Y V_i$. So it suffices to show that the morphism

$$X \to \bigcup_i V_i \times_Y V_i$$

is a closed immersion. Again, since closed immersions are local on the base, it's enough to show that

$$V_i \to V_i \times_Y V_i$$

is a closed immersion. This reduces, by (1.1.8) to showing that the ring map

$$S_i \otimes_R S_i \to S_i$$

is surjective. But this is clear! □

The last part of the proof also shows the following.

s-affine-diagonal-closed

Corollary 2.9.3. *If in the above Proposition, if f is an affine morphism, then the diagonal is a closed immersion.*

Proof. Indeed, in this case, $f^{-1}(W_i)$ is itself affine, for every affine open $W_i \subset Y$. So, by the proof above, the morphism

$$f^{-1}(W_i) \to f^{-1}(W_i) \times_{W_i} f^{-1}(W_i)$$

is a closed immersion for every i, which, since closed immersions are local on the base, shows that the diagonal is itself closed. □

Example 2.9.4. The diagonal need not in general be closed. Consider the affine line X over k with a doubled origin. More precisely, take two copies of $\mathbb{A}^1_k$, say X_1 and X_2, and glue them along the open set $\mathbb{A}^1_k \setminus (x)$. Then, we see that $X \times_k X$ is the affine plane over k but with four origins, one for each ordered pair (i, j), with $i, j = 1, 2$. What is the diagonal? X_1 maps to the usual diagonal in $X_1 \times_k X_1$, and X_2 maps to its usual diagonal, and the two diagonals coincide except for the two distinct origins corresponding to the pairs $(1, 1)$ and $(2, 2)$. In other words, the diagonal in $X \times_k X$ is just the usual diagonal on the plane, but with a doubled origin. Now, the two remaining origins are still in the closure of this diagonal, and so the diagonal can't be closed, but it is *locally* closed, since it's closed inside the open subscheme $\bigcup_{i=1,2} X_i \times_k X_i$.

The above example is symptomatic of what can go wrong: the points fail to be far enough apart. Recall now that a topological space X is Hausdorff if and only if the diagonal in $X \times X$ is closed. The Zariski topology is not fine enough to naïvely transpose the usual Hausdorff condition onto our geometric situation: we cannot hope to end up with anything useful. But we do have the notions of products and diagonals, and the equivalent property that we gave above is the one that will be most appropriate here. So without further ado...

DEFINITION 2.9.5. A Y-scheme $f : X \to Y$ is *separated* if the diagonal $\Delta_f : X \to X \times_Y X$ is a closed immersion. We may also call f a *separated morphism* in this case.

A scheme X is *separated* if it is separated as a $\mathbb{Z}$-scheme.

Corollary (2.9.3) above immediately gives us the following result.

mos-affine-separated

PROPOSITION 2.9.6. *An affine morphism is separated.*

PROOF. Clear from the definition and the Corollary. ☐

Here's a nice characterization of separatedness

arated-affine-int-affine

PROPOSITION 2.9.7. *Let $f : X \to \operatorname{Spec} R$ be an R-scheme. Then, f is separated if and only if for some affine cover $\{U_i = \operatorname{Spec} R_i\}$ of X and any pair of indices (i, j), $U_i \cap U_j$ is also affine, with its ring of global sections generated by the restrictions of $\Gamma(U_i, \mathscr{O}_X)$ and $\Gamma(U_j, \mathscr{O}_X)$.*

PROOF. From our Very Useful Fiber Diagram (7.4) (henceforth referred to as VUFD), we have a fiber diagram

$$\begin{array}{ccc} U_i \times_X U_j & \xrightarrow{\phi} & U_i \times_{\operatorname{Spec} R} U_j \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_f} & X \times_{\operatorname{Spec} R} X \end{array}$$

Now, $U_i \times_X U_j$ is canonically isomorphic to $U_i \cap U_j$, and the arrow on the bottom is just the diagonal. Now, suppose f is separated; then Δ_f is a closed immersion, and hence ϕ is a closed immersion. But $U_i \times_{\operatorname{Spec} R} U_j = \operatorname{Spec} R_i \otimes_R R_j$ is affine, and so by (2.1.3), we see that $U_i \cap U_j$ is also affine with its ring of global sections generated by the image of $R_i \otimes_R R_j$, which gives us one implication. For the other, suppose $U_i \cap U_j$ is affine, and $\Gamma(U_i \cap U_j, \mathscr{O}_X)$ is generated by the restrictions of $\Gamma(U_i, \mathscr{O}_X)$ and $\Gamma(U_j, \mathscr{O}_X)$, for all pairs i, j. Then ϕ is induced by a surjection, and is thus a closed immersion for all i, j. Since $\{U_i \times_{\operatorname{Spec} R} U_j\}$ is an open cover for $X \times_{\operatorname{Spec} R} X$, and closed immersions are local on the base, we see that Δ_f must also be a closed immersion. ☐

The above property can be used, with a slight modification, to characterize another useful class of morphisms.

DEFINITION 2.9.8. A Y-scheme $f : X \to Y$ is *quasi-separated* if the diagonal Δ_f is quasi-compact. As always, we may also say that f is a quasi-separated morphism.

uasisep-characterization

PROPOSITION 2.9.9. *A morphism $f : X \to Y$ is quasi-separated if and only if for every affine open $W \subset Y$, and every pair of affine opens $U, V \subset f^{-1}(W)$, $U \cap V$ can be covered with finitely many affine opens.*

Proof. Let U, V, W be as in the statement. As before, we have the fiber diagram:

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \times_Y V \\ \Big\downarrow & & \Big\downarrow \\ X & \xrightarrow{\Delta_f} & X \times_Y X \end{array}$$

If Δ_f is quasi-compact, then $U \cap V$ is quasi-compact, and so can be covered with finitely many affine opens. Conversely, if $U \cap V$ can be covered with finitely many affine opens for every such pair $U, V \subset f^{-1}(W)$, then we see that we can cover $X \times_Y X$ with affine opens whose pullbacks in X can be covered with finitely many affine opens and are thus quasi-compact. Since quasi-compactness is local on the base, we're done. □

incipal-open-quasisep-qc

Corollary 2.9.10. *Let $f : X \to \operatorname{Spec} R$ be a quasi-separated, quasi-compact morphism. Let $A = \Gamma(X, \mathscr{O}_X)$; then for every $a \in A$, we have*

$$A_a \cong \Gamma(X_a, \mathscr{O}_X)$$

Proof. Just observe that the hypotheses of part (3) in (1.4.2) are satisfied in this case. □

Example 2.9.11. Take two copies X_1 and X_2 of $Y = \operatorname{Spec} k[x_1, x_2, \ldots]$, and glue them along the open set $Y \setminus \{(x_1, x_2, \ldots)\}$ to get infinite dimensional affine k-space X with a doubled origin. The intersection of the two affine schemes X_1 and X_2 inside X is just $\mathbb{A}_k^\infty$ with the origin removed. This is the complement of the closed set $V((x_1, x_2, \ldots))$, and is certainly not quasi-compact. So we see that X is not quasi-separated. Some sort of non-Noetherian pathology was essential in this example, since we get quasi-compactness for free in the Noetherian situation.

mos-separated-aff-loc

Proposition 2.9.12. *Separatedness and quasiseparatedness are local on the base and stable under base change. The composition of two (quasi)separated morphisms is again (quasi)separated.*

Proof. For a change, we'll prove stability under base change first. Let $g : Y' \to Y$ and $f : X \to Y$ be Y-schemes. If $W = X \times_Y Y'$, then we have the following fiber diagram.

$$\begin{array}{ccc} W & \longrightarrow & W \times_{Y'} W \\ \Big\downarrow & & \Big\downarrow \\ X & \longrightarrow & X \times_Y X \end{array}$$

We actually get this from VUFD (7.4) by observing that

$$(X \times_Y Y') \times_{Y'} (X \times_Y Y') \cong X \times_Y (X \times_Y Y') \text{ and}$$
$$(X \times_Y Y') \times_X X \cong X \times_Y Y'$$

Given this diagram, we see that if the diagonal on the bottom is closed (resp. quasi-compact) then so is the diagonal on the top. This shows exactly that separatedness (resp. quasiseparatedness) is stable under base change. Observe that we used the fact that closed immersions (2.1.12) and quasi-compact morphisms (2.4.4) are stable under base change.

Now we do localness on the base.

(1) Suppose $f : X \to Y$ is a Y-scheme, and $U \subset Y$ is open. Then we have the following fiber diagram

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & f^{-1}(U) \times_U f^{-1}(U) \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_Y X \end{array}$$

This shows that the diagonal on the top is a closed immersion (resp. quasi-compact) whenever the diagonal on the bottom is.

(2) Conversely, if we have an open cover $\{U_i\}$ of Y such that the diagonal $f^{-1}(U_i) \to f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$ is a closed immersion (resp. quasi-compact), then, since closed immersions (resp. quasi-compact morphisms) are local on the base, we find that the diagonal $X \to X \times_Y X$ is also a closed immersion (resp. quasi-compact).

For the last statement, suppose we have two morphisms $f : X \to Y$ and $g : Y \to Z$. We have the following diagram from VUFD (7.4).

$$\begin{array}{ccccc} X & \longrightarrow & X \times_Y X & \longrightarrow & X \times_Z X \\ & & \downarrow & & \downarrow \\ & & Y & \longrightarrow & Y \times_Z Y \end{array}$$

Now, since both closed immersions and quasi-compact morphisms are stable under base changes and compositions, the composition $X \to X \times_Z X$ on the top row is a closed immersion (resp. quasi-compact) whenever the diagonal on the bottom row is a closed immersion (resp. quasi-compact). □

mos-separated-criteria

COROLLARY 2.9.13. *Let X be a scheme. The following are equivalent:*

(1) *X is separated.*
(2) *There is a separated morphism $f : X \to \operatorname{Spec} R$, for some ring R.*
(3) *Every morphism of schemes $f : X \to Y$ with domain X is separated.*

PROOF. We'll use the criterion from (2.9.7). According to this a morphism $f : X \to \operatorname{Spec} R$ is separated if and only if there is an open affine cover $\{U_i\}$ of X such that $U_i \cap U_j$ is affine, with the natural map

$$\Gamma(U_i, \mathscr{O}_X) \otimes_{\mathbb{Z}} \Gamma(U_j, \mathscr{O}_X) \to \Gamma(U_i \cap U_j, \mathscr{O}_X)$$

a surjection, for all pairs i, j. But this criterion is completely independent of R! So we immediately get the equivalence (1) ⇔ (2). It is clear that (3) ⇒ (1), so we'll finish the proof by showing (2) ⇒ (3): but this follows immediately from the fact that separatedness is local on the base, and the fact that any open subscheme of a separated scheme is also separated. □

The next Proposition is very useful.

mos-property-P

Proposition 2.9.14. *Let Ξ be a property of morphisms of schemes that's stable under base changes and compositions.*

(1) *Suppose we have a diagram such as this:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_{h}\searrow \quad \swarrow_{g} & \\ & Z & \end{array}$$

If the diagonal $\Delta_g : Y \to Y \times_Z Y$ and h have property Ξ, then so does f. In particular, if closed immersions have property Ξ and g is separated, then h has property Ξ if and only if f also does.

(2) *If $X \to X'$ and $Y \to Y'$ are morphisms of Z-schemes with property Ξ, then $X \times_Z Y \to X' \times_Z Y'$ also has property Ξ. Equivalently, Ξ is stable under products.*

(3) *If every closed immersion has property Ξ, and $X \to Y$ has property Ξ, then so does $X_{red} \to Y_{red}$.*

Proof. (1) If we prove the first part of (1), the second will follow, since Δ_g will be a closed immersion and will thus have property Ξ. Consider the following fiber diagram.

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & X \times_Z Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1_Y} \\ Y & \xrightarrow[\Delta_g]{} & Y \times_Z Y \end{array}$$

We obtain this from VUFD (7.4) via the isomorphism

$$X \times_Y Y \cong Y.$$

Since Ξ is stable under base change, we see that Γ has property Ξ. Observe now that if $p_2 : X \times_Z Y \to Y$ is the canonical projection, then $p_2 \circ \Gamma = f$. So to show that f has property Ξ, it's enough to show that p_2 does. But we get that from the following base change diagram:

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Z \end{array}$$

using the fact that h has property Ξ.

(2) Since Ξ is stable under composition, it suffices to show that $X \times_Z Y \to X \times_Z Y'$ has property Ξ. But observe that

$$X \times_Z Y \cong (X \times_Z Y') \times_{Y'} Y,$$

and so we have the base change diagram

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X \times_Z Y' & \longrightarrow & Y' \end{array}$$

Since Ξ is stable under base change, we see that the vertical map on the left also has property Ξ.

(3) We have the following diagram

$$\begin{array}{ccccc} X_{\mathrm{red}} & \longrightarrow & X \times_Y Y_{\mathrm{red}} & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & Y_{\mathrm{red}} & \longrightarrow & Y \end{array}$$

Now, since the morphism $X_{\mathrm{red}} \to X$ is a closed immersion, it has property Ξ. The morphism $X \times_Y Y_{\mathrm{red}} \to X$ is the base change of $Y_{\mathrm{red}} \to Y$ and is thus also a closed immersion; hence, it's separated. By part (1), we see that $X_{\mathrm{red}} \to X \times_Y Y_{\mathrm{red}}$ must also have property Ξ. Since $X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is the composition

$$X_{\mathrm{red}} \to X \times_Y Y_{\mathrm{red}} \to Y_{\mathrm{red}},$$

it must also have property Ξ (the second map in the composition has property Ξ, since it's the base change of a morphism with property Ξ). □

10. The Graph of a Morphism and the Locus of Agreement

DEFINITION 2.10.1. The *graph* of a morphism $f : X \to Y$ of Z-schemes is the morphism $\Gamma_f : X \to X \times_Z Y$ whose projections onto X and Y are the identity and f respectively.

Observe that, if $p_1 : X \times_Z Y \to X$ is the projection onto X, then by definition $p_1 \circ \Gamma_f = 1_X$. In other words, Γ_f is a section for p_1 over X. Hence, the nature of the graph of a morphism is described by the next Proposition.

s-section-locally-closed

PROPOSITION 2.10.2. *For any morphism $p : X \to Y$, a section $s : Y \to X$ of p is a locally closed immersion. If p is separated, then s is in fact a closed immersion.*

PROOF. We have the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ s\ } & X \\ & \searrow^{1_Y} & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

Now, 1_Y is an immersion, and Γ_p is also an immersion. So we can apply part (1) of (2.9.14) to conclude that s is also an immersion. If p is separated, then 1_Y and Γ_p are both closed immersions, and the assertion follows. □

mos-graph-separated

COROLLARY 2.10.3. *For any morphism $f : X \to Y$ of Z-schemes, the graph $\Gamma_f : X \to X \times_Z Y$ is a locally closed immersion. If Y is a separated Z-scheme, then Γ_f is in fact a closed immersion.*

PROOF. Note that Γ_f is a section of the natural projection $p_1 : X \times_Z Y \to X$. Since p_1 is the base change of $Y \to Z$, it's separated whenever Y is separated over Z. Now apply the Proposition. □

DEFINITION 2.10.4. Given two morphisms of Z-schemes $f, g : X \to Y$, the *locus of agreement* of f and g is a locally closed subscheme $V \to X$ that satisfies the following universal property: Every morphism of Z-schemes $h : W \to X$ with $f \circ h = g \circ h$ factors uniquely through V.

mos-locus-of-agreement

PROPOSITION 2.10.5. *The locus of agreement $h : V \to X$ of any two morphisms between the Z-schemes X and Y exists, and is a locally closed immersion. If $Y \to Z$ is separated, then the locus of agreement is in fact a closed immersion. If X is reduced, then V has the reduced induced subscheme structure.*

PROOF. We define $V \to X$ to be the pullback in the following diagram

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{(f,g)} & Y \times_Z Y \end{array}$$

Since it's the base change of the locally closed diagonal morphism, it is itself locally closed. It's also easy to check that the universal property that it satisfies by virtue of being the pullback is precisely the property we need for it to be the locus of agreement. When Y is separated over Z, the diagonal is a closed immersion, and so $V \to X$ is also a closed immersion.

If X is reduced, then every morphism $Z \to X$ will factor through $Z_{\text{red}} \to X$, and so will factor also through V_{red}. This implies that $V = V_{\text{red}}$, and so V is reduced. □

s-agree-dense-everywhere

COROLLARY 2.10.6. *Suppose $f, g : X \to Y$ are morphisms of Z-schemes that agree on a dense open subset of X. If X is reduced and Y is separated over Z, then $f = g$; i.e. the closed subscheme $V \to X$ is in fact an isomorphic to the identity morphism on X.*

PROOF. By the Proposition, we see that $V \to X$ is a closed subscheme with the reduced induced subscheme structure. But now, if $f = g$ on a dense open subset $U \subset X$, the inclusion $U \hookrightarrow X$ must factor through V by its universal property. Hence, the closed set underlying V is all of X. But since V has the reduced induced subscheme structure and X is reduced, we see that $V \to X$ is in fact an isomorphism, and hence $f = g$ on all of X. □

EXAMPLE 2.10.7. This is not true if either X is not reduced or Y is not separated. For the latter, just take the affine line with the doubled origin, and consider two different maps to itself: the identity and the map that switches the two origins. For the former, consider two morphisms $\operatorname{Spec} k[x]/(x^2) \to \operatorname{Spec} k[x]/(x^2)$: the identity and the map induced by the zero map. It's easy to see the locus of agreement is the closed subscheme $\operatorname{Spec} k[x]/(x) \to \operatorname{Spec} k[x]/(x^2)$, which is not an isomorphism.

The basic issue with nonreduced schemes is that two morphisms can agree as maps on the underlying topological space, but be completely different as morphisms of schemes, which is exactly what is happening in the situation here. Both x and 0 are 'zero' functions on $\operatorname{Spec} k[x]/(x^2)$, but x carries more information than 0 in the scheme theoretic sense.

11. Universally Closed and Proper Morphisms

Recall that a morphism is universally Ξ for some property Ξ if all its base changes also have property Ξ.

mos-univ-closed-aff-loc

Proposition 2.11.1. *The following statements are true*

1. *Every closed immersion is universally closed.*
2. *Universally closed morphisms are stable under base change.*
3. *Universally closed morphisms are stable under composition.*
4. *Universally closed morphisms are local on the base.*
5. *Let $f : X \to Y$ be a surjective morphism of S-schemes, and suppose $g : X \to S$ is universally closed. Then $h : Y \to S$ is also universally closed.*

Proof.

(1) Closed immersions are stable under base change, and are closed morphisms. The result is immediate.

(2) Suppose $f : X \to Y$ is universally closed and $Y' \to Y$ is any Y-scheme. Then, we see that for any Y'-scheme Z, we have
$$(X \times_Y Y') \times_{Y'} Z \cong X \times_Y Z,$$
and so $X \times_Y Y' \to Y'$ is also universally closed.

(3) Suppose $f : X \to Y$ and $g : Y \to Z$ are universally closed. Let $W \to Z$ be any Z-scheme. Then we see that
$$W \times_Z X \cong (W \times_Z Y) \times_Y X.$$
So the morphism $W \times_Z X \to W$ factors as
$$(W \times_Z Y) \times_Y X \to W \times_Z Y \to W,$$
and the conclusion follows from the stability under composition of closed morphisms.

(4) Suppose $f : X \to Y$ is a morphism, and suppose $\{V_i\}$ is an open cover for Y such that $f^{-1}(V_i) \to V_i$ is universally closed for every i. Then, for any Y-scheme $g : Z \to Y$, the morphisms $g^{-1}(V_i) \times_{V_i} f^{-1}(V_i) \to g^{-1}(V_i)$ are closed. The result now follows from the fact that closed morphisms are local on the base, which is a purely topological statement.

(5) Surjective morphisms are stable under base change, and so we find that, for all S-schemes $T \to S$, the morphism of T-schemes
$$X \times_S T \to Y \times_S T$$
is surjective. Moreover $X \times_S T \to T$ is universally closed, by part (2). Hence, it suffices to show that $h : Y \to S$ is closed. But if $Z \subset Y$ is a closed subset, then $Z = f(f^{-1}(Z))$, and so
$$h(Z) = h(f(f^{-1}(Z))) = g(f^{-1}(Z))$$
is closed.

□

We will define now what are probably the most important class of morphisms in the elementary study of algebraic geometry.

DEFINITION 2.11.2. A morphism $f : X \to Y$ is *proper* if it is separated, of finite type and universally closed.

The next Proposition is immediate from (2.9.12), (**??**) and (2.11.1).

mos-proper-aff-loc

PROPOSITION 2.11.3. *The following statements are true*

(1) *Proper morphisms are stable under base change.*
(2) *Proper morphisms are stable under composition.*
(3) *Proper morphisms are local on the base.*
(4) *Suppose $f : X \to Y$ is a surjective morphism of S-schemes, where $X \to S$ is universally closed. If $Y \to S$ is separated and of finite type, then $Y \to S$ is proper.*

12. Summary of the Chapter

We've studied all kinds of morphisms in this section. In the next Proposition, we record the relationships between the different classes or morphisms.

relations-prps-morphisms

PROPOSITION 2.12.1. *The following statements hold for morphisms of schemes.*

Closed immersions are: *Monomorphisms, Finite and Proper.*
Open immersions are: *Monomorphisms, Open, Locally of finite type and Separated.*
Locally closed immersions are: *Monomorphisms, Locally of finite type and Separated.*
Affine morphisms are: *Quasi-compact and Separated.*
Integral morphisms are: *Affine.*
Morphisms of finite type are: *Locally of finite type and Quasi-compact.*
Finite Morphisms are: *Closed, Affine, Quasifinite, Integral, Of Finite Type and Separated.*
Quasifinite Morphisms are: *Of Finite Type.*
Separated Morphisms are: *Quasi-separated.*
Proper Morphisms are: *Universally closed, of Finite Type and Separated.*

PROOF. Most of these have been proved before. So we'll just give references.

(1) That closed immersions are finite basically follows from the fact that R/I is a finite R-module for any ring R and any ideal $I \subset R$. That they're monomorphisms follows immediately from their definition. Now, since they are affine, they are automatically separated, and since they're stable under base change, they're universally closed, and hence proper.
(2) That open immersions are locally of finite type follows from the fact that any localization R_a for a ring R and an element $a \in R$ is a finitely generated R-algebra. To see that an open immersion is separated, just note that for any open subscheme $U \subset X$, the morphism $U \to U \times_X U$ is a homeomorphism of the underlying topological spaces.
(3) Follows from (1) and (2).
(4) Quasi-compactness follows from its definition, and separatedness follows from (2.9.6).
(5) Follows from definition.

(6) See (2.7.4).
(7) That finite morphisms are affine (and thus separated) follows from their definition. They're clearly of finite type; for closedness and quasifiniteness, see [HPII, 3.5].
(8) From the definition.
(9) Ditto.
(10) Like I said.

□

Every property of morphisms we've encountered has been local on the base and stable under base change. Let's record that here.

orph-aff-loc-base-change

PROPOSITION 2.12.2. *The following classes of morphisms of schemes are local on the base and stable under base change.*

(1) *Surjections and monomorphisms.*
(2) *Closed, open and locally closed immersions.*
(3) *Affine morphisms.*
(4) *Quasi-compact morphisms.*
(5) *Morphisms of locally finite type and morphisms of finite type.*
(6) *Finite and quasifinite morphisms.*
(7) *Separated and Quasi-separated morphisms.*
(8) *Universally closed morphisms.*
(9) *Proper morphisms.*

mos-morph-closed-comp

PROPOSITION 2.12.3. *The following classes of morphisms are stable under composition.*

(1) *Surjections and monomorphisms.*
(2) *Closed and open morphisms.*
(3) *Closed, open and locally closed immersions.*
(4) *Affine and quasi-compact morphisms.*
(5) *Morphisms of locally finite type and morphisms of finite type.*
(6) *Finite and quasifinite morphisms.*
(7) *Separated and quasi-separated morphisms.*
(8) *Universally closed morphisms.*
(9) *Proper morphisms.*

PROOF. Most of this has been done before.

(1) Immediate.
(2) Trivial.
(3) See above.
(4) This is straightforward from the definition.
(5) See [HPII, 3.11].
(6) Easy. The quasifinite case follows partly from (3) right above.
(7) See (2.9.12).
(8) See (2.11.1).
(9) See (2.11.3).

□

CHAPTER 3

The Proj Construction

chap:proj

1. Proj of a Graded Ring

In these notes, we fix a graded ring $S = \bigoplus_{n\in\mathbb{Z}} S_n$ with the irrelevant ideal $S^+ = \bigoplus_{n\neq 0} S_n$.

DEFINITION 3.1.1. We set

$$\operatorname{spc}(\operatorname{Proj} S) = \{P \subset R : P \text{ a homogeneous prime}, P \not\supseteq S^+\} \subset \operatorname{spc}(\operatorname{Spec} S).$$

We will use the letter X to denote $\operatorname{spc}(\operatorname{Proj} S)$ for the remainder of this section.

For a homogeneous ideal $I \subset S$, we set $V_+(I) = \{P \in X : P \supset I\} \subset X$, and for a homogeneous element $f \in S$, we set $X_{(f)} = \{P \in X : f \notin P\} \subset X$.

proj-vplus-prps

PROPOSITION 3.1.2. *The assignment $I \mapsto V_+(I)$ satisfies the following properties:*

1. $V_+(IJ) = V_+(I \cap J) = V_+(I) \cup V_+(J)$, *for two homogeneous ideals* I, J.
2. $V_+(\sum_k I_k) = \cap_k V_+(I_k)$, *for an arbitrary collection of homogeneous ideals* $\{I_k\}$. *Thus we can equip X with a topology where $\{V_+(I) : I \subset R \text{ homogeneous}\}$ is the collection of closed sets.*
3. $V_+(I) \subset V_+(J)$ *if and only if* $J \cap S^+ \subset \operatorname{rad}(I)$.
4. $V_+(I) = \emptyset$, *if and only if* $\operatorname{rad} I \supset S^+$. *In particular,* $X = \emptyset$ *if and only if* $\operatorname{Nil} S \supset S^+$.
5. $V_+(I) = X$ *if and only if* $I \cap S^+ \subset \operatorname{Nil} S$.
6. *The topology on X is the topology induced from* $Y = \operatorname{spc}(\operatorname{Spec} S)$.
7. $V_+(I)$ *is homeomorphic to* $\operatorname{spc}(\operatorname{Proj} S/I)$.

PROOF. (1) Clear.

(2) Immediate. Observe that $V_+((0)) = X$ and $V_+(S^+) = \phi$.

(3) $V_+(I) = V_+(\operatorname{rad}(I))$.

(4) Follows from the fact that $\operatorname{rad} I$ is homogeneous and is hence the intersection of homogeneous primes containing I.

(5) First suppose $J \cap S^+ \subset \operatorname{rad} I$; then every prime $P \in V_+(I) = V_+(\operatorname{rad} I)$ also contains $J \cap S^+$ and thus JS^+. Since P doesn't contain S^+, it must contain J. Conversely, suppose $V_+(I) = V_+(\operatorname{rad} I) \subset V_+(J)$. Let $P \in V_+(I)$ be a prime containing I (not necessarily homogeneous); then P^* is a homogeneous prime containing I. If $P^* \in X$, then $P^* \in V_+(I)$, and so $P^* \in V_+(J)$, which means that $J \subset P$. If $P^* \supset S^+$, then $J \cap S^+ \subset P$. So we see that

$$J \cap S^+ \subset \bigcap_{P \in V_+(I)} P = \operatorname{rad} I$$

(6) From (3): take $J = S^+$. For the second statement, take $I = 0$.

(7) Ditto: take $J = (0)$.
(8) Follows from the fact that $V_+(I) = X \cap V(I)$.
(9) The homeomorphism from $V(I)$ to $\operatorname{spc}(\operatorname{Spec} S/I)$ restricts to our required homeomorphism from $V_+(I)$ to $\operatorname{spc}(\operatorname{Proj} S/I)$.

□

DEFINITION 3.1.3. In the topology on X given to us by the Proposition, the sets of the form $X_{(f)}$ are open. We'll call them *principal open sets* of X.

The next Proposition does most of the work in the construction of the natural scheme structure on X.

-loc-homogeneous-element

PROPOSITION 3.1.4. *Let $f \in S$ be a homogeneous element of non-zero degree s, and let M be a graded S-module. Then*

(1) *There is a one-to-one correspondence between homogeneous primes in S not containing f, and the primes in $S_{(f)}$.*
(2) *For any homogeneous ideal $I \subset S$, we have*
$$(S/I)_{(f)} \cong S_{(f)}/I_{(f)}.$$
(3) *If $g \in S$ is another homogeneous element, then the correspondence in* (2) *induces a one-to-one correspondence between primes in S not containing fg and primes in $S_{(f)}$ not containing $u = g^s f^{-\deg g}$.*
(4) *If $g \in S$ is another homogeneous element such that $g^k = af$, for some homogeneous $a \in S$, then the natural map $S_f \to S_g$ induces a canonical isomorphism ${M_{(f)}}_u \to M_{(g)}$, where $u = g^s f^{-\deg g}$.*
(5) *If $P \subset S$ is a homogeneous prime not containing f, then we have a natural isomorphism*
$${M_{(f)}}_{PS_f \cap S_{(f)}} \cong M_{(P)_0}.$$

PROOF. (1) First, let $Q \subset S_{(f)}$ be a prime. Consider the ideal $\operatorname{rad} QS_f \subset S_f$. We claim that this is a homogeneous prime. That it's homogeneous follows from [CA, 1.1.5]. To check primeness, suppose we have two homogeneous elements $a, b \in S_f$ such that $ab \in \operatorname{rad} QS_f$. There is some $k \in \mathbb{N}$ such that $a^{ks}b^{ks} \in QS_f$. Let $\deg a = r$ and $\deg b = t$; then $f^{-kr}a^{ks}f^{-kt}b^{ks} \in Q$, and since Q is prime, we can assume without loss of generality that $f^{-kr}a^{ks} \in Q$, and so $a^{ks} \in QS_f$. This gives us $a \in \operatorname{rad} QS_f$, and so $\operatorname{rad} QS_f$ is indeed prime. If $P = \phi^{-1}(\operatorname{rad} QS_f)$, then it's a homogeneous prime in S. It's clear that P doesn't contain f. Moreover, by the properties of localization, $PS_f = \operatorname{rad} QS_f$, and so $PS_f \cap Q = \operatorname{rad}(QS_f) \cap Q = \operatorname{rad} Q = Q$. Conversely, suppose $P' \subset S$ is another prime not containing f such that $P'S_f \cap Q = Q$. Then for every $a \in P'S_f$ with $\deg a = r$, we see that $f^{-r}a^s \in P'S_f \cap Q \subset PS_f$, and so $a \in PS_f$. This shows $P'S_f \subset PS_f$, and by symmetry we have $PS_f = P'S_f$. Thus the correspondence $Q \mapsto \phi^{-1}(\operatorname{rad} QS_f)$ gives us the one-to-one mapping that we claimed.
(2) Follows immediately from the fact that $(S/I)_f \cong S_f/I_f$, and the definition of the grading on the quotient ring.
(3) Suppose $fg \in P \subset S$ and $f \notin P$. Then $g \in P$, and so $u \in PS_f \cap S_{(f)}$. Conversely, if $fg \notin P$, then $f, g \notin P$, but if $u \in PS_f$, then $g^s \in P$, implying $g \in P$, which is a contradiction.

(4) The canonical homomorphism from $M_f \to M_g$ is given by $\frac{x}{f^m} \mapsto \frac{a^m x}{g^{mk}}$, for $x \in M$. See (1.1.4) to see that this is well-defined. This is in fact a homomorphism of degree 0 of graded modules, since $\deg x + m(\deg a - k \deg g) = \deg x - m \deg f$. So it induces a homomorphism $M_{(f)} \to M_{(g)}$ on the zeroth degree subrings. Observe that $u \in S_{(f)}$ acts invertibly on $M_{(g)}$; hence the homomorphism above factors uniquely through a homomorphism ${M_{(f)}}_u \to M_{(g)}$. We'll show injectivity of this map first. Suppose $\frac{x}{f^n} \in M_{(f)}$ goes to zero under this map. Then, there is $l \in \mathbb{N}$ such that $g^l s a^n x = 0 \in M$. Multiplying by suitable powers of a and f, we find that $g^{(l+n)s} x = 0 \in M$; but then $u^{(l+n)s} x = 0 \in M_{(f)}$, and so $\frac{x}{f^n} = 0 \in {M_{(f)}}_u$. For surjectivity, we'll write down this map explicitly.
$$\frac{x}{f^n u^m} \mapsto \frac{a^n f^{mt} x}{g^{nk+ms}} = \frac{f^{tm-n} x}{g^{sm}},$$
where $t = \deg g$. Now, every element in $M_{(g)}$ is of the form $\frac{y}{g^l}$, where $\deg y = tl$. Let $m \geq 0$ be large enough so that $sm \geq l$. Consider $x = \frac{g^{sm-l} y}{f^{tm-n}} \in M_{(f)}$: $\deg x = ns$, and so $\frac{x}{f^n u^m} \in {M_{(f)}}_u$. We see immediately that $\frac{x}{f^n u^m}$ goes to $\frac{y}{g^l}$ under the map above. This shows surjectivity, and finishes our proof.

(5) Observe that $M_{(P)}$ is a localization of M_f, and the natural homomorphism $M_f \to M_{(P)}$ induces a homomorphism $M_{(f)} \to {M_{(P)}}_0$. Since every element in $PS_f \cap S_{(f)}$ acts invertibly on ${M_{(P)}}_0$, this map factors through a homomorphism
$$(M_{(f)})_{PS_f \cap S_{(f)}} \to {M_{(P)}}_0$$
$$\frac{m/f^t}{a/f^r} \mapsto \frac{f^r m}{a f^t},$$
where $m \in M$, $a \notin P$ are homogeneous, and $\deg m = ts$, $\deg a = rs$. This map is surjective: suppose $\frac{n}{g^k} \in {M_{(P)}}_0$, where $g \notin P$ and $k \deg g = \deg n$. Then, consider
$$x = \frac{g^{k(s-1)n} / f^{k \deg g}}{g^{ks} / f^{k \deg g}};$$
we see that $x \mapsto \frac{n}{g^k}$ under the homomorphism above. It remains to see that this map is injective. So suppose $\frac{f^r m}{a f^t} = 0$; then there is a homogeneous element $b \notin P$ such that $b f^r m = 0$. This implies that $bm = 0$ in M_f, which means that $\frac{b f^t}{f^{t+(\deg b/s)}} \frac{m}{f^t} = 0$, and so $\frac{m}{f^t} = 0$ to begin with.

□

proj-principal-open-prps

Proposition 3.1.5. *We can say the following things about principal open sets:*

(1) $X_{(f)} = X$, *if and only if* $\mathrm{rad}(f) \supset S^+$.
(2) $X_{(f)} = \emptyset$ *if and only if* $f \in \mathrm{Nil}\, S$.
(3) $X_{(fg)} = X_{(f)} \cap X_{(g)}$.
(4) *Given any open set* $U \subset X$, *and a prime* $P \in U$, *there is a homogeneous* $f \in S$ *such that* $P \in X_{(f)} \subset U$.
(5) *The principal open sets form an open base for the topology on* X.
(6) $X_{(f)} \subset X_{(g)}$ *if and only if* $f^k = ag$, *for some homogeneous* $a \in S$, $k \in \mathbb{N}$.
(7) $X_{(f)}$ *is homeomorphic to* $\mathrm{spc}(\mathrm{Spec}\, S_{(f)})$.

(8) $\{X_{(f_i)} : i \in I\}$ *is an open cover for* X *if and only if* $S^+ \subset \mathrm{rad}((f_i : i \in I))$.

PROOF. (1) Follows from part (5) of the (3.1.2)
(2) Follows from part (6) of the same Proposition.
(3) A prime P doesn't contain fg if and only if it doesn't contain both f and g.
(4) If $U = X \setminus V_+(I)$, then take f to be any homogeneous element of I.
(5) Clear.
(6) We see that $V_+((g)) \subset V_+((f))$ if and only if $(f) = (f) \cap S^+ \subset \mathrm{rad}(g)$ if and only if $f^k = ag$, for some $a \in S$, $k \in \mathbb{N}$. Since f^k and g are homogeneous, it follows that we can discard all but the homogeneous component of a with degree $ks - \deg g$.
(7) Observe that $X_{(f)} = Y_f \cap X$, where $Y = \mathrm{spc}(\mathrm{Spec}\, S)$. So the restriction of the homeomorphism from Y_f to $\mathrm{spc}(\mathrm{Spec}\, S_f)$ is a homeomorphism from $X_{(f)}$ to its image in $\mathrm{spc}(\mathrm{Spec}\, S_f)$. If we now compose this with the contraction map from $\mathrm{spc}(\mathrm{Spec}\, S_f)$ to $\mathrm{spc}(\mathrm{Spec}\, S_{(f)})$, we see that from (3.1.4) that we have a continuous bijection from $X_{(f)}$ to $\mathrm{spc}(\mathrm{Spec}\, S_{(f)})$. Again from (3.1.4) $X_{(fg)}$ goes to a principal open set in $\mathrm{spc}(\mathrm{Spec}\, S_{(f)})$, and so the map is in an open map and thus a homeomorphism.
(8) This can be an open cover if and only if $V_+((f_i : i \in I)) = \emptyset$, which can happen if and only if $S^+ \subset \mathrm{rad}((f_i : i \in I))$.

□

Finally we're ready to define $\mathrm{Proj}\, S$.

proj-construction-proj

THEOREM 3.1.6. *There is a unique* S_0*-scheme* $\mathrm{Proj}\, S$ *with underlying topological space* $X = \mathrm{spc}(\mathrm{Proj}\, S)$*, for which the following statements are true:*
(1) *For every homogeneous element* $f \in S$*, the open subscheme* $X_{(f)}$ *is isomorphic to* $\mathrm{Spec}\, S_{(f)}$.
(2) $\mathrm{Proj}\, S$ *is separated.*
(3) *For every point* $P \in \mathrm{spc}(\mathrm{Proj}\, S)$*, the stalk* $\mathscr{O}_{\mathrm{Proj}\, S, P}$ *is isomorphic to* $S_{(P)_0}$.
(4) *Every irreducible component of* $\mathrm{Proj}\, S$ *is of the form* $V_+(P)$*, for some minimal prime* $P \subset S$.
(5) *If* S *is finitely generated over* S_0*, then the structure morphism* $\mathrm{Proj}\, S \to \mathrm{Spec}\, S_0$ *is of finite type. If, in addition,* S_0 *is Noetherian, then* $\mathrm{Proj}\, S$ *is in fact a Noetherian scheme.*

PROOF. For a homogeneous element $f \in S$, let $\phi_f : \mathrm{Spec}\, S_{(f)} \to X_{(f)}$ be the homeomorphism that we found in the Proposition above. Let $\mathscr{O}_{X_{(f)}} = \phi_{f*}\mathscr{O}_{\mathrm{Spec}\, S_{(f)}}$; then $(X_{(f)}, \mathscr{O}_{X_{(f)}})$ equips $X_{(f)}$ with the structure of an affine scheme isomorphic to $\mathrm{Spec}\, S_{(f)}$. We'll be done if we show that we can glue these open subschemes together.

For this, suppose $X_{(g)}$ is another open subscheme. Then $X_{(fg)}$ has an intrinsic structure of an affine scheme arising from its homeomorphism with $\mathrm{Spec}\, S_{(g)}$, but it also inherits affine subscheme structures as a principal open subset inside both $\mathrm{Spec}\, S_{(f)}$ and $\mathrm{Spec}\, S_{(g)}$. Then, part (4) of (3.1.4) tells us that these structures are isomorphic via a canonical isomorphism. Thus, we can indeed glue them together to obtain a global scheme structure, which we call $\mathrm{Proj}\, S$.

That this makes it an S_0-scheme follows because each affine open $X_{(f)}$ has a unique morphism to $\mathrm{Spec}\, S_0$ induced by the natural map $S_0 \to S_{(f)}$. To see that it's separated, we use the criterion from (2.9.7). We take our open cover to be $\{X_{(f)}\}$, for $f \in S$ homogeneous. We see that $X_{(f)} \cap X_{(g)}$ is affine for $f, g \in S$; so it only remains to show that its ring of global sections $S_{(fg)}$ is generated by the restrictions of $S_{(f)}$ and $S_{(g)}$. We see from (3.1.4) that $S_{(fg)} \cong S_{(f)_u}$, where $u = g^{\deg f} f^{-\deg g}$. Clearly, $f^{\deg g} u$ is in the image of the natural map $S_{(g)} \to S_{(fg)}$, thus showing that $S_{(fg)}$ is generated by the images of $S_{(f)}$ and $S_{(g)}$.

For the statement about stalks, we use the isomorphism

$$S_{(f)\,PS_f \cap S_{(f)}} \cong S_{(P)_0},$$

from (3.1.4), and the fact that for any affine scheme $\operatorname{Spec} R$, and any prime $Q \subset R$, $\mathscr{O}_{\operatorname{Spec} R, Q} \cong R_Q$.

If now S is finitely generated over S_0, we will show that $S_{(f)}$ is also finitely generated over S_0 for every homogeneous $f \in S$. This will show that $\operatorname{Spec} S_{(f)}$ is of finite type over $\operatorname{Spec} S$, which will prove the statement. Suppose $\deg f = d$; then we see from [CA, 1.6.5] that $S^{(d)}$ is finitely generated over S_0, and thus $S_{(f)} = S^{(d)}_f/(f-1)$ is also finitely generated over S_0. If S_0 is Noetherian, then S is also Noetherian by [CA, 1.3.4], and so by the same argument $S_{(f)}$ is Noetherian, which finishes our proof, since Noetherianness is local on the base. See (1.3.3). □

Now, we investigate the generic points of a projective scheme.

proj-generic-points

PROPOSITION 3.1.7. *Let S be a graded ring.*

(1) *A closed subset $V_+(I) \subset \operatorname{Proj} S$ is irreducible if and only if $\operatorname{rad}(I) \cap S^+$ is prime.*
(2) *The irreducible components of $\operatorname{Proj} S$ are of the form $V_+(P)$, where P is minimal among homogeneous primes not containing S^+. In particular, $\operatorname{Proj} S$ is irreducible if and only if there is a unique homogeneous prime minimal among those not containing S^+.*
(3) *The generic points of $\operatorname{Proj} S$ are in bijective correspondence with the homogeneous primes of S that are minimal among primes not containing S^+.*

PROOF. (1) Consider $V(I) \subset \operatorname{Spec} S$: this is the union of closed irreducible subsets $V(P)$, where P is a minimal prime over I. Since I is homogeneous, we see from [CA, 1.4.2] that every minimal prime over I is also homogeneous. Now, $V_+(P) = V(P) \cap \operatorname{Proj} S$ is empty if and only if $P \supset S^+$. Therefore, $V_+(I)$ is irreducible if and only if there is at most one minimal prime P over I such that $P \not\supseteq S^+$. Equivalently, $V_+(I)$ is irreducible if and only if $\operatorname{rad}(I) \cap S^+$ is prime.
(2) Follows from (1), and the fact that if $P \subsetneq Q$, then $V_+(P) \supsetneq V_+(Q)$.
(3) Immediate.

□

2. Functorial Properties of Proj

Naturally, we would now like to investigate what kind of morphisms of schemes are induced by homomorphisms between graded rings.

proj-ring-map-proj-map

PROPOSITION 3.2.1. *Let R and S be two graded rings, and let $\phi : R \to S$ be a homomorphism of rings. Suppose there is $e \in \mathbb{Z}$ such that $\phi(R_n) \subset S_{ne}$, for all*

$n \in \mathbb{Z}$. Let $G(\phi)$ be the open subscheme $\operatorname{Proj} S \setminus V_+(\varphi(R^+)S)$. Then there is a natural affine morphism

$$\operatorname{Proj}(\phi) : G(\phi) \to \operatorname{Proj} R$$

such that, for all $f \in R^+$,

$$\operatorname{Proj}(\phi)^{-1}((\operatorname{Proj} R)_{(f)}) = (\operatorname{Proj} S)_{(\phi(f))}.$$

Proof. The restriction of the induced map $\operatorname{spc}(\operatorname{Spec} S) \to \operatorname{spc}(\operatorname{Spec} R)$ (which is just contraction of primes under ϕ) to $G(\phi)$ gives us a map, which we'll call ϕ^*, of the underlying topological spaces. Let $f \in R$ be any homogeneous element; consider the affine open $\operatorname{Spec} R_{(f)} \subset \operatorname{Proj} R$. As a subspace, it consists of all homogeneous primes not contained in R^+ and not containing f. Now, any prime in S not containing $\phi(f)$ also does not contain $\phi(R^+)S$. Hence

proj-eqn-affineness

$$(\phi^*)^{-1}(\operatorname{Spec} R_{(f)}) = \operatorname{Spec} S_{(\phi(f))} \subset G(\phi). \tag{1}$$

Moreover, the induced map on the localizations $\phi_f : R_f \to S_{\phi(f)}$ takes $R_{(f)}$ to $S_{(\phi(f))}$. To see this, note that if $\deg a - k \deg f = 0$, then

$$\deg(\phi(a/f^k)) = \deg \phi(a) - k \deg(\phi(f)) = e(\deg a - k \deg f) = 0.$$

This induces a morphism of schemes

$$\operatorname{Spec} S_{(\phi(f))} \longrightarrow \operatorname{Spec} R_{(f)}.$$

We only need to check now that these morphisms glue together nicely. Suppose $\operatorname{Spec} R_{(g)} \subset \operatorname{Spec} R_{(f)}$; then $g^k = af$, for some homogeneous element $a \in R$. It suffices to check that the following diagram commutes.

$$\begin{array}{ccc} R_{(f)} & \longrightarrow & S_{(\phi(f))} \\ \downarrow & & \downarrow \\ R_{(g)} & \longrightarrow & S_{(\phi(g))}. \end{array}$$

This is easy. Suppose $\frac{x}{f^n} \in R_{(f)}$; we look at where this goes via the two different paths it can take to $S_{(\phi(g))}$.

$$\frac{x}{f^n} \mapsto \frac{a^n x}{g^{nk}} \mapsto \frac{\phi(a^n x)}{\phi(g^{nk})}$$
$$\frac{x}{f^n} \mapsto \frac{\phi(x)}{\phi(f)^n} \mapsto \frac{\phi(a)^n \phi(x)}{\phi(g)^{nk}}.$$

Since ϕ is a ring homomorphism, we see that the diagram does commute, and we're done with the proof, except for the part that $\operatorname{Proj}(\phi)$ is affine; but this follows from equation (1) above. □

Example 3.2.2. Consider $Y = \operatorname{Proj} k[t_0, \ldots, t_n]$ with $\deg t_i = e_i$, for some positive integers e_i, and let $X = \operatorname{Proj} k[x_0, \ldots, x_n]$, where $\deg x_i = 1$, for all i Then we have a natural morphism $\operatorname{Proj}(\phi)$ from X to Y induced by the homomorphism $\phi : k[t_0, \ldots, t_n] \to k[x_0, \ldots, x_n]$ that takes t_i to $x_i^{e_i}$. For $0 \leq i \leq n$, we have an induced morphism on affine opens

$$\operatorname{Spec} k[x_0 x_i^{-1}, \ldots, x_n x_i^{-1}] \to \operatorname{Spec} k[t_0 t_i^{-1}, \ldots, t_n t_i^{-1}].$$

This morphism is evidently finite, since for every j, $(x_j/x_i^{-1})^{e_j} \in \operatorname{im} \phi_{(t_i)}$.

oj-quotient-graded-rings

COROLLARY 3.2.3. *Let $I \subset R$ be a homogeneous ideal, then the natural homomorphism $\phi : R \to R/I$ induces a closed immersion* $\operatorname{Proj}(\phi) : \operatorname{Proj}(R/I) \to \operatorname{Proj} R$, *whose topological image is $V_+(I)$.*

PROOF. Since $\phi(R^+)(R/I) = (R/I)^+$, we see that $G(\phi)$ is the whole of $\operatorname{Proj}(R/I)$. It's enough to show now that $\operatorname{Spec}(R/I)_{\phi(f)} \to \operatorname{Spec} R_{(f)}$ is a closed immersion (since closed immersions are local on the base). But this follows from the fact that $(R/I)_{\phi(f)} \cong R_{(f)}/I_{(f)}$ (see [CA, 1.6.3]). The second statement follows from part (8) of (3.1.2). ☐

proj-veronese-embedding

PROPOSITION 3.2.4. *Suppose R is a graded ring. Let $e \in \mathbb{Z}$, and let $R^{(e)}$ be the d^{th} Veronese subring of R. Then we have a natural isomorphism of R_0-schemes:*

$$\operatorname{Proj} R^{(d)} \cong \operatorname{Proj} R.$$

PROOF. We have a natural map of rings $\varphi : R^{(e)} \to R$, which is just the inclusion map. By the definition of $R^{(e)}$, this map satisfies the condition $\varphi(R_n^{(e)}) = R_{en}$. Let $X = \operatorname{Proj} R$, and let $X^{(e)} = \operatorname{Proj} R^{(e)}$. We claim that there is a cover of X by affine opens $X_{(f)}$ with $f \in R^{(e)}$. Indeed, we observe that $X_{(g)} = X_{(g^e)}$, for any homogeneous element g. Given this, from part (2) of [CA, 1.6.5], we know that the natural map φ induces an isomorphism $R_{(f)}^{(e)} \cong R_{(f)}$, for every $f \in R^{(e)}$. This finishes our proof, since

$$(\varphi^*)^{-1}(X_{(f)}^{(e)}) = X_{(f)}.$$

☐

REMARK 3.2.5. This is a generalization of the result from classical algebraic geometry that says that the Veronese embedding is an isomorphism onto its image.

DEFINITION 3.2.6. Any scheme isomorphic to $\operatorname{Proj} S$ for some graded ring S finitely generated by S_1 over $S_0 = R$ is called a *projective scheme over R*. In this case, we say that the R-algebra S is *projective over R*.

proj-affine-scheme-proj

EXAMPLE 3.2.7. Any affine scheme is naturally a projective scheme. For any ring R, we check immediately that $X = \operatorname{Proj} R[t] \cong \operatorname{Spec} R$. To see this, simply observe that $\{X_{(t)}\}$ is an open cover for X, and so $X \cong X_{(t)} = \operatorname{Spec} R$.

-proj-polynomial-algebra

EXAMPLE 3.2.8 (Proj of a Polynomial Algebra). Let $S = R[x_0, \ldots, x_n]$ be a polynomial algebra over a ring R with the usual grading. Then, since $(x_0, \ldots, x_n) = S^+$, we see by part (8) of (3.1.5) that $\{X_{(x_i)} : 0 \le i \le n\}$ is an open affine cover for $\operatorname{Proj} S$, with $X_{(x_i)} \cong \operatorname{Spec} R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. These glue together to give us the scheme structure on $\operatorname{Proj} S$. We denote this scheme by $\mathbb{P}_R^n$, and we'll call it *projective n-space over R*.

The next Corollary says that projectivity is not really that drastic a restriction.

every-fintype-projective

COROLLARY 3.2.9. *Let S be a positively graded ring finitely generated over $R = S_0$ as an R-algebra. Then* $\operatorname{Proj} S$ *is isomorphic to a projective scheme over R.*

PROOF. By part (4) of [CA, 1.6.5], we can find $d \ge 1$ such that $S^{(d)}$ is generated by $S_1^{(d)}$ over R. Now apply (3.2.4). ☐

proj-artinian-proj-empty

COROLLARY 3.2.10. *Suppose R is a positively graded ring, and let $I \subset R$ be a homogeneous ideal. Let $I' = \oplus_{n \ge m} I_n$, for some integer $m \in \mathbb{N}$. Then* $\operatorname{Proj} R/I \cong \operatorname{Proj} R/I'$ *as subschemes of* $\operatorname{Proj} R$.

Proof. Let $S' = R/I'$ and let $S = R/I$; then we have a natural quotient map $S' \to S$. If $e \geq m$, then this induces an isomorphism from $S'^{(e)}$ to $S^{(e)}$. This finishes our proof by the last corollary. □

Now, we'll study the behavior of Proj under base change.

proj-segre-embedding

Proposition 3.2.11. *Let R and R' be two graded rings with $R_0 = R'_0 = S$. Let $T = \bigoplus_{n\in\mathbb{Z}}(R_n \otimes_S R'_n)$. Then*

$$\operatorname{Proj} T \cong \operatorname{Proj} R \times_{\operatorname{Spec} S} \operatorname{Proj} R'.$$

Remark 3.2.12. In the proof below, we'll implicitly assume that all our elements have positive degree to make the arguments simpler. The modification for the general case is easy, but tedious and unenlightening.

Proof. For convenience, let $X = \operatorname{Proj} R$, $Y = \operatorname{Proj} R'$; we'll denote their fiber product over $\operatorname{Spec} S$ by $X \times_S Y$. Set $Z = \operatorname{Proj} T$. Let $f \in R$ be any homogeneous element; we define

$$Z_f = \bigcup_{g\in R'} \operatorname{Spec} T_{(f^{\deg g} \otimes g^{\deg f})}.$$

Now, we claim that there is a natural isomorphism

$$\begin{aligned} T_{(f'\otimes g')} &\xrightarrow{\cong} R_{(f)} \otimes_S R'_{(g)} \\ \frac{a\otimes b}{(f'\otimes g')^s} &\mapsto \frac{a}{f'^s} \otimes \frac{b}{g'^s}, \end{aligned}$$

where $f' = f^{\deg g}$, $g' = g^{\deg f}$. One checks immediately that this is well defined. This map has an inverse given by

$$\begin{aligned} R_{(f)} \otimes_S R'_{(g)} &\xrightarrow{T}{}_{(f\otimes g)} \\ \frac{a}{f^r} \otimes \frac{b}{g^t} &\mapsto \frac{a^{t\deg g} \otimes b^{r \deg f}}{(f'\otimes g')^{rt}}. \end{aligned}$$

Hence we see that

$$Z_f = \bigcup_{g\in R'} \operatorname{Spec}(R_{(f)} \otimes_S R'_{(g)}),$$

with

$$\begin{aligned} \operatorname{Spec}(R_{(f)} \otimes_S R'_{(g)}) \cap \operatorname{Spec}(R_{(f)} \otimes_S R'_{(h)}) &\cong \operatorname{Spec} T_{(f^{\deg g + \deg h} \otimes (gh)^{\deg f})} \\ &\cong \operatorname{Spec}(R_{(f)} \otimes_S R'_{(gh)}). \end{aligned}$$

From this, it's clear that

$$Z_f = X_{(f)} \times_S Y.$$

Moreover, for any other homogeneous element $f' \in R$, it's immediate that $Z_f \cap Z_{f'} = Z_{ff'} = X_{(ff')} \times_S Y$. This shows that

$$Z = \bigcup_{f\in R} Z_f = X \times_S Y.$$

□

hemes-affine-base-change

COROLLARY 3.2.13. *Let* $f : Y' = \operatorname{Spec} S' \to Y = \operatorname{Spec} S$ *be a morphism of affine schemes, and let* R *be a graded ring with* $R_0 = S$. *Then*

$$\operatorname{Proj} T = \operatorname{Proj} R \times_Y Y',$$

where $T := R \otimes_S S'$ *is given the grading given by the decomposition*

$$T = \bigoplus_{n \in \mathbb{Z}} (R_n \otimes_S S').$$

PROOF. Let $R' = S'[t, t^{-1}]$ in the Proposition above, and then observe that $R'_n = S'$, for all $n \in \mathbb{Z}$. Moreover, $\operatorname{Proj} R' = \operatorname{Spec} S'$, as one sees for example from the fact that the only homogeneous primes in R' are the ones entirely contained in S' (for all homogeneous elements of non-zero degree are units). Then our conclusion follows immediately from the Proposition. □

3. Projective and Proper Morphisms: Chow's Lemma

The next result can be proven using the sledgehammer of Chow's Lemma (**??**), but here's a fine elementary proof that I got from Qing Liu.

roj-proper-affine-finite

PROPOSITION 3.3.1. *Suppose* $f : \operatorname{Spec} R \to \operatorname{Spec} S$ *is a proper morphism of affine schemes. Then* f *is finite.*

PROOF. Let $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$; we can assume that X is reduced. Indeed, since R is of finite type over S it suffices to show that R is integral over S. If X_{red} is finite over Y, then $R/\operatorname{Nil} R$ is integral over S. But this implies that R is integral over S: Every element in R is of the form $a + n$, with $a \notin \operatorname{Nil} R$ and n nilpotent. Since $R/\operatorname{Nil} R$ is integral over S, there is some monic polynomial $p(t) \in S[t]$ such that $p(a) \in \operatorname{Nil} R$. But then $p^n(a) = 0$, for some $n \in \mathbb{N}$. Nilpotent elements are trivially integral over S, and so we see that R is itself integral over S.

Since f is of finite type, R is a finitely generated S-algebra. Hence we have a surjection

$$S[x_1, \ldots, x_n] \to R,$$

for some $n \in \mathbb{Z}$. Our proof will be by induction on n. If $n = 0$, then f is a closed immersion and is hence proper. If $n = 1$, then the surjection $S[t] \to R$ gives a closed immersion of S-schemes $g' : X \to \mathbb{A}^1_S$. We can identify $\mathbb{A}^1_S$ with the open subscheme $Z_{(x)}$ of the projective scheme $Z = \operatorname{Proj} S[x, y]$, using the map $\frac{y}{x} \mapsto t$, and we can compose f with this open immersion to obtain an immersion $g : X \to \mathbb{P}^1_S$. Now, if $\pi : \mathbb{P}^1_S \to Y$ is the structure morphism, then $f = \pi \circ g$ is proper, and so, since π is separated, we see that g must also be proper (2.9.14).

In particular, g is closed, and we see that $g(X) = V_+(I)$, for some radical homogeneous ideal $I \subset S[x, y]$. Since $g(X) \subset Z_{(x)}$, we find that

$$V_+(I + (x)) = V_+(I) \cap V_+((x)) = \emptyset.$$

By part (5) of (3.1.2), this means that $(x, y) \subset \operatorname{rad}(I + (x))$. In particular, there exist $m \in \mathbb{N}$, and homogeneous polynomials $q(x, y) \in I$ and $p(x, y) \in S[x, y]$ such that

$$y^m = q(x, y) + xp(x, y).$$

Now, $J = I_{(x)} \subset k[x, y]_{(x)} = k[t]$ is such that $V(J) = g'(X)$. But now observe that

$$t^m - p(1, t) \in J,$$

implying that J contains a monic polynomial, since $\deg p \leq m-1$. Since R is reduced, it follows that $R \cong S[t]/J$ is integral over S, and is thus finite.

For the general case, suppose $R = S[a_1, \ldots, a_n]$, and, for $1 \leq j \leq n$, let $X_j = \operatorname{Spec} S[a_1, \ldots, a_j]$. Now, observe that the natural morphism of S-schemes $X \to X_{n-1}$ is proper by (2.9.14), and hence by the $n = 1$ case, we see that X is finite over X_{n-1}. By induction, X_{n-1} is finite over Y, and so we see that X is finite over Y. □

DEFINITION 3.3.2. A morphism $f : X \to Y$ is *projective* if there exists an affine open cover $\{U_i = \operatorname{Spec} R_i : i \in I\}$ such that, for every $i \in I$, $f^{-1}(U_i)$ is a projective R_i-scheme.

REMARK 3.3.3. This is an unsatisfactory definition, but it comes with localness built in, which is excellent. We'll give a much better (but equivalent) definition in Chapter **??**.

proj-fin-gen-closed

PROPOSITION 3.3.4. *Let S be a graded ring finitely generated over $R = S_0$. Then the structure morphism $\pi : \operatorname{Proj} S \to \operatorname{Spec} R$ is closed.*

PROOF. Let $Z = V_+(I)$ be a closed subset of $\operatorname{Proj} S$, and let $Y = \pi(Z)$ be its image. We will show that $V = \operatorname{Spec} R \setminus Y$ is open. Let $x \in \operatorname{Spec} R$: then, considering the inclusion $j : Z \hookrightarrow \operatorname{Proj} S$ as a closed immersion corresponding to the ideal I, we see that $(\pi \circ j)^{-1}(x)$ is homeomorphic to $Z \times_R k(x)$ (1.7.8). But if $Z = \operatorname{Proj} S/I$, then $Z \times_R k(x)$ is isomorphic to $\operatorname{Proj}((S/I) \otimes_R k(x))$. Now, this is empty if and only if the irrelevant ideal of $(S/I) \otimes_R k(x)$ is nilpotent, but that can happen if and only if $(S/I) \otimes_R k(x)$ has only finitely many non-zero graded components (this is where we use the finite generation bit). In other words, $x \in V$ if and only if $(\pi \circ j)^{-1}(x)$ is empty, if and only if, for large enough m,

$$(S/I)_m \otimes_R k(x) = 0.$$

By Nakayama's lemma, this can only happen if

$$(S/I)_m \otimes_R \mathscr{O}_{X,x} = 0.$$

By [CA, 7.1.5], we then have a principal open neighborhood $U = (\operatorname{Spec} R)_a$ of x, such that, for all $y \in U$,

$$(S/I)_m \otimes_R \mathscr{O}_{X,y} = 0,$$

for m large enough, and so $U \subset V$, showing that V is an open set. □

The next Theorem is of great importance.

-projective-space-proper

THEOREM 3.3.5. *Every projective morphism is proper.*

PROOF. Properness is a local condition (2.11.3), and so it suffices to show that the map $\operatorname{Proj} S \to \operatorname{Spec} R$ is proper, for any affine scheme $\operatorname{Spec} R$, and any graded R-algebra S finitely generated by S_1 over $R = S_0$. But if S is generated by $n+1$ elements $\{a_0, \ldots, a_n\}$ of S_1, for some $n \in \mathbb{N}$, then we have a natural surjection of graded R-algebras

$$R[T_0, \ldots, T_n] \to S$$
$$T_i \mapsto a_i.$$

This gives us a closed immersion of $\operatorname{Proj} S$ into $\mathbb{P}^n_R$, by (3.2.3). Since closed immersions are proper, and proper morphisms are stable under composition (2.11.3), it now suffices to show that the natural morphism $\pi : \mathbb{P}^n_R \to R$ is proper.

From (3.1.6), we see that π is separated and of finite type. To show that it is universally closed, we must show that

$$\mathbb{P}^n_R \times_R X \to X$$

is a closed morphism, for all R-schemes X. It's enough to prove this for the case where $X = \operatorname{Spec} T$ is affine; but in this case, we see by (3.2.13) that $\mathbb{P}^n_R \times_R \operatorname{Spec} T = \mathbb{P}^n_T$. Hence we'll be done if we show that the morphism $\mathbb{P}^n_T \to \operatorname{Spec} T$ is closed, for any ring T. That follows from the previous Proposition. □

CHAPTER 4

Sheaves of Modules over Schemes

chap:sms

This would be a good time to go back to [RS, 1] and review stuff about modules over ringed spaces. Just for the record, a module over a scheme X is just a module of sheaves over the ringed space $(X, \mathscr{O}_X)$.

1. Quasi-coherent Sheaves over an Affine Scheme

-coherent-affine-schemes

1.1. The Tilde Correspondence. In this section, we'll build an equivalence between the category of R-modules and the category of quasi-coherent $\mathscr{O}_{\operatorname{Spec} R}$-modules.

Note on Notation 2. For the rest of the section, we fix an affine scheme $X = \operatorname{Spec} R$.

In one direction, suppose we have an R-module M. Then we can construct a quasi-coherent $\mathscr{O}_X$-module $\widetilde{M}$ with $\Gamma(X_a, \widetilde{M}) = M_a$, for every $a \in R$. The construction is entirely straightforward: just as in the construction of the structure sheaf on $\operatorname{Spec} R$, we define a presheaf $\widetilde{M}$ on the base of principal open sets, by setting $\widetilde{M}(X_f) = M_f$, with the restriction maps defined just as they are in the case of $\operatorname{Spec} R$ (1.1.4). Now, the verification that the presheaf associated to this presheaf on a base is actually a sheaf proceeds precisely as in the proof of (1.1.5). Moreover, for every point $x \in \operatorname{Spec} R$, we see that $\widetilde{M}_x$ is just the localization M_P at the prime P corresponding to x. The proof is again identical the one in (1.1.6).

Suppose now that we have a map of R-modules $\phi : M \to N$. Then, for every element $a \in R$, we get a map of R_a-modules $\phi_a : M_a \to N_a$, and if $X_a \subset X_b$, then the following diagram commutes:

$$\begin{array}{ccc} M_a & \longrightarrow & N_a \\ \downarrow & & \downarrow \\ M_b & \longrightarrow & N_b \end{array}$$

where the vertical maps are given by $\frac{m}{a^n} \mapsto \frac{a^{n(k-1)}c^n m}{b^n}$, where $c \in R$ and $k \in \mathbb{N}$ are such that $b = ac^k$ (1.1.2). So we have a morphism between presheaves on a base, which we can extend to a morphism of sheaves $\widetilde{\phi} : \widetilde{M} \to \widetilde{N}$. See the argument in [RS, 1.17].

In particular, for any R-module M, this gives us a morphism $\mathscr{O}_X \to \widetilde{\operatorname{End}_R(M)}$ induced by the natural map $R \to \operatorname{End}_R(M)$. Now, for every $a \in A$, we have a natural inclusion $\widetilde{\operatorname{End}_R(M)} \hookrightarrow \underline{\operatorname{End}}\, \widetilde{M}$, given by

$$\begin{aligned} \Gamma(X_a, \widetilde{\operatorname{End}_R(M)}) &\to \underline{\operatorname{End}}(\widetilde{M}|_{X_a}) \\ \phi &\mapsto \widetilde{\phi}. \end{aligned}$$

Observe that we're treating ϕ as an element of $\operatorname{End}_{R_a}(M_a)$ under the natural map $\operatorname{End}_R(M)_a \to \operatorname{End}_{R_a}(M_a)$. Also, we're using the fact that $\widetilde{M}|_{X_a} = \widetilde{M_a}$, which is also easily checked.

This gives rise to a morphism $\mathscr{O}_X \to \underline{\operatorname{End}}(\widetilde{M})$, which makes $\widetilde{M}$ into an $\mathscr{O}_X$-module in a natural way. More concretely, this is just the structure morphism obtained from the natural maps $R_a \times M_a \to M_a$ on each principal open X_a.

It's easy to check now that the assignment $M \mapsto \widetilde{M}$ gives a functor from R-mod to $\mathscr{O}_X$-mod. Also, since localization preserves exact sequences, and exactness of morphisms of sheaves only needs to be checked on stalks, we see immediately that this is an *exact* functor.

Observe that the same argument as above now gives us the following proposition.

sms-tilde-hom-sheaf-hom

PROPOSITION 4.1.1. *For any pair of R-modules M, N, we have a natural morphism*

$$\widetilde{\operatorname{Hom}_R(M, N)} \to \underline{\operatorname{Hom}}_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}).$$

PROOF. We already know by the above argument that we have a map

$$\widetilde{\operatorname{Hom}_R(M, N)} \to \underline{\operatorname{Hom}}(\widetilde{M}, \widetilde{N}).$$

It remains to check that under this map an element on the left goes to a morphism of $\mathscr{O}_X$-modules on the right. It's enough to check that on a principal open set X_a, the map

$$\operatorname{Hom}_R(M, N)_a \to \operatorname{Hom}_{R_a}(M_a, N, a) \to \operatorname{Hom}(\widetilde{M_a}, \widetilde{N_a})$$

takes an element on the left hand side to a morphism of $\mathscr{O}_{X_a}$-modules on the right. Again, since the condition for checking if a morphism is a morphism of $\mathscr{O}_X$-modules can be checked on each open set, it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} R_a \times M_a & \longrightarrow & M_a \\ \downarrow & & \downarrow \\ R_a \times N_a & \longrightarrow & N_a. \end{array}$$

But this is immediate from the definition of a map of R_a-modules! □

les-over-principal-opens

LEMMA 4.1.2. *Let X be a quasi-compact scheme, let $\mathscr{M}$ be an $\mathscr{O}_X$-module. Suppose there is a finite affine open cover $\{U_i\}$ of X such that $\mathscr{M}|_{U_i} \cong \widetilde{M}_i$, for some $\mathscr{O}_X(U_i)$-module M_i. Let $M = \mathscr{M}(X)$, and let $b \in R$ be any element of R.*

(1) *Suppose $m \in M$ is such that $\operatorname{res}_{X, X_b}(m) = 0$. Then, there is an integer $n \geq 0$ such that $b^n m = 0 \in M$.*
(2) *Suppose further that $U_i \cap U_j$ is quasi-compact for each pair i, j. Let $m \in \Gamma(X_b, \mathscr{M})$; then there is an integer $n \geq 0$ such that $b^n m$ is the restriction of an element in M.*

PROOF. (1) For each i, $\Gamma(U_i \cap X_b, M) = (M_i)_b$. Let $m_i = \mathrm{res}_{X,U_i}(m)$, then we can find some integer $n \geq 0$ such that $b^n m_i = 0 \in M_i$ (where we're treating b as a section over U_i). But then, since the X_{a_i} are a cover for X, $b^n m$ must be 0.

(2) There is an integer $s \geq 0$ such that for each i, we have $p_i \in M_i$ such that p_i restricts to $b^s m$ on $U_i \cap X_b$. For each pair i, j, $p_i - p_j$ on $U_i \cap U_j$ restricts to 0 on $U_i \cap U_j \cap X_b$. If we cover $U_i \cap U_j$ by finitely many principal open sets, we can, by part (1), find an integer $k_{ij} \geq 0$, such that over each such open set $b^{k_{ij}}(p_i - p_j)$ vanishes. So $b^{k_{ij}}(p_i - p_j)$ vanishes on $U_i \cap U_j$. If we take $n = s + k$, where k is the supremum of all integers k_{ij}, then we see that $\{b^k p_i \in M_i\}$ defines a coherent collection of elements that we can glue together to give a global section p that restricts to $b^n m$ on $U_i \cap X_b$, for each i, and therefore restricts to $b^n m$ on X_b.

□

coherent-module-R-module

PROPOSITION 4.1.3. *The assignment $M \mapsto \widetilde{M}$ is an equivalence of categories between R-mod and the category $\mathscr{O}_X$-qcoh of quasi-coherent $\mathscr{O}_X$-modules.*

PROOF. Observe that if

$$R^I \to R^J \to M \to 0$$

is a free presentation for M, then this gives rise to a free presentation

$$\mathscr{O}_X^I \to \mathscr{O}_X^J \to \widetilde{M} \to 0$$

for $\widetilde{M}$. This shows that $\widetilde{M}$ is quasi-coherent.

We need to check that this functor is full, faithful and essentially surjective. We'll do this one at a time.

Full and faithful: There is evidently a natural map

$$\mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_R(M, N)$$

that just extracts the map on global sections. For every map of R-modules $\phi : M \to N$, we built above a corresponding map $\widetilde{\phi} : \widetilde{M} \to \widetilde{N}$ of $\mathscr{O}_X$-modules. It's easily checked that this gives the inverse we need.

Essentially surjective: This is the most involved part of the proof. Suppose we have a quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, and $M = \Gamma(X, \mathscr{M})$. We must show that $\mathscr{M} \cong \widetilde{M}$. Since $\mathscr{M}$ is quasi-coherent, we can find a finite principal open cover $\{X_{a_i}\}$, and a free presentation

$$\mathscr{O}_{X_{a_i}}^{I_i} \to \mathscr{O}_{X_{a_i}}^{J_i} \to \mathscr{M}|_{X_{a_i}} \to 0,$$

for each i. But this means that $\mathscr{M}|_{X_{a_i}} = \widetilde{M_i}$, for some R_{a_i}-module M_i. Now, for any $b \in R$, we have the following commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M_b \\ \downarrow & \swarrow & \\ \Gamma(X_b, \mathscr{M}) & & \end{array}$$

where we get the diagonal map from the universal property of localization. Observe that we're now in a position to apply the results of the previous

Lemma. Part (1) of that lemma tells us that the vertical map, and hence the diagonal map, is injective, and part (2) of the lemma tells us that the diagonal map is surjective. So we see that $M_b \cong \Gamma(X_b, \mathscr{M})$, for all $b \in R$. Moreover, these isomorphisms are natural, and so give us an isomorphism from $\widetilde{M}$ to $\mathscr{M}$.

□

In fact, more can be said about this functor. Compare the next proposition with (1.2.2).

ms-tilde-globsec-adjoint

Proposition 4.1.4. *The assignment $M \mapsto \widetilde{M}$ is a left adjoint to the global sections functor from $\mathscr{O}_X$-mod to R-mod. Equivalently, for every R-module M and $\mathscr{O}_X$-module $\mathscr{N}$, there is a natural isomorphism of R-modules*

$$\operatorname{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{N}) \cong \operatorname{Hom}_R(M, \Gamma(X, \mathscr{N})).$$

Proof. There is evidently a natural map in one direction just as in the proof of the last Proposition. Suppose now that we have an R-module map $\phi : M \to \Gamma(X, \mathscr{N})$. Then, for every $a \in X$, we have the composition

$$M_a \xrightarrow{\phi_a} \Gamma(X, \mathscr{N})_a \to \Gamma(X_a, \mathscr{N}),$$

which gives us a morphism $\widetilde{M} \to \mathscr{N}$. It's easy to check that this gives us an inverse to the natural map in the other direction. □

When we get to cohomology of quasi-coherent sheaves in Chapter 9, we'll see that the next Proposition generalizes to the assertion that quasi-coherent sheaves over an affine scheme have trivial cohomology.

-quasi-coherent-no-cohom

Proposition 4.1.5. *Suppose we have an exact sequence of $\mathscr{O}_X$-modules*

$$0 \to \mathscr{M}_1 \xrightarrow{\phi} \mathscr{M}_2 \xrightarrow{\psi} \mathscr{M}_3 \to 0.$$

If $\mathscr{M}_1$ is quasi-coherent, then the sequence of global sections

$$0 \to \Gamma(X, \mathscr{M}_1) \to \Gamma(X, \mathscr{M}_2) \to \Gamma(X, \mathscr{M}_3) \to 0$$

is also exact.

Proof. Let $s \in \Gamma(X, \mathscr{M}_3)$ be a global section, and let $t \in \Gamma(X_a, \mathscr{M}_2)$ be such that $\psi(t) = \operatorname{res}_{X,X_a}(s)$. By [NOS, 4.8], we can find a principal open cover $\{U_i = X_{a_i}\}$ of X, and sections $t_i \in \Gamma(U_i, \mathscr{M}_2)$ such that $\phi(t_i) = \operatorname{res} X, U_i(s)$. Now, $t - t_i \in \Gamma(U_i \cap X_a, \mathscr{M}_2)$ goes to 0 in $\mathscr{M}_3$, for every i, and so we can find $p_i \in \Gamma(U_i \cap X_a, \mathscr{M}_1)$ such that $\phi(p_i) = t - t_i$. Since $\mathscr{M}_1$ is quasi-coherent, we can find $r \geq 0$ such that, for every i, $a^r p_i$ is the restriction of some section $u_i \in \Gamma(U_i, \mathscr{M}_1)$. Consider now the sections $q_i = a^r t_i + \phi(u_i)$ of $\mathscr{M}_2$ over U_i. Over $U_i \cap X_a$, we see that q_i restricts to $a^r t$. Moreover, over $U_i \cap U_j$, we see that $\psi(q_i - q_j) = a^r \psi(t_i - t_j) = 0$. Hence, $q_i - q_j = \phi(p_{ij})$ for some section p_{ij} of $\mathscr{M}_1$ over $U_i \cap U_j$. But $q_i - q_j$ restricts to 0 over $U_i \cap U_j \cap X_a$, and so we can find $k \geq 0$ large enough such that $a^k p_{ij} = 0$, for all pairs i, j. Now, consider the sections $a^k q_i$ of $\mathscr{M}_2$ over U_i. We see that they agree over the intersections $U_i \cap U_j$, and so they glue together to give a global section that maps to $a^n s \in \Gamma(X, \mathscr{M}_3)$, where $n = r + k$. In particular, for every i, we can find a global section s_i of $\mathscr{M}_2$ that maps to $a_i^n s$ in $\mathscr{M}_3$. But of course, as always, we can find $c_i \in R$ such that $\sum_i c_i a_i^n = 1$. Consider now, the global section $\sum_i c_i s_i$; this will map to $\sum_i c_i a_i^n s = s$. □

1.2. Coherent Sheaves over a Noetherian Affine Scheme. In general, coherent sheaves have no such nice correspondence with any subcategory of R-mod, but in the case where R is Noetherian, we do have a good description.

herent-noetherian-affine

PROPOSITION 4.1.6. *Suppose R is a Noetherian ring, and M is an R-module. Then the following are equivalent:*

(1) *M is finitely presented.*
(2) *M is finitely generated.*
(3) *$\widetilde{M}$ is a coherent $\mathscr{O}_X$-module.*
(4) *$\widetilde{M}$ is a finitely presented $\mathscr{O}_X$-module*
(5) *$\widetilde{M}$ is a $\mathscr{O}_X$-module of finite type.*

In particular, the assignment $M \mapsto \widetilde{M}$ gives an equivalence between the category of finitely generated R-modules and the category $\mathscr{O}_X$-coh of coherent $\mathscr{O}_X$-modules.

PROOF. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. (2) $\Rightarrow$ (1) follows from the Noetherian hypothesis.

We'll do (5) $\Rightarrow$ (2) first. Since $\widetilde{M}$ is of finite type, we can find a principal open cover $\{X_{a_i}\}$ of X, and a surjection

$$R_{a_i}^{m_i} \to M_{a_i} \to 0,$$

for all i. Equivalently, M is locally finitely generated. From this, it's easy to see that M is finitely generated, with the usual partition of unity argument. More precisely, take a finite collection of elements $\{m_j \in M\}$ such that their images generate M_{a_i} for every i. Then, for every $m \in M$, we can find an integer $N \geq 0$ such that $a_i^N m$ is in the module generated by the m_j, for all i. Since we can find c_i such that $\sum_i c_i a_i^N = 1$, we're now done.

It remains to prove (2) $\Rightarrow$ (3). It's immediate that if M is finitely generated, then $\widetilde{M}$ is of finite type. So it's enough to verify the second condition for coherence. Suppose we have a morphism $\phi : \mathscr{O}_X|_V^n \to \widetilde{M}|_V$, then on every principal open set $X_f \subset V$, this corresponds to a map of rings $R_f^n \to M_f$, whose kernel is again finitely generated, since R_f is Noetherian. This implies that $\ker \phi|_{X_f}$ is of finite type for every $X_f \subset V$, which of course implies that $\ker \phi$ is of finite type. □

1.3. Sheaf Hom, Tensor Product and other Categorical considerations. We go back now to the natural map defined in (4.1.1), and ask: when is this is an isomorphism? In general, it's not one, but under some finite presentation conditions the situation's brighter.

in-pres-tilde-shfhom-iso

PROPOSITION 4.1.7. *If M is finitely presented, then, for any R-module N, the natural map*

$$\widetilde{\mathrm{Hom}_R(M,N)} \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\widetilde{M},\widetilde{N})$$

is an isomorphism. In particular, $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\widetilde{M},\widetilde{N})$ is quasi-coherent.

PROOF. Note that if M is finitely presented, then so is $\widetilde{M}$. So it follows from [RS, 4.15] that $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\widetilde{M},\widetilde{N})$ is quasi-coherent (we'll show below that our Standing Assumption [RS, **??**] there is valid in our situation). Now the result follows from (4.1.3) and the fact that

$$\begin{aligned} \Gamma(X, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\widetilde{M},\widetilde{N})) &= \mathrm{Hom}_{RegX}(\widetilde{M},\widetilde{N}) \\ &\cong \mathrm{Hom}_R(M,N). \end{aligned}$$

□

Remark 4.1.8. Observe that we can now use [RS, 4.16] to give an alternate proof of [CA, 3.1.12].

uasi-coherent-tensor-iso

Proposition 4.1.9. *Let M, N be R-modules. Then, we have a natural isomorphism*

$$\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}$$

In particular, $\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}$ is quasi-coherent.

Proof. For every principal open set $X_a \subset X$, we have a natural isomorphism

$$(M \otimes_R N)_a \cong M_a \otimes_{R_a} N_a$$

This gives us an isomorphism from $\widetilde{M \otimes_R N}$ to the presheaf of which $\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}$ is the sheafification. But since $\widetilde{M \otimes_R N}$ is already a sheaf, this tells us that this is in fact an isomorphism with $\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}$. □

The next Proposition will be important for later applications.

ms-tilde-commute-colimit

Proposition 4.1.10. *Let $\{M_i : i \in I\}$ be a filtered system of R-modules. Then there is a natural isomorphism*

$$\widetilde{\varinjlim M_i} \cong \varinjlim \widetilde{M_i}.$$

In particular,

$$\widetilde{\oplus_i M_i} \cong \oplus_i \widetilde{M_i},$$

for any collection $\{M_i : i \in I\}$ of R-modules.

Proof. Since the filtered colimit (or direct limit) is the cokernel of a morphism between direct sums, and since the tilde functor is exact, it suffices to show that the tilde functor preserves direct sums. This we do using Yoneda's Lemma, and (4.1.4). In fact it now follows immediately from the observation that $\oplus_i \widetilde{M_i}$ represents the functor $\prod_i \mathrm{Hom}_R(M_i, \Gamma(X, \_\_))$ and $\widetilde{\oplus_i M_i}$ represents the functor $\mathrm{Hom}_R(\oplus_i M_i, \Gamma(X, \_\_))$, both of which functors are canonically isomorphic. □

1.4. Direct and Inverse Images. Suppose we now have a map of rings $\phi : S \to R$ inducing a morphism $(f, f^\sharp) : \operatorname{Spec} R \to \operatorname{Spec} S$. What are the direct and inverse image morphisms induced by this map? The next Proposition has the answer.

sms-dirinvimg-affine

Proposition 4.1.11. *If $(f, f^\sharp) : \operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of affine schemes. Then*

1. *For any R-module M, $f_* \widetilde{M} = \widetilde{{}_S M}$, where, by ${}_S M$ we denote M viewed as an S-module.*
2. *For any S-module N, $f^* \widetilde{N} = \widetilde{N \otimes_S R}$.*

In particular, both f_* and f^* carry quasi-coherent modules to quasi-coherent modules. Moreover, if R and S are Noetherian, f^* also preserves coherent $\mathscr{O}_{\operatorname{Spec} R}$-modules.

Proof. The last conclusion follows from Proposition (4.1.6), using the fact that if N is a finitely generated S-module, then $N \otimes_S R$ is a finitely generated R-module. The statement before that is immediate from the rest of the proposition. In the following, we set $Y = \operatorname{Spec} S$ and $X = \operatorname{Spec} R$.

(1) Just observe that for each $a \in S$, we have

$$(f_*\widetilde{M})(Y_a) = \widetilde{M}(X_{\phi(a)}) = M_{\phi a} = ({}_S M)_a.$$

(2) We use Yoneda's Lemma. For every $\mathscr{O}_X$-module $\mathscr{F}$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathscr{O}_X}(f^*\widetilde{N}, \mathscr{F}) &\cong \operatorname{Hom}_{\mathscr{O}_Y}(\widetilde{N}, f_*\mathscr{F}) \\ &\cong \operatorname{Hom}_S(N, {}_S\Gamma(X, \mathscr{F})) \\ &\cong \operatorname{Hom}_R(N \otimes_S R, \Gamma(X, \mathscr{F})) \\ &\cong \operatorname{Hom}_{\mathscr{O}_X}(\widetilde{N \otimes_S R}, \mathscr{F}),\end{aligned}$$

where the first isomorphism follows from the adjunction [RS, 2.24], the second follows from (4.1.4), the third is the usual base change isomorphism, and the last is again (4.1.4).

□

1.5. Quasi-coherent Sheaves of Algebras. It's immediate from all the work we've done that any quasi-coherent $\mathscr{O}_X$-algebra will be of the form $\widetilde{B}$ for some R-algebra B. The following Proposition is the main result.

sms-qc-algebras-affine

PROPOSITION 4.1.12. *Let $\widetilde{B}$ and $\widetilde{C}$ be two quasi-coherent $\mathscr{O}_X$-algebras. Then we have a natural bijection*

$$\operatorname{Hom}_{\mathscr{O}_X\text{-}alg}(\widetilde{A}, \widetilde{B}) \cong \operatorname{Hom}_{R\text{-}alg}(A, B).$$

PROOF. There's the obvious map in one direction (left to right) that specializes to the global sections. In the other direction, we observe that, by (4.1.3), any R-algebra homomorphism induces a morphism of $\mathscr{O}_X$-*modules* from $\widetilde{A}$ to $\widetilde{B}$. But it's immediate that this induced morphism gives us R_a-algebra homomorphisms from $\widetilde{A}(X_a) = A_a$ to $\widetilde{B}(X_a) = B_a$, for any element $a \in A$. From this it follows that the map in question actually sends R-algebra homomorphisms to $\mathscr{O}_X$-algebra morphisms. □

2. Quasi-coherent Sheaves over General Schemes

With all these results in hand, it's easy now to describe quasi-coherent sheaves over a general scheme X.

2.1. First Properties.

-coherent-general-scheme

PROPOSITION 4.2.1. *Let X be a scheme. Then the following statements are equivalent for an $\mathscr{O}_X$-module $\mathscr{M}$.*

(1) *For all affine opens $U = \operatorname{Spec} R \subset X$, $\mathscr{M}|_U \cong \widetilde{M}$, for some R-module M.*
(2) *There exists an affine open cover $\{U_i = \operatorname{Spec} R_i\}$ of X such that for each i, there is an R_i-module M_i with $\mathscr{M}|_{U_i} \cong \widetilde{M_i}$.*
(3) *$\mathscr{M}$ is quasi-coherent.*

PROOF. (1) ⇒ (2) ⇒ (3) is easy. For (3) ⇒ (1), just observe that if $\mathscr{M}$ is quasi-coherent, then so is $\mathscr{M}|_U$, for any open set $U \subset X$. See [RS, 4.8]. Now apply Proposition (4.1.3). □

s-quasi-coherent-abelian

COROLLARY 4.2.2. *Let X be a scheme. If $\phi : \mathscr{M} \to \mathscr{N}$ is a morphism of quasi-coherent $\mathscr{O}_X$-modules, then $\ker \phi$, $\operatorname{coker} \phi$ and $\operatorname{im} \phi$ are all quasi-coherent. The category of quasi-coherent $\mathscr{O}_X$-modules is thus abelian.*

Proof. Since quasicoherence is a local condition, this can be checked locally. But there, this is clear, since the assignment $M \mapsto \widetilde{M}$ is an exact functor. □

uasi-coherent-closed-ext

Corollary 4.2.3. *Let X be a scheme. Then the category of quasi-coherent $\mathscr{O}_X$-modules is closed under extensions. That is, if we have an exact sequence*

$$0 \to \mathscr{M}_1 \to \mathscr{M}_2 \to \mathscr{M}_3 \to 0$$

of $\mathscr{O}_X$-modules, then, if any two of the modules are quasi-coherent, so is the third. In particular, $\mathscr{O}_X$-qcoh is a Serre sub-category of $\mathscr{O}_X$-mod.

Proof. Since quasicoherence is a local condition, we can assume that X is affine. Let $M_i = \Gamma(X, \mathscr{M}_i)$, and consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M_1} & \longrightarrow & \widetilde{M_2} & \longrightarrow & \widetilde{M_3} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{M}_1 & \longrightarrow & \mathscr{M}_2 & \longrightarrow & \mathscr{M}_3 & \longrightarrow & 0 \end{array}$$

We get the vertical maps from Proposition (4.1.4). In general, the top row will be left exact, but not necessarily exact on the right. First suppose $\mathscr{M}_1$ and $\mathscr{M}_2$ are quasi-coherent. Then the two vertical maps on the left are isomorphisms, by (4.1.3). Hence, the cokernel $\mathscr{M}_3$ will also be quasi-coherent, since it'll be isomorphic to $\widetilde{M_2/M_1}$. Similarly, if $\mathscr{M}_2$ and $\mathscr{M}_3$ are quasi-coherent, then $\mathscr{M}_1 = \ker(\widetilde{M_2 \to M_3})$ will also be quasi-coherent. It remains to consider the case where $\mathscr{M}_1$ and $\mathscr{M}_3$ are quasi-coherent. In this case, we have isomorphisms on the right and on the left, and the top row's actually exact by Proposition (4.1.5). Hence, the middle arrow is also an isomorphism by the 5-lemma. □

Remark 4.2.4. The above results show that our Standing Assumption from [RS, **??**] is true for quasi-coherent modules over a scheme X.

s-quasi-coherent-lattice

Corollary 4.2.5. *Let X be a scheme. If $\mathscr{N}_1$ and $\mathscr{N}_2$ are two quasi-coherent subsheaves of a quasi-coherent sheaf $\mathscr{M}$, then so are $\mathscr{N}_1 \cap \mathscr{N}_2$ and $\mathscr{N}_1 + \mathscr{N}_2$. That is, the quasi-coherent subsheaves of $\mathscr{M}$ form a sublattice of the lattice of subsheaves of $\mathscr{M}$.*

Proof. Observe that $\mathscr{N}_1 \cap \mathscr{N}_2$ is the kernel of the natural morphism $\mathscr{M} \to \mathscr{M}/\mathscr{N}_1 \oplus \mathscr{M}/\mathscr{N}_2$ and that $\mathscr{N}_1 + \mathscr{N}_2$ is the image of the natural morphism $\mathscr{N}_1 \oplus \mathscr{N}_2 \to \mathscr{M}$. □

herent-noetherian-scheme

Proposition 4.2.6. *If X is a locally Noetherian scheme, then the following are equivalent for an $\mathscr{O}_X$-module $\mathscr{M}$.*

(1) *For every affine open $U = \operatorname{Spec} R \subset X$, $\mathscr{M}|_U = \widetilde{M}$, for some finitely generated R-module M.*
(2) *There exists an affine open cover $\{V_i = \operatorname{Spec} R_i\}$ of X such that $\mathscr{M}|_{V_i} = \widetilde{M_i}$, for some finitely generated R_i-module M_i.*
(3) *$\mathscr{M}$ is coherent.*

In particular, if X is locally Noetherian, then $\mathscr{O}_X$ is a coherent sheaf of rings.

PROOF. (1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (3), observe that coherence is a local condition and employ (4.1.6). For (3) $\Rightarrow$ (1), just observe that $\mathscr{M}|_U$ is coherent as an $\mathscr{O}_U$-module and is thus isomorphic to $\widetilde{M}$ for some finitely generated R-module M, by (4.1.6). ☐

t-tensor-products-shfhom

PROPOSITION 4.2.7. *Let X be a scheme (resp. locally Noetherian scheme), and let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent (resp. coherent) $\mathscr{O}_X$-modules.*

(1) *$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ is also quasi-coherent (resp. coherent).*
(2) *If $\mathscr{M}$ is finitely presented, then $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is also quasi-coherent (resp. coherent).*

PROOF. The assertions in the general quasi-coherent case follow from (4.1.9) and (4.1.7). For the locally Noetherian case, we use these results in conjunction with (4.2.6). ☐

2.2. Nakayama's Lemma.

sms-nakayamas-lemma

PROPOSITION 4.2.8. *Let X be a scheme and let $\mathscr{M}$ be a quasi-coherent sheaf of finite type over X. Consider the function*

$$e(x) = \dim_{k(x)} \left(\mathscr{M}_x \otimes_{\mathscr{O}_{X,x}} k(x) \right)$$

on X.

(1) *e is upper semicontinuous on X.*
(2) *If X is reduced, then $\mathscr{M}$ is free in a neighborhood of $x \in X$ if and only if e is constant in a neighborhood of x.*

PROOF. (1) $e(x)$ is simply the minimal number of generators required to generate $\mathscr{M}_x$ over $\mathscr{O}_{X,x}$. Suppose $e(x) = n$, and let $s_1, \dots, s_n$ be sections of $\mathscr{M}$ over some neighborhood U of x such that their images in $\mathscr{M}_x$ generate $\mathscr{M}_x$ over $\mathscr{O}_{X,x}$. Let $\varphi : \mathscr{O}_U^n \to \mathscr{M}$ be the morphism defined by these sections, and let $\mathscr{C} = \operatorname{coker} \varphi$. We find that $\mathscr{C}_x = 0$, which implies that $\mathscr{C} = 0$ in a neighborhood of x. This shows upper semicontinuity.

(2) If $\mathscr{M}$ is free in a neighborhood of x, then clearly its rank is constant in a neighborhood of x. Conversely, suppose $e \equiv n$ in a neighborhood of x, and let $s_1, \dots, s_n$ be sections of $\mathscr{M}$ over an affine neighborhood U of x that generate $\mathscr{M}_x$. Let $\varphi : \mathscr{O}_U^n \to \mathscr{M}$ be the morphism determined by these sections; then, as above, φ is surjective in a neighborhood of x, which we might as well take to be U. Suppose $\mathscr{K} = \ker \varphi$ is non-zero, and let s be a non-zero section of $\mathscr{K}$ over some affine neighborhood V of x contained in U. Then, since X is reduced, there is some generic point $\xi \in V$ such that $s_\xi \neq 0$. But then $K_\xi \neq 0$, contradicting the fact that $\mathscr{M}_\xi$ has dimension n over $k(\xi) = \mathscr{O}_{X,\xi}$.

☐

REMARK 4.2.9. This will be greatly generalized in Chapter **??**.

As a corollary to this, we prove a useful lemma from commutative algebra.

-ring-freeness-criterion

COROLLARY 4.2.10. *Let $(R, \mathfrak{m})$ be a local domain, and let M be a finitely generated R-module. Let $k = R/\mathfrak{m}$ and $K = K(R)$ be the residue field and the quotient field of R respectively. Then M is free over R of rank r if and only if we have*

$$\dim_K(M \otimes_R K) = \dim_k(M \otimes_R k) = r.$$

Proof. Necessity is obvious; for sufficiency, consider the affine scheme $X = \operatorname{Spec} R$, and let ξ denote the generic point of R and x the closed point. Now, the subset $\{e \leq r\}$ is open and contains the closed point x; hence it must be the entire space X. Similarly, $\{e \geq r\}$ is closed and contains the generic point ξ, and must hence also be the entire space X. Thus we find that e has constant value r on all of X. Now, from part (2) of the Proposition, we conclude that $\widetilde{M}$ is free of rank r in a neighborhood of x; since the only neighborhood of x is X itself, we conclude that M must be a free R-module of rank r. □

2.3. Direct and Inverse Images.

dirinvimg-quasi-coherent

Proposition 4.2.11. *Let $f : X \to Y$ be a morphism of schemes.*

1. *If $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_Y$-module, then so is $f^*\mathscr{M}$.*
2. *If X is locally Noetherian, and $\mathscr{M}$ is in fact coherent, then so is $f^*\mathscr{M}$.*
3. *If f is quasi-compact and quasi-separated, and $\mathscr{N}$ is a quasi-coherent $\mathscr{O}_X$-module, then $f_*\mathscr{N}$ is quasi-coherent.*
4. *If Y is locally Noetherian, f is finite and $\mathscr{N}$ is coherent, then $f_*\mathscr{N}$ is also coherent.*

Remark 4.2.12. The conclusion of part (4) is valid for arbitrary proper morphisms; but this generalization will have to await coherent cohomology.

Proof. Most of this has been done.

1. See [RS, 4.8].
2. See [RS, 4.22], and use (4.2.6) for the fact that $\mathscr{O}_X$ is coherent.
3. The question is local, so we can assume Y is affine. Since f is quasi-compact and quasi-separated, we can find a finite affine open cover $\{U_i\}$ of X such that $U_i \cap U_j$ is also covered by finitely many affine opens $\{U_{ijk}\}$, for every pair of indices (i, j) (2.9.9). Now, for every affine open $V = U_i$ or $V = U_{ijk}$ we see that $f_*(\mathscr{N}|_V)$ is again quasi-coherent, by (4.1.11). But, by the sheaf axiom, we have an exact sequence
$$0 \to \mathscr{M} \to \oplus_i \mathscr{M}|_{U_i} \to \oplus_{i,j,k} \mathscr{M}|_{U_{ijk}}.$$
That is, a section of $\mathscr{M}$ can be given by giving sections over the U_i that agree on the intersections $U_i \cap U_j = \bigcup_k U_{ijk}$. Applying f_* to this sequence and using the fact that it's left exact (since it's right adjoint to f^* [RS, 2.24]), we get an exact sequence
$$0 \to f_*\mathscr{M} \to \oplus_i f_*(\mathscr{M}|_{U_i}) \to \oplus_{i,j,k} f_*(\mathscr{M}|_{U_{ijk}}).$$
Therefore, $f_*\mathscr{M}$ is the kernel of a map between quasi-coherent modules, and is thus quasi-coherent, by (4.2.2).
4. Again, since the question is local, we can assume $Y = \operatorname{Spec} R$ is affine, which implies that $X = \operatorname{Spec} S$ is also affine, with S finite over R. Let N be an S-module such that $\mathscr{N} = \widetilde{N}$; then since $\mathscr{N}$ is coherent, N must be finitely generated. By (4.1.11), $f_*\mathscr{N} = \widetilde{{}_R N}$. Since S is a finite R-module, we see that ${}_R N$ must also be a finite R-module, and hence $f_*\mathscr{N}$ is of finite type, and thus must be coherent, by (4.1.6).

□

Remark 4.2.13. In general, it's not true that the pushforward of a coherent sheaf is coherent, even when X and Y are of finite type over a field k. Consider for example the natural morphism $\operatorname{Spec} k[x] \to \operatorname{Spec} k$ given by the inclusion $k \hookrightarrow k[x]$. The pushforward of the structure sheaf $\mathscr{O}_{\operatorname{Spec} k[x]}$ is the constant sheaf $\underline{k[x]}$, which is not finitely generated over k, and is thus not a coherent $\mathscr{O}_{\operatorname{Spec} k}$-module. This map is of course not proper.

nvimg-commute-tensor-hom

PROPOSITION 4.2.14. *Let $f : X \to Y$ be a morphism of schemes, and let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{O}_Y$-modules. We have natural isomorphisms:*

(1) $f^*(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}) \cong f^*\mathscr{M} \otimes_{\mathscr{O}_X} f^*\mathscr{N}$.
(2) *If $\mathscr{M}$ is locally free, then*

$$f^*(\underline{\operatorname{Hom}}_{\mathscr{O}_Y}(\mathscr{M}, \mathscr{N})) \cong \underline{\operatorname{Hom}}_{\mathscr{O}_X}(f^*\mathscr{M}, f^*\mathscr{N}).$$

PROOF. (1) See [RS, 2.26].
(2) See [RS, 3.5].

□

This is a stronger version of [RS, **??**].

nit-dirinvimg-surjective

PROPOSITION 4.2.15. *Let $f : X \to Y$ be a quasi-compact, quasi-separated Y-scheme. Then, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, the following statements are equivalent:*

(1) *There is a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a surjective morphism $\gamma : f^*\mathscr{E} \to \mathscr{M}$.*
(2) *The natural morphism $\varepsilon_{\mathscr{M}} : f^*f_*\mathscr{M} \to \mathscr{M}$ is surjective.*
(3) *For every affine open $U \subset Y$, $\mathscr{M}|_{f^{-1}(U)}$ is generated by sections over $f^{-1}(U)$.*

PROOF. Since any such γ must factor through $\varepsilon_{\mathscr{M}}$, it's immediate that (1) and (2) are equivalent. That (3) ⇒ (1) is clear (take $\mathscr{E} = \mathscr{O}_Y^I$, for some suitable indexing set I). To show that (2) ⇒ (3), we might as well assume that $Y = \operatorname{Spec} R$ is affine. In this case, given any section $s \in \mathscr{F}(U)$, and any $x \in U$, we can find $r \in R$, such that $x \in X_g$, where $g = \varphi(r)$, where $\varphi : R \to \Gamma(X, \mathscr{O}_X)$ is the map induced by f. Moreover, we can make X_g small enough that we can use [RS, **??**] to find finitely many sections $t_i \in \mathscr{F}(X_g)$ such that

$$s|_W = \sum_i a_i \cdot (t_i|_W),$$

where $a_i \in \Gamma(W, \mathscr{O}_X)$, and where $W \subset X_g \cap U$ is also of the form X_h, for some $h \in \Gamma(X, \mathscr{O}_X)$. Now, by (4.1.2), $\mathscr{F}(X_g) = \mathscr{F}(X)_g$. From this, it's clear that up to units, $s|_W$ can be described as a linear combination of restrictions of global sections. □

DEFINITION 4.2.16. With the notation of the Proposition, if $\mathscr{M}$ satisfies any of its three equivalent conditions, we say that $\mathscr{M}$ is *relatively generated by global sections*. The rationale behind this definition should be clear from the Proposition.

2.4. External Tensor Products.

DEFINITION 4.2.17. Let $f : X \to S$ and $g : Y \to S$ be S-schemes, and let $\mathscr{F}$ be a sheaf over X, and let $\mathscr{G}$ be a sheaf over Y. Let $W = X \times_S Y$, and let $\pi_1 : W \to X$ and $\pi_2 : W \to Y$ be its two natural projections. The *external tensor product* of $\mathscr{F}$ and $\mathscr{G}$ over $\mathscr{O}_S$, denoted by $\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{G}$, is the sheaf $\pi_1^*\mathscr{F} \otimes_{\mathscr{O}_W} \pi_2^*\mathscr{G}$ over W.

REMARK 4.2.18. We find from (4.2.7) and (4.2.11) that, if $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent, then so is $\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{G}$.

2.5. Extensions of Sheaves of Finite Type. We'll now prove a technical result that'll be useful later.

-extension-fingen-affine

LEMMA 4.2.19. *Let* $X = \operatorname{Spec} R$ *be an affine scheme, and let* $U \subset X$ *be an open, quasi-compact subscheme. Suppose* $\mathscr{F}$ *is a module of finite type over* $\mathscr{O}_U$*, and suppose* $\mathscr{G}$ *is a quasi-coherent* $\mathscr{O}_X$*-module over* X *such that* $\mathscr{F} \subset \mathscr{G}|_U$*. Then there is a* $\mathscr{O}_X$*-submodule* $\mathscr{F}' \subset \mathscr{G}$ *of finite type such that* $\mathscr{F}'|_U \cong \mathscr{F}$*.*

PROOF. Let $i : U \to X$ be the inclusion map. Then, i is separated and quasi-compact, and so $i_*\mathscr{F}$ is quasi-coherent, by (4.2.11). Now, there is a natural isomorphism $\eta : \mathscr{G} \to i_*(\mathscr{G}|_U)$. Let $\mathscr{E} = \eta^{-1}(i_*\mathscr{F})$; we see immediately that $\mathscr{E}$ is quasi-coherent, and so $\mathscr{E} = \widetilde{E}$, for some R-module E. Since $(i_*\mathscr{F})|_U$, we see that $\mathscr{E}|_U = \mathscr{F}$. Since U is quasi-compact and $\mathscr{F}$ is of finite type, we can find finitely many elements $f_1, \ldots, f_n \in R$ such that $U = \bigcup_i X_{f_i}$ and E_{f_i} is a finitely generated R_{f_i}-module, for each i. Since E is the direct limit of its finitely generated R-submodules, we can find a finitely generated submodule $F' \subset E$ such that $F'_{f_i} = E_{f_i}$, for each i. This shows that $\mathscr{F}' = \widetilde{F} \subset \mathscr{G}$ is a submodule of finite type such that $\mathscr{F}'|_U \cong \mathscr{F}$. □

nsions-of-fingen-sheaves

THEOREM 4.2.20. *Let* X *be a quasi-compact scheme,* $U \subset X$ *an open subscheme such that the inclusion map* $i : U \to X$ *is quasi-compact, and* $\mathscr{F}$ *a* $\mathscr{O}_U$*-module of finite type. Let* $\mathscr{G}$ *be a quasi-coherent* $\mathscr{O}_X$*-module such that* $\mathscr{F} \subset \mathscr{G}|_U$*. Then, there is a finitely generated subsheaf* $\mathscr{F}' \subset \mathscr{G}$ *such that* $\mathscr{F}'|_U \cong \mathscr{F}$*.*

PROOF. Cover X with finitely many affine opens $\{U_1, \ldots, U_n\}$; then, for each i, $U \cap U_i$ is either empty or a quasi-compact open subscheme of U_i. We do induction on n: the base case is the one where X is affine, and this was dealt with in the lemma above. So suppose $n > 1$, and let $X' = \bigcup_{i=1}^{n-1} U_i$; then, by induction, there is a quasi-coherent subsheaf $\mathscr{F}_1 \subset \mathscr{G}|_{X'}$ of finite type such that $\mathscr{F}_1 \cap_{U \cap X'} \cong \mathscr{F}|_{U \cap X'}$. Now, if $X' \cap U_n = \emptyset$, then we can use the Lemma again to extend $\mathscr{F}|_{U \cap U_n}$ to any submodule of finite type of $\mathscr{G}|_{U_n}$ and thus finish our proof. If not, then let $\mathscr{F}_2 \subset \mathscr{G}|_{U_n}$ be a quasi-coherent submodule of finite type such that $\mathscr{F}_2|_{U_n \cap X'} \cong \mathscr{F}_1|_{U_n \cap X'}$. Now we can glue $\mathscr{F}_2$ and $\mathscr{F}_1$ along $U_n \cap X'$ to find our $\mathscr{F}'$. □

sions-of-fintype-sheaves

COROLLARY 4.2.21. *Let* X *be a quasi-compact scheme,* $U \subset X$ *an open subscheme such that the inclusion map* $i : U \to X$ *is quasi-compact, and* $\mathscr{F}$ *a quasi-coherent* $\mathscr{O}_U$*-module of finite type. Then there exists a quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{F}'$ *of finite type such that* $\mathscr{F}'|_U \cong \mathscr{F}$*.*

PROOF. Take $\mathscr{G} = i_*\mathscr{F}$ in the Theorem above. □

coherent-dirlimit-fingen

COROLLARY 4.2.22. *Let* X *be a Noetherian scheme. Then every quasi-coherent* $\mathscr{O}_X$*-module is the direct limit of its coherent* $\mathscr{O}_X$*-submodules.*

PROOF. Let $\mathscr{G}$ be a quasi-coherent sheaf over X, and let $s \in \Gamma(\mathscr{G}, U)$ be a section of $\mathscr{G}$ over some open subscheme $U \subset X$, and let $\mathscr{F}$ be the subsheaf of $\mathscr{G}|_U$ generated by s. Now, since X is Noetherian, its underlying topological space is also Noetherian (1.5.4), and so the inclusion $i : U \to X$ satisfies the hypotheses of the Theorem. Hence we can find a submodule $\mathscr{F}' \subset \mathscr{G}$ of finite type such that $\mathscr{F}'|_U \cong \mathscr{F}$. Now let $\mathscr{G}' \subset \mathscr{G}$ be the submodule of $\mathscr{G}$ that is the direct limit of the submodules of $\mathscr{G}$ of finite type. Then we have shown that $\mathscr{G}'_x = \mathscr{G}_x$, for all $x \in X$, and so $\mathscr{G}' = \mathscr{G}$. □

Let R be a Noetherian ring. Recall the definition of an associated prime of R: It's a prime $P \subset R$ such that R/P embeds into R. How would we rephrase this geometrically in terms of $\operatorname{Spec} R$? Let $(s) \subset R$ be the principal ideal whose annihilator is P; it follows that s maps to 0 in R_Q, for some prime $Q \subset R$ if and only if $P \not\subseteq Q$. In other words, treating s as a global section of the structure sheaf on $\operatorname{Spec} R$, $\operatorname{Supp} s$ is precisely the closed subset $V(P) = \overline{\{y\}}$, where $y \in \operatorname{Spec} R$ is the point corresponding to P.

2.6. Associated Points. Recall that, for a finitely generated module M over a Noetherian ring R, the associated primes of M are the ones that are annihilators of some element in M. Now we want to globalize this notion.

Definition 4.2.23. Let X be a locally Noetherian scheme, and let $\mathscr{M}$ be a coherent sheaf over X. A point $x \in X$ is an *associated point of* $\mathscr{M}$ if there is some open subscheme $U \subset X$ containing x, and some section $s \in \Gamma(U, \mathscr{M})$ such that $\operatorname{Supp} s = \overline{\{x\}} \cap U$.

ated-point-affine-scheme

Proposition 4.2.24. *Let $X = \operatorname{Spec} R$ be an affine scheme, and let M be a finite R-module; then $x \in X$ is an associated point of $\widetilde{M}$ if and only if it corresponds to an associated prime of M.*

Proof. Suppose $x \in X$ is an associated point of $\widetilde{M}$; let $U \subset X$ be an open neighborhood of x, and let $s \in \Gamma(U, \widetilde{M})$ be a section such that $\operatorname{Supp} s = \overline{\{x\}} \cap U$. By replacing U with a smaller principal affine neighborhood X_a, we can assume that $U = \operatorname{Spec} R_a$ is affine, and we can consider s to be an element of M_a. In this case,

$$\operatorname{Supp} s = \{Q \subset R_a : (s)_Q \neq 0\} = V(\operatorname{ann}(s)).$$

Let $P \subset R_a$ be the prime corresponding to x; then we see that $\operatorname{rad}(\operatorname{ann}(s)) = P$, since $V(\operatorname{ann}(s)) = V(P)$. Now, by a standard argument from Atiyah-Macdonald, we conclude that P is in fact an associated prime of M_a, and hence is the localization of some associated prime of M. In particular, x corresponds to an associated prime of M.

Now, conversely, suppose $x \in X$ corresponds to an associated prime P of M; then by definition there exists $s \in M$ such that $P = \operatorname{ann}(s)$, and so $\operatorname{Supp} s = V(P) = \overline{\{x\}}$. □

Definition 4.2.25. The associated points of the structure sheaf $\mathscr{O}_X$ are called the *associated points of X*

sms-associated-points

Proposition 4.2.26. *Let X be a locally Noetherian scheme, and let $\mathscr{M}$ be a coherent sheaf over X.*

(1) *The generic points of X are associated points of X.*
(2) *If X is reduced, then the generic points are precisely the associated points of X.*
(3) *If X is quasi-compact, then $\mathscr{M}$ has only finitely many associated points.*
(4) *$x \in X$ is an associated point for $\mathscr{M}$ if and only if every non-unit in $\mathscr{O}_{X,x}$ is a zero divisor for $\mathscr{M}_x$. That is, if and only if* $\operatorname{depth} \mathscr{M}_x = 0$.

Proof. We fix an affine open cover $\{U_i = \operatorname{Spec} R_i\}$ for X.

(1) Any generic point ξ of X is also a generic point for one of the U_i, and so corresponds to a minimal prime in R_i. But every minimal prime in R_i is also an associated prime. From this, and (4.2.24), the result follows.

(2) Let $x \in X$ be an associated point; then x is also an associated point for some U_i, and hence corresponds to an associated prime of R_i. But R_i is reduced, and hence all the associated primes of R_i are minimal [CA, 4.3.4].
(3) In this case, the affine open cover is finite, and each U_i has only finitely many associated points of $\mathscr{M}$, since any finite module over a Noetherian ring has only finitely many associated primes.
(4) Again, we can assume $X = \operatorname{Spec} R$ is affine; in this case $\mathscr{O}_{X,x} = R_P$, and $P \in \operatorname{Ass} M$ if and only if $P_P \in \operatorname{Ass} M_P$ if and only if every non-unit in R_P is a zero-divisor of M_P. For the final equivalence, just observe that $P_P \subset \mathcal{Z}(R_P)$, and so P_P is contained in some associated prime of R_P, which of course implies that $P_P \in \operatorname{Ass} R_P$.

□

Definition 4.2.27. An associated point of a scheme X is called an *embedded point* if it is not a generic point.

2.7. Flatness.

Definition 4.2.28. Let X be a scheme, and let $\mathscr{M}$ be a quasi-coherent sheaf over X. We say that $\mathscr{M}$ is *flat over* X if $\mathscr{M}_x$ is flat over $\mathscr{O}_{X,x}$, for all $x \in X$.

More generally, if $f : X \to Y$ is a morphism of schemes and $\mathscr{M}$ is a quasi-coherent sheaf over X, we say that $\mathscr{M}$ is *flat over* Y if $\mathscr{M}_x$ is flat over $\mathscr{O}_{Y,f(x)}$, for all $x \in X$.

sms-flatness-equiv-prps

Proposition 4.2.29. *Let $f : X \to Y$ be a morphism of schemes; then the following are equivalent for a quasi-coherent sheaf $\mathscr{M}$ over X:*
(1) *$\mathscr{M}$ is flat over Y.*
(2) *For every affine open $U = \operatorname{Spec} R$ of Y, we have $f_*\mathscr{M}|_U \cong \widetilde{N}$, for some flat R-module N.*

If, in addition, $\mathscr{M}$ is of finite type, then the two statements above are equivalent to the following:
(1) *$f_*\mathscr{M}$ is locally free over Y.*
(2) *The function*

$$e(y) = \dim_{k(y)}\big((f_*\mathscr{M})_y \otimes_{\mathscr{O}_{Y,y}} k(y)\big)$$

is locally constant.

Proof. Clearly (2) implies (1); so we'll show the converse. Assume that $\mathscr{M}$ is flat over Y; we can suppose $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$ are both affine, and then show that $\mathscr{M} = \widetilde{M}$, for some S-module M flat over R. But this follows immediately from [CA, 3.1.10].

For the final pair of equivalences, we use [CA, 3.3.8] in conjunction with (4.2.8).
□

Now, with some Noetherian hypotheses, we can give more interesting properties of flatness.

atness-associated-points

Proposition 4.2.30. *Let $f : X \to Y$ be a morphism of locally Noetherian schemes, and suppose $\mathscr{M}$ is a coherent sheaf over X that is flat over Y. A point $x \in X$ is associated to $\mathscr{M}$ if and only if the following conditions hold:*
(1) *$f(x)$ is associated to Y.*

(2) *x is associated to the sheaf $\mathscr{M} \otimes_{\mathscr{O}_Y} k(y)$ over X_y.*

Proof. Since $\mathscr{M}$ is flat over Y, we have, by [CA, 10.5.2], the identity:

$$\operatorname{depth} \mathscr{M}_x = \operatorname{depth} \mathscr{O}_{Y,f(x)} + \operatorname{depth}(\mathscr{M}_x \otimes_{\mathscr{O}_{Y,f(x)}} k(y)).$$

From this the equivalence follows immediately. □

The next result will be useful in the study of flat families in Chapter **??**.

atness-criterion-one-dim Corollary 4.2.31. *Suppose that in the Proposition above Y is a Dedekind scheme. Then $\mathscr{M}$ is flat over Y if and only if every associated point of $\mathscr{M}$ in x maps to the generic point of Y. In particular, if X is reduced, then f is a flat morphism if and only if every irreducible component of X dominates Y.*

Proof. The question is local, and so follows immediately from [CA, 7.5.7]. □

sms-generic-flatness Theorem 4.2.32 (Generic Flatness). *Let $f : X \to Y$ be a dominant morphism of finite type between two integral, locally Noetherian schemes, and let $\mathscr{M}$ be a coherent sheaf over X. Then there is an open subscheme $V \subset Y$ such that $\mathscr{M}|_{f^{-1}(V)}$ is flat over V.*

Proof. There is no harm in assuming that $Y = \operatorname{Spec} R$ is affine, with R some Noetherian domain and thus that X is in fact Noetherian, covered by finitely many affine opens $\{U_1, \ldots, U_n\}$, with $U_i = \operatorname{Spec} S_i$, where S_i is a finitely generated R-algebra. Now, by [CA, 8.2.1], there is for each i an open subscheme (and in fact a principal open subscheme) $V_i \subset Y$ such that $\mathscr{M}|_{f^{-1}(V_i)}$ is flat over V_i. We finish the proof by taking $V = \cap_i V_i$. □

3. Global Spec

3.1. The Construction. Suppose we're given an affine morphism $f : Y \to X$; then since affine morphisms are quasi-compact and separated (2.12.1), we see from (4.2.11) that $f_* \mathscr{O}_Y$ is a quasi-coherent $\mathscr{O}_X$-module. Thus it is a quasi-coherent $\mathscr{O}_X$-algebra by the following definition.

Definition 4.3.1. An $\mathscr{O}_X$-algebra $\mathscr{A}$ is *quasi-coherent* if it's quasi-coherent as an $\mathscr{O}_X$-algebra.

Note on Notation 3. In a slight change of terminology, we will now refer to any affine morphism $Y \to X$ as an affine X-scheme.

This process of obtaining quasi-coherent algebras is reversible.

-affine-morphisms-qc-alg Proposition 4.3.2 (Definition). *For every $\mathscr{O}_X$-algebra $\mathscr{A}$, there is, upto isomorphism of X-schemes, a unique affine morphism $f : \operatorname{Spec} \mathscr{A} \to X$ such that $f_* \mathscr{O}_{\operatorname{Spec} \mathscr{A}} = \mathscr{A}$.* Spec $\mathscr{A}$ *is called the* global Spec *associated to the algebra $\mathscr{A}$.*

Proof. First, suppose $f : Y \to X$ and $g : Y' \to X$ are two affine morphisms such that $f_* \mathscr{O}_Y \cong g_* \mathscr{O}_{Y'}$ as $\mathscr{O}_X$-algebras. Then, for every affine open $V = \operatorname{Spec} R$, we have $f_* \mathscr{O}_V = g_* \mathscr{O}_V$. But of course $f_* \mathscr{O}_V = \mathscr{O}_Y(\widetilde{f^{-1}(V)})$, and $g_* \mathscr{O}_V = \mathscr{O}_{Y'}(\widetilde{g^{-1}(V)})$. Since both f, g are affine, we have rings S and S' such that $f^{-1}(V) = \operatorname{Spec} S$ and $g^{-1}(V) = \operatorname{Spec} S'$. Since

$$S = (f_* \mathscr{O}_V)(X) \cong (g_* \mathscr{O}_V)(X) = S',$$

we see that S and S' are isomorphic as R-algebras, and so there is a unique isomorphism $f^{-1}(V) = \operatorname{Spec} S' \to \operatorname{Spec} S = g^{-1}(V)$ of V-schemes (and hence a unique isomorphism of X-schemes). We glue these together to get a unique isomorphism $Y \to Y'$ of X-schemes. This shows the uniqueness of the X-scheme $\operatorname{Spec} \mathscr{A} \to X$. Now we'll show existence.

Step 1: First, suppose we've defined $f : \operatorname{Spec} \mathscr{A} \to X$. Then, for any open subset $V \subset X$, $f : f^{-1}(V) \to V$ is precisely the morphism $\operatorname{Spec}(\mathscr{A}|_V) \to V$. It's easy to check that it satisfies $f_* \mathscr{O}_{f^{-1}(V)} = \mathscr{A}|_V$, and by the proof of uniqueness above, we have our result.

Step 2: Now, suppose we have an open cover $\{V_i\}$ of X, and global Specs $f_i : \operatorname{Spec}(\mathscr{A}|_{V_i}) \to V_i$ for each i. By uniqueness $f_i^{-1}(V_i \cap V_j) \to V_i \cap V_j$ and $f_j^{-1}(V_i \cap V_j) \to V_i \cap V_j$ are both global Specs associated to $\mathscr{A}|_{V_i \cap V_j}$ and are thus uniquely isomorphic over X. This lets us glue together all the f_i along these unique isomorphisms to get a global Spec $f : \operatorname{Spec} \mathscr{A} \to X$ associated to $\mathscr{A}$.

Step 3: So it only remains to construct the global Spec for a quasi-coherent algebra $\mathscr{A}$ over an affine scheme $\operatorname{Spec} R$. But this is easy: since the algebra is quasi-coherent, it corresponds to $\widetilde{S}$ for some R-algebra S. It's immediate then that $\operatorname{Spec} S \to \operatorname{Spec} R$ defines the global Spec associated to $\mathscr{A}$.

□

uiv-qc-alg-aff-morphisms PROPOSITION 4.3.3. *The assignment* $\mathscr{A} \mapsto \operatorname{Spec} \mathscr{A}$ *gives a contravariant equivalence from the category of quasi-coherent* $\mathscr{O}_X$*-algebras to the category of affine* X*-schemes.*

REMARK 4.3.4. If $X = \operatorname{Spec} \mathbb{Z}$, this is just the usual contravariant equivalence from Ring to Sch.

PROOF. First we should show that this assignment actually gives us a functor. Let $f : \operatorname{Spec} \mathscr{A} \to X$ and $g : \operatorname{Spec} \mathscr{B} \to X$ be global Specs associated to $\mathscr{O}_X$-algebras $\mathscr{A}$ and $\mathscr{B}$. Suppose $\phi : \mathscr{A} \to \mathscr{B}$ is a morphism of $\mathscr{O}_X$-algebras. Let $V = \operatorname{Spec} R \subset X$ be an affine open; then $f^{-1}(V) = \operatorname{Spec} S$ and $g^{-1}(V) = \operatorname{Spec} S'$, where $S = \mathscr{A}(V)$ and $S' = \mathscr{B}(V)$. The ring map ϕ thus gives us a morphism $h_V : g^{-1}(V) \to f^{-1}(V)$. It's clear that for any principal open $V_a \subset V$, the restriction of h_V to $g^{-1}(V_a)$ will be the same as the morphism h_{V_a}, since they're both induced by the R-algebra maps $S_a \to S'_a$. Hence, we can glue together these h_V to get a morphism $\operatorname{Spec} \phi : \operatorname{Spec} \mathscr{B} \to \operatorname{Spec} \mathscr{A}$. From the local definitions, it's immediate that this behaves functorially under composition.

We also have a functor in the other direction that takes any affine morphism $f : Y \to X$ to the $\mathscr{O}_X$-algebra $f_* \mathscr{O}_Y$. Now, by the uniqueness of the global Spec we see that $\operatorname{Spec}(f_* \mathscr{O}_Y)$ is isomorphic to Y as an X-scheme, and by definition we have $f_* \mathscr{O}_{\operatorname{Spec} \mathscr{A}} = \mathscr{A}$. Hence the two functors give us the contravariant equivalence we seek. □

Global Specs behave well under base change.

-global-spec-base-change PROPOSITION 4.3.5. *Suppose* $\operatorname{Spec} \mathscr{A} \to X$ *is an affine* X*-scheme, and let* $f : Y \to X$ *be any other* X*-scheme. Then we have the following isomorphism of affine* Y*-schemes.*

$$\operatorname{Spec} \mathscr{A} \times_X Y \cong \operatorname{Spec} f^* \mathscr{A}$$

PROOF. Since affine morphisms are stable under base change, we know that $\operatorname{Spec}\mathscr{A}\times_X Y\to Y$ is of the form $\operatorname{Spec}\mathscr{B}\to Y$ for some quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{B}$. Now it's enough to see what $\mathscr{B}$ should be. This we can do locally: so let $X=\operatorname{Spec}R$, $\mathscr{A}=\widetilde{A}$, for some R-algebra A, and $Y=\operatorname{Spec}S$. Then, we see that

$$\mathscr{B}=\widetilde{A\otimes_R S}=f^*\widetilde{A}=f^*\mathscr{A},$$

where we used (4.1.11). □

e-space-polynomial-gspec

EXAMPLE 4.3.6. Consider affine space $\mathbb{A}^n_X$ over a scheme X. It's defined as the fiber product $\mathbb{A}^n_{\mathbb{Z}}\times_{\mathbb{Z}}X$. Observe now that $\mathbb{A}^n_{\mathbb{Z}}\to\operatorname{Spec}\mathbb{Z}$ is just the morphism

$$\operatorname{Spec}\mathbb{Z}[x_1,\dots,x_n]\to\operatorname{Spec}\mathbb{Z},$$

which is of course an affine morphism induced by the quasi-coherent algebra $\operatorname{Sym}(\mathscr{O}^n_{\operatorname{Spec}\mathbb{Z}})$ on $\operatorname{Spec}\mathbb{Z}$. By the Proposition, we immediately see that $\mathbb{A}^n_X$ is $\operatorname{Spec}\mathscr{B}$, where

$$\mathscr{B}=f^*(\operatorname{Sym}(\mathscr{O}^n_{\operatorname{Spec}\mathbb{Z}}))=\operatorname{Sym}(f^*\mathscr{O}^n_{\operatorname{Spec}\mathbb{Z}})=\operatorname{Sym}(\mathscr{O}^n_X).$$

In sum we see that

$$\mathbb{A}^n_X=\operatorname{Spec}(\operatorname{Sym}(\mathscr{O}^n_X)).$$

Global Specs behave very much like regular affine schemes. In fact, one should just think of them as relativized affine schemes (which is precisely what they are, in any case), i.e. affine schemes in the category Sch_X. The following Proposition is analogous to (1.2.2).

ushforward-gspec-adjoint

PROPOSITION 4.3.7. *For every X-scheme Y, and every quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}$, we have a natural bijection*

$$\operatorname{Hom}_{\mathrm{Sch}_X}(Y,\operatorname{Spec}\mathscr{A})\cong\operatorname{Hom}_{\mathscr{O}_X\text{-}alg}(\mathscr{A},f_*\mathscr{O}_Y).$$

PROOF. Let $\{V_i\}$ be an affine open cover for X. Then, by (1.2.2) and (4.1.12), we have a natural bijection

$$\operatorname{Hom}_{\mathrm{Sch}_{V_i}}(f^{-1}(V_i),g^{-1}(V_i))\cong\operatorname{Hom}_{\mathscr{O}_{V_i}\text{-alg}}(\mathscr{A}|_{V_i},f_*\mathscr{O}_Y|_{V_i}),$$

where $g:\operatorname{Spec}\mathscr{A}\to X$ is the structure morphism. We're using the fact that $g^{-1}(V_i)\to V_i$ is an affine morphism. Now, given an $\mathscr{O}_X$-algebra morphism $\mathscr{A}\to f_*\mathscr{O}_Y$, we can define morphisms of X-schemes $f^{-1}(V_i)\to g^{-1}(V_i)$ that glue together to give a morphism $Y\to\operatorname{Spec}\mathscr{A}$. The fact that this assignment is a bijection on each affine open V_i ensures that it remains a bijection when globalized thus. Or, one can also observe that there is already a natural map in the other direction that just specializes a morphism of X-schemes to the maps on the global section algebras over $\mathscr{O}_X(X)$. □

phisms-into-affine-space

EXAMPLE 4.3.8. Take $X=k$ and $\mathscr{A}=\widetilde{\operatorname{Sym}(k^n)}$, for some $n\in\mathbb{N}$. Then, what we see above is that homomorphisms $\operatorname{Sym}(k^n)\to\Gamma(Y,\mathscr{O}_Y)$ are in one-to-one correspondence with morphisms of k-schemes $Y\to\mathbb{A}^n_k$. Now, consider the set $Y(k)=\operatorname{Hom}_{\mathrm{Sch}_k}(k,Y)$ of k-valued points. Observing that $\mathbb{A}^n_k(k)=k^n$, we want to figure out what the map induced from $Y(k)$ to k^n is. So looking at $\operatorname{Sym}(k^n)$ as the polynomial ring $k[x_1,\dots,x_n]=:S$, a map $S\to\Gamma(Y,\mathscr{O}_Y)$ is given by n global sections $s_1,\dots,s_n$ over Y, and let $f:Y\to\mathbb{A}^n_k$ be the morphism induced by that map. Then, the map $Y(k)\to k^n$ is given by $\varphi\mapsto f\circ\varphi$. Now, φ corresponds simply to a point $y\in Y$, with $k(y)=k$. To figure out what $f(y):=f\circ\varphi$ is, we can assume that $Y=\operatorname{Spec}A$ is affine, and consider $s_1,\dots,s_n$ as elements of A.

Set $s_i(y) = (s_i)_y$; we claim that $f(y)$ corresponds to the n-tuple $(s_1(y), \dots, s_n(y))$. For this, it's enough to show that under the homomorphism $\psi : k[x_1, \dots, x_n] \to A$, which takes x_i to s_i, the prime $P \subset A$ corresponding to y contracts to the maximal ideal generated by the elements $x_i - s_i(y)$. But this follows from the fact that $\psi(x_i - s_i(y)) = s_i - s_i(y) \in P$, for all i.

sms-linear-projections

EXAMPLE 4.3.9. Let $Y = \mathbb{A}^n_k$; consider any r-tuple $(s_1, \dots, s_r)$ of linear polynomials in $k[x_1, \dots, x_n]$. Treating this as an r-tuple of global sections, we get a morphism from Y to $\mathbb{A}^r_k$. Such a morphism is known as a *linear map*. If we had chosen the r-tuple to be linearly independent, then the morphism will induce a surjective map from $Y(k)$ to $\mathbb{A}^r_k(k)$. Thus, this is a *linear projection*, with center the hyperplane $H \subset Y$ cut out by the ideal $(s_1, \dots, s_r) \subset k[x_1, \dots, x_n]$. In fact, if we complete $(s_1, \dots, s_r)$ to a basis $(s_1, \dots, s_n)$ of the space of linear functions on Y, then every point y in $Y(k)$ is uniquely determined by the n-tuple $(s_1(y), \dots, s_n(y)) \in k^n$. The projection map then simply takes this n-tuple to $(s_1(y), \dots, s_r(y)) \in \mathbb{A}^r_k(k) = k^r$.

In general, given any affine scheme X of finite type over k, there is a closed embedding of X in $\mathbb{A}^n_k$ into Y. A morphism $f : X \to \mathbb{A}^r_k$ is *linear* if it is the restriction of a linear map $\mathbb{A}^n_k \to \mathbb{A}^r_k$.

3.2. Modules over Quasi-coherent Algebras. Now, we study modules over the scheme $\operatorname{Spec} \mathscr{A}$. Given an $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$-module $\mathscr{M}$, we can push it forward to get an $\mathscr{O}_X$-module $f_*\mathscr{M}$. Since f is affine, and is, in particular, quasi-compact and separated, we see that the push forward of a quasi-coherent $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$-module will again be quasi-coherent as an $\mathscr{O}_X$-module. But in fact $f_*\mathscr{M}$ has the natural structure of an $\mathscr{A}$-module, since $\mathscr{A} = f_*\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$. So the push forward gives us a functor $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$-mod to $\mathscr{A}$-mod.

Conversely, if we're given an $\mathscr{A}$-module $\mathscr{N}$, then first observe that $\{f^{-1}(V) : V \subset X \text{ open}\}$ is an open base for the topology on $\operatorname{Spec} \mathscr{A}$. So we can define what for now we'll call an $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$-module presheaf on a base, by setting $\widetilde{N}(f^{-1}(V)) = \mathscr{N}(V)$, and observing that from the $\mathscr{A}$-module structure on $\mathscr{N}$, we get the module structure morphisms of presheaves on a base

$$\mathscr{A}(f^{-1}(V)) \times \widetilde{N}(f^{-1}(V)) \to \widetilde{N}(f^{-1}(V)).$$

Now, we can first extend $\widetilde{N}$ to an honest sheaf, and then extend the module structure morphisms also to morphisms of sheaves to get an $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$-module $\widetilde{N}$.

It's clear that these two assignments (the push forward and the tilde functor) are inverse to each other. This gives us the following Proposition.

chemes-tilde-equivalence

PROPOSITION 4.3.10. *For any affine X-scheme $\operatorname{Spec} \mathscr{A}$, where $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_X$-algebra, the assignment $\mathscr{M} \to \widetilde{\mathscr{M}}$ induces an equivalence of categories between the category of quasi-coherent $\mathscr{A}$-modules and the category of quasi-coherent $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$-modules. Now, let $\mathscr{M}$ be a quasi-coherent $\mathscr{A}$-module, and let $f : \operatorname{Spec} \mathscr{A} \to \operatorname{Spec} \mathscr{B}$ be a morphism of X-schemes, where $\mathscr{B}$ is another quasi-coherent $\mathscr{O}_X$-algebra. Observe that f is also affine, and that it corresponds naturally to a morphism of $\mathscr{O}_X$-algebras $\mathscr{B} \to \mathscr{A}$.*

(1) *If $\mathscr{N}$ is any $\mathscr{O}_{\operatorname{Spec} \mathscr{A}}$-module then we have a natural isomorphism*

$$\operatorname{Hom}_{\mathscr{O}_{\operatorname{Spec} \mathscr{A}}}(\widetilde{\mathscr{M}}, \mathscr{N}) \cong \operatorname{Hom}_{\mathscr{A}\text{-}mod}(\mathscr{M}, f_*\mathscr{N}).$$

(2) *If $\mathscr{N}$ is another quasi-coherent $\mathscr{A}$-module, then we have natural morphisms of $\mathscr{O}_{\operatorname{Spec} A}$-modules:*

$$\widetilde{\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}} \longrightarrow \widetilde{\mathscr{M}} \otimes_{\mathscr{O}_{\operatorname{Spec}\mathscr{A}}} \widetilde{\mathscr{N}};$$

$$\widetilde{\underline{\operatorname{Hom}}_{\mathscr{A}\text{-}mod}(\mathscr{M}, \mathscr{N})} \longrightarrow \underline{\operatorname{Hom}}_{\mathscr{O}_{\operatorname{Spec}\mathscr{A}}}(\widetilde{\mathscr{M}}, \widetilde{\mathscr{N}}).$$

The first morphism is always an isomorphism; the second morphism is an isomorphism whenever $\mathscr{M}$ is finitely presented.

(3) *If $\mathscr{N}$ is now a quasi-coherent $\mathscr{B}$-module, then we have natural isomorphisms*

$$f_*\widetilde{\mathscr{M}} \cong \widetilde{{}_{\mathscr{B}}\mathscr{M}};$$

$$f^*\widetilde{\mathscr{N}} \cong \widetilde{\mathscr{N} \otimes_{\mathscr{B}} \mathscr{A}},$$

where ${}_{\mathscr{B}}\mathscr{M}$ is $\mathscr{M}$ looked upon as a $\mathscr{B}$-module under the natural morphism $\mathscr{B} \to \mathscr{A}$. In particular, if $\mathscr{B} = \mathscr{O}_X$, and $\mathscr{N}$ is a quasi-coherent $\mathscr{O}_X$-module, then

$$f^*\mathscr{N} \cong \widetilde{\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{A}}.$$

rphism-pushforward-exact

COROLLARY 4.3.11. *Let $f : X \to Y$ be an affine morphism; then $f_* : \mathscr{O}_X$-qcoh $\to \mathscr{O}_Y$-qcoh is an exact functor.*

PROOF. Indeed, the last Proposition tells us that f_* actually gives us an equivalence with the subcategory $f_*\mathscr{O}_X$-qcoh of $\mathscr{O}_Y$-qcoh □

3.3. Quasi-coherent Ideal Sheaves. Many constructions that we encountered before can be clarified through the consistent use of ideal sheaves. For example, suppose we're given a closed immersion $i : Z \to X$. Then, we have a short exact sequence

$$0 \to \ker i^\sharp \to \mathscr{O}_X \xrightarrow{i^\sharp} i_*\mathscr{O}_Z \to 0.$$

Since a closed immersion is affine (and hence separated and quasi-compact) (2.12.1), we see by (4.2.11) that $i_*\mathscr{O}_Z$ is quasi-coherent. Thus $\mathscr{I} = \ker i^\sharp$ is also quasi-coherent. Moreover, we also know that $Z = \operatorname{Supp} i_*\mathscr{O}_Z$ (see [NOS, 8.3]). Hence, the ideal sheaf $\mathscr{I}$ completely determines the closed subscheme corresponding to $i : Z \to X$. This association can be reversed. In fact, suppose $\mathscr{I} \hookrightarrow \mathscr{O}_X$ is a quasi-coherent sheaf of ideals; then $\mathscr{O}_X/\mathscr{I}$ is a quasi-coherent sheaf of algebras over X, and we can associate to it the affine morphism $\operatorname{Spec}(\mathscr{O}_X/I) \to \operatorname{Spec}\mathscr{O}_X$. We check easily that this is a closed immersion.

DEFINITION 4.3.12. Let X be a scheme; we will denote by $\mathscr{N}_X$ the presheaf that assigns to every open subset $U \subset X$ the ideal $\operatorname{Nil}(\mathscr{O}_U)$. This is the *nilradical* of $\mathscr{O}_X$. It's easy to see that this is in fact a quasi-coherent ideal sheaf over X.

DEFINITION 4.3.13. Let $f : X \to Y$ be a morphism of schemes, and let $\mathscr{I}$ be an ideal sheaf over Y. Then we denote by $f^{-1}\mathscr{I}$ the ideal sheaf over X that's the image of the natural morphism $f^*\mathscr{I} \to \mathscr{O}_X$ induced by pulling back the inclusion $\mathscr{I} \hookrightarrow \mathscr{O}_Y$.

c-ideal-closed-immersion

PROPOSITION 4.3.14. *Let X be a scheme.*

(1) *The assignment $\mathscr{I} \mapsto \operatorname{Spec}(\mathscr{O}_X/\mathscr{I})$ induces an anti-isomorphism of lattices between the lattice of quasi-coherent ideal sheaves of $\mathscr{O}_X$ and the lattice of closed subschemes of X.*

(2) *Under this assignment the nilradical* $\mathscr{N}_X$ *maps to* X_{red}.
(3) *If* $f : X \to Y$ *is a morphism of schemes, and* Z *is a closed subscheme of* Y*, then* $f^{-1}(Z) = \operatorname{Spec}\left(\mathscr{O}_X / f^{-1}\mathscr{I}\right)$.
(4) *If, in addition,* X *is Noetherian, then every quasi-coherent sheaf of ideals of* $\mathscr{O}_X$ *is in fact coherent.*

PROOF. Suppose that Z is a closed subscheme of X and W is a closed subscheme of Z.

(1) LATER

□

3.4. Sheaves with Local Support. Let $X = \operatorname{Spec} R$ be a Noetherian affine scheme. Recall from [RS, 4.5], the definition of the support of an $\mathscr{O}_X$-module. We see immediately that if $\widetilde{M}$ is a quasi-coherent $\mathscr{O}_X$-module, then

$$\operatorname{Supp} \widetilde{M} = \{P \subset R : M_P \neq 0\} = \operatorname{Supp} M.$$

If now, M is finitely generated, then we know that $\operatorname{Supp} M = V(\operatorname{ann}(M))$, and is thus a closed set.

For $I \subset R$, and M any R-module, consider the submodule

$$\Gamma_I(M) = \{m \in M : I^n m = 0, \text{ for some } n \geq 0\} \subset M.$$

Now, suppose $m \in M$; then, if $m \in \Gamma_I(M)$, we see that $\operatorname{Supp} m \subset V(I)$, for if $I^n m = 0$, for some n, and $I \subsetneq P$, then m is annihilated by some element outside of P and is thus zero in M_P. Conversely, if $\operatorname{Supp} m \subset V(I)$, then $I \subset \operatorname{rad}(\operatorname{ann}(m))$, and so there is some $n \geq 0$ such that $I^n m = 0$. In sum, if $Z = V(I)$, then $\Gamma_Z(X, \widetilde{M}) = \Gamma_I(M)$.

Now, by [NOS, 8.13], we have an exact sequence

$$0 \to \underline{H}^0_Z(\mathscr{M}) \to \mathscr{M} \to j_*(\mathscr{M}|_U),$$

where $U = X \setminus Z$, and $j : U \hookrightarrow X$ is the inclusion. Now, j is an open immersion and is thus separated (see 2.12.1); since R is Noetherian, U is also quasi-compact, and so we're in a position to apply (4.2.11) to conclude that $j_*(\mathscr{M}|_U)$ is quasi-coherent. Then, by (4.2.2), we see that $\underline{H}^0_Z(\mathscr{M})$ is also quasi-coherent. This implies

$$\underline{H}^0_Z(\mathscr{M}) = \widetilde{\Gamma_I(M)}.$$

Now, suppose X is any locally Noetherian scheme. Let $Z \subset X$ be some closed set, and let $\mathscr{M}$ be a coherent $\mathscr{O}_X$-module. Then, for any affine open $U = \operatorname{Spec} R \subset X$, we have

$$\underline{H}^0_Z(\mathscr{M})|_U = \underline{H}^0_{Z \cap U}(\mathscr{M}|_U) \cong \widetilde{\Gamma_I(M)},$$

where M is an R-module such that $\mathscr{M}|_U = \widetilde{M}$.

Let us record this in the next Proposition.

eaves-with-local-support

PROPOSITION 4.3.15. *Let* X *be a locally Noetherian scheme, and let* $Z \subset X$ *be a closed subscheme with ideal sheaf* $\mathscr{I}_Z \subset \mathscr{O}_X$*. Then, for any quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{M}$*, the sheaf* $\underline{H}^0_Z(\mathscr{M})$ *is again quasi-coherent. Moreover, for every affine open* $U = \operatorname{Spec} R \subset X$*, we have*

$$\underline{H}^0_Z(\mathscr{M})|_U = \widetilde{\Gamma_I(M)},$$

where $I \subset R$ *is an ideal such that* $\widetilde{I} = \mathscr{I}_Z|_U$*, and* M *is an* R*-module such that* $\widetilde{M} = \mathscr{M}|_U$*. If* $\widetilde{M}$ *is coherent, then so is* $\underline{H}^0_Z(\mathscr{M})$*.*

REMARK 4.3.16. Observe that $\underline{H}_Z^0(M)$ depends only on the topology of Z, while $\Gamma_I(M)$ appears to take into account its geometry; this is illusory, since $\Gamma_I(M)$ depends only on the $\mathrm{rad}(I)$.

CHAPTER 5

Local Properties of Schemes and Morphisms

chap:local

1. Local Determination of Morphisms

In this section, we'll look at how local information about morphisms determines their global behavior and also prove some local extension results.

ocal-local-determination

Proposition 5.1.1. *Let $\alpha : X \to S$ and $\beta : Y \to S$ be two S-schemes, and suppose that Y is of finite type over S.*

(1) *Let $f, g : X \to Y$ be two morphisms of S-schemes, and let $x \in X$ be a point such that $f(x) = g(x) = y$ and such that the induced maps $f_x^\sharp$ and $g_x^\sharp$ agree as maps from $\mathscr{O}_{Y,y}$ to $\mathscr{O}_{X,x}$. Then there exists an open neighborhood U of x such that $f|_U = g|_U$.*

(2) *Suppose now that S is locally Noetherian, and that we are given a map of rings $f : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. Then we can extend this to an S-morphism $U \to Y$ so that the following diagram commutes:*

$$\begin{array}{ccc} U & \longrightarrow & Y \\ \uparrow & & \uparrow \\ \operatorname{Spec} \mathscr{O}_{X,x} & \longrightarrow & \operatorname{Spec} \mathscr{O}_{Y,y} \end{array}$$

(3)

Proof. The questions are local; so, for these, we reduce at once to the case where $S = \operatorname{Spec} R$ and $Y = \operatorname{Spec} T$ are affine, with $T = R[T_1, \ldots, T_n]/I$, where $R[T_1, \ldots, T_n]$ is the polynomial ring in n variables over R.

(1) There is a closed immersion of Y into $\mathbb{A}^n_S$. So, by composing with this closed immersion, we can replace Y with $\mathbb{A}^n_S$ (we can do this, since closed immersions are monomorphisms). By (4.3.8), both f and g are given by n-tuples of global sections $(s_i)_{1 \le i \le n}$ and $(t_i)_{1 \le i \le n}$ of $\mathscr{O}_X$. To say that they induce the same map on $\operatorname{Spec} \mathscr{O}_{X,x}$ is to say that $(s_i - t_i)_x = 0$, for $1 \le i \le n$. Now, we can find a neighborhood U of x such that $(s_i - t_i)|_U = 0$, for $1 \le i \le n$, and so $f|_U = g|_U$.

(2) Now, a morphism from $\operatorname{Spec} \mathscr{O}_{X,x}$ to Y is given by an n-tuple $(a_i)_{1 \le i \le n}$ of elements in $\mathscr{O}_{X,x}$ such that, for every $p \in I$, $p(a_1, \ldots, a_n) = 0$. Since R is Noetherian, so is $R[T_1, \ldots, T_n]$, and so the ideal I is generated by finitely many elements $p_1, \ldots, p_r$. Now, we can find a neighborhood W of x and sections $s_i \in \Gamma(W, \mathscr{O}_X)$ such that $(s_i)_x = a_i$, for $1 \le i \le n$. Moreover, since $p_j(s_1, \ldots, s_n)_x = 0$, for $1 \le j \le r$, we can find a neighborhood $U \subset W$ such that $p_j(s_1, \ldots, s_n)|_U = 0$, for $1 \le j \le r$. But then the n-tuple of sections $(s_i|_U)$ over U now defines a morphism of U into Y.

□

ocal-valuative-criterion

COROLLARY 5.1.2. *Let $\mathscr{O}$ be a reduced local ring, and let K be its field of fractions. For any separated $\mathscr{O}$-scheme X, the natural map*

$$\operatorname{Hom}_{\mathrm{Sch}_{\mathscr{O}}}(\operatorname{Spec}\mathscr{O}, X) \to \operatorname{Hom}_{\mathrm{Sch}_{\mathscr{O}}}(\operatorname{Spec} K, X)$$

is injective. If X is proper over $\mathscr{O}$ and $\mathscr{O}$ is a valuation ring, then the map is bijective.

PROOF. Let $f, g : \operatorname{Spec}\mathscr{O} \to X$ be two sections of the structure morphism of X. If f and g agree on $\operatorname{Spec} K$, then, since $K = \mathscr{O}_{\mathscr{O},\xi}$, where ξ is the generic point of $\operatorname{Spec}\mathscr{O}$, we can use part (1) of the Proposition above to conclude that f and g agree on some open neighborhood U of ξ, which is of course dense in X. But then, since X is separated and $\mathscr{O}$ is reduced, it follows from (2.10.5), that f and g agree everywhere on X. This finishes the proof of the first assertion.

Now suppose that X is proper and that $\mathscr{O}$ is a valuation ring. A morphism $f : \operatorname{Spec} K \to X$ corresponds to a point $x \in X$ and a field extension $K/k(x)$. Let $Z \to X$ be the scheme theoretic image of f. Then Z is also proper over $\mathscr{O}$, since proper morphisms are stable under composition. Moreover, Z is reduced (2.5.5) and irreducible (since $Z = \overline{\{x\}}$), and is therefore integral. The image of Z in $\operatorname{Spec}\mathscr{O}$ contains the generic point ξ and is closed; therefore it must be all of $\operatorname{Spec}\mathscr{O}$. Let $s \in \mathscr{O}$ be the closed point, and let $z \in Z$ be a point lying over s. Then $\mathscr{O}_{Z,z}$ is a local ring dominating $\mathscr{O}$; since $\mathscr{O}$ is a valuation ring, this implies that $\mathscr{O}_{Z,z} = \mathscr{O}$. Consider the morphism

$$g : \operatorname{Spec}\mathscr{O} = \operatorname{Spec}\mathscr{O}_{Z,z} \to Z \to X.$$

We find now that g induces the morphism f on $\operatorname{Spec} K$, thus showing the bijectivity of our map. □

Still assuming that S is locally Noetherian, pick $s \in S$, and suppose we have a morphism of S-schemes

$$\varphi : X \times_S \operatorname{Spec}\mathscr{O}_{S,s} \to Y \times_S \operatorname{Spec}\mathscr{O}_{S,s}.$$

If X is also of finite type over S, then we can find an open neighborhood U of s such that φ is the base change of an S-morphism

$$f : X \times_S U \to Y \times_S U.$$

Moreover, if φ is an isomorphism, then we can choose f to be an isomorphism.

-secn:rational-functions

2. Rational Maps and Rational Functions

2.1. Rational Maps.

DEFINITION 5.2.1. Let X and Y be two S-schemes. A *rational pair over X* is a pair (f, U), where $U \subset X$ is a dense open subscheme and $f : U \to Y$ is an S-morphism. We say that two such pairs (f, U) and (g, V) are equivalent, if $f|_W = g|_W$, for some dense open subscheme $W \subset U \cap V$. A *rational S-map* from X to Y is an equivalence class of rational pairs. If S is clear from the context, then we will refer to such a class as simply a rational map. The set of rational S-maps from X to Y is denoted by $R_S(X, Y)$.

A *rational function* on X is an element of $R_X(X, \mathbb{A}^1_X)$ ($\mathbb{A}^1_X = \mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} X$, where $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]$). Observe, that for every open subscheme $U \subset X$, we have

$$\begin{aligned} \operatorname{Hom}_{\mathrm{Sch}_X}(U, \mathbb{A}^1_X) &= \operatorname{Hom}_{\mathrm{Sch}}(U, \operatorname{Spec} \mathbb{Z}[t]) \\ &= \operatorname{Hom}_{\mathrm{Ring}}(\mathbb{Z}[t], \Gamma(U, \mathscr{O}_X)) \\ &= \Gamma(U, \mathscr{O}_X). \end{aligned}$$

So the set of rational functions on X is simply the set of equivalence classes of sections of the structure sheaf over dense open subschemes of X, and thus has a natural ring structure. We denote this by $K(X)$ and call it the *ring of rational functions* on X.

Remark 5.2.2. More explicitly, we see that two pairs (f, U) and (g, V), with f and g sections of the structure sheaf over U and V, respectively, are equivalent if $f|_W = g|_W$, for some dense open $W \subset U \cap V$. The ring structure is simply obtained by taking, for the sum, the class of $(f|_{U\cap V} + g|_{U\cap V}, U \cap V)$, and, for the product, the class of $(f|_{U\cap V} g|_{U\cap V}, U \cap V)$.

Remark 5.2.3. Observe that we have a natural map $\operatorname{Hom}_{\mathrm{Sch}_S}(X, Y) \to R_S(X, Y)$. In particular, $K(X)$ is always an $\mathscr{O}_X(X)$-algebra.

In the affine Noetherian case, we can compute the ring of rational functions quite explicitly.

noetherian-affine-scheme

Proposition 5.2.4. *Let $X = \operatorname{Spec} R$ be an affine scheme, where R is a Noetherian ring, and let Q be the complement of the union of the minimal primes of R. Then $K(X)$ is naturally isomorphic to $Q^{-1}R$.*

Proof. First, observe that a principal open subscheme $X_f \subset X$ is dense in X if and only if $f \in Q$. Indeed, X_f is dense if and only if it contains every generic point of X, which of course are in bijective correspondence with the minimal primes of R. Moreover, if $U \subset X$ is any dense open subscheme with complement $V(I)$, for some ideal $I \subset R$, we see that $V(I)$ does not contain any generic points of X, and so I is not contained in any minimal prime of P. Since R is Noetherian, X has only finitely many generic points, and so, by prime avoidance, we can find an element $f \in I$ not contained in any minimal prime of R. Now we see that $X_f \subset U$ is a dense principal open subscheme.

Given this, we see that the principal open subschemes $\{X_f : f \in Q\}$ form a co-final subset of the directed set of dense open subschemes of X. So we clearly have

$$\begin{aligned} K(X) &= \varinjlim_{x \in Q} \Gamma(X_f, \mathscr{O}_X) \\ &= \varinjlim_{x \in Q} R_f \\ &= Q^{-1}R. \end{aligned}$$

□

ocal-rational-open-dense

Proposition 5.2.5. *Let X and Y be two S-schemes, and let $U \subset X$ be an open subscheme.*

(1) *There is a natural restriction map $R_S(X, Y) \to R_U(X, Y)$, making $U \mapsto R_U(X, Y)$ a presheaf of sets over X.*

(2) *If $U \subset X$ is dense, then this map is a bijection.*

PROOF. Suppose (f, V) and (g, W) is an equivalent pair on X; then, since V and W are dense in X, both $V \cap U$ and $W \cap U$ are open dense subsets of U. Thus $(f|_{V\cap U}, V \cap U)$ and $(g|_{W\cap U}, W \cap U)$ are equivalent pairs over U. This gives us the natural restriction map. If, now, U is in fact dense, then any pair (f, V) with $V \subset U$ dense in U and $f : V \to Y$ an S-morphism in fact determines a pair over X, since V will also be dense in X. In particular, the natural restriction map is a surjection. By a similar argument, it's easy to see that the restriction map is also injective. □

Now, suppose that X is irreducible. Then an open dense subscheme of X is simply an open subscheme of X containing the generic point ξ of X. In particular, if $\varphi = [(f, U)]$ is a rational function over X, then (g, V) is another representative of φ if and only if f and g define the same germ at ξ. So we get a natural map $K(X) \to \mathscr{O}_{X,\xi}$ that is clearly bijective. This gives us the next Lemma.

local-rational-functions

LEMMA 5.2.6. *Let X be an irreducible scheme with generic point ξ. Then the natural map*

$$K(X) \to \mathscr{O}_{X,\xi}$$

is an isomorphism. In particular, if X is integral, then we recover the field of rational functions on X, and, if X is Noetherian, then $K(X)$ is a local Artin ring.

ional-map-fin-irred-comp

PROPOSITION 5.2.7. *Let X and Y be two S-schemes, and suppose that X has only finitely many irreducible components $X_1, \ldots, X_r$, with generic points $\xi_1, \ldots, \xi_r$.*

(1) *The natural map:*

$$R_S(X, Y) \to \prod_{i=1}^{r} R_S(X_i, Y)$$

is a bijection. In particular, we have an isomorphism of rings

$$\begin{aligned} K(X) &\cong \prod_{i=1}^{r} R(X_i) \\ &\cong \prod_{i=1}^{r} \mathscr{O}_{X,\xi_i}. \end{aligned}$$

(2) *If Y is of finite type over S, the natural map*

$$R_S(X, Y) \to \prod_{i=1}^{r} \mathrm{Hom}_{\mathrm{Sch}_S}(\mathrm{Spec}\, \mathscr{O}_{X_i,\xi_i}, Y)$$

is injective. If, in addition, S is locally Noetherian, then it is in fact bijective.

PROOF. For each $1 \le i \le r$, let $U_i = X - \bigcup_{j \ne i} X_j$. Then U_i is an open subset of X contained in X_i, and is thus dense in X_i; and, moreover, the collection

$\{U_1, \ldots, U_r\}$ is pairwise disjoint, with $U = \bigcup_i U_i$ dense in X. Now we have:

$$\begin{aligned} R_S(X,Y) &= R_S(U,Y) \\ &= \prod_{i=1}^{r} R_S(U_i, Y) \\ &= \prod_{i=1}^{r} R_S(X_i, Y), \end{aligned}$$

where we have made repeated use of (5.2.5). The second equality in the string above follows immediately from the fact that the U_i are pairwise disjoint. This finishes the proof of (1). (2) follows immediately from (1) and the Lemma above. For (3), first observe that any rational map from X_i to Y determines a unique morphism of S-schemes from $\operatorname{Spec} \mathscr{O}_{X_i,\xi_i}$ to Y. Given this, both assertions in (3) follow immediately from (5.1.1). □

ational-maps-over-points

COROLLARY 5.2.8. *Let X and Y be S-schemes, with S locally Noetherian, X irreducible with generic point ξ, and Y of finite type over S. Suppose ξ lies over $s \in S$.*

(1) *Giving a rational S-map from X to Y is equivalent to giving a point $y \in Y$ lying over S and a homomorphism of rings $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,\xi}$.*
(2) *For any point $z \in X$, there is a bijection*

$$R_S(X,Y) \cong R_S(\operatorname{Spec} \mathscr{O}_{X,z}, Y).$$

(3) *If X is in addition integral, then there is a natural isomorphism*

$$R_S(X,Y) \cong \operatorname{Hom}_{\mathrm{Sch}_{k(s)}}(\operatorname{Spec} K(X), Y_s).$$

(4) *In particular, if $S = \operatorname{Spec} k$, for some field k, then we have*

$$\begin{aligned} R_k(X,Y) &\cong \operatorname{Hom}_{\mathrm{Sch}_k}(\operatorname{Spec} K(X), Y) \\ &\cong \operatorname{Hom}_k(\Gamma(Y, \mathscr{O}_Y), K(X)). \end{aligned}$$

If Y is also integral, this gives us a natural isomorphism

$$R_k(X,Y) \cong \operatorname{Hom}_k(K(Y), K(X)).$$

PROOF. For (1), simply note that the data of a point $y \in Y$ lying over s and a homomorphism $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,\xi}$ are in bijective correspondence with morphisms of S-schemes $\operatorname{Spec} \mathscr{O}_{X,\xi} \to Y$, and use the Proposition above, specializing it to the case where X is itself irreducible. Statement (2) follows at once, since the local rings of both X and $\operatorname{Spec} \mathscr{O}_{X,z}$ at ξ are naturally isomorphic.

For (3), note that we have

$$\begin{aligned} \operatorname{Hom}_S(\operatorname{Spec} \mathscr{O}_{X,\xi}, Y) &= \operatorname{Hom}_S(\operatorname{Spec} k(\xi), Y) \\ &= \operatorname{Hom}_{k(s)}(\operatorname{Spec} k(\xi), Y_s). \end{aligned}$$

(4) follows immediately from this. □

2.2. The Domain of Definition of a Rational Map.

l-maps-reduced-separated

LEMMA 5.2.9. *Let X and Y be S-schemes, with X reduced and Y separated. Let (f, U) and (g, V) be two equivalent rational pairs over X, then $f|_{U \cap V} = g|_{U \cap V}$.*

PROOF. Replacing X with $U \cap V$ and f and g with their restrictions to $U \cap V$, we have two S-morphisms $f, g : X \to Y$ that agree on an open dense subscheme of X. Now the result follows from (2.10.6). □

DEFINITION 5.2.10. Let $\varphi \in R_S(X, Y)$ be a rational S-map between S-schemes X and Y. We say that φ is *defined at x*, for a point $x \in X$, if there exists a representative (f, U) of φ, with $x \in U$. The subset of X consisting of the points where φ is defined is called the *domain of definition* of φ and is denoted $\mathrm{dom}(\varphi)$. Clearly, $\mathrm{dom}(\varphi)$ is an open dense subset of X.

onal-maps-domain-of-defn

PROPOSITION 5.2.11. *Let X and Y be S-schemes, with X reduced and Y separated. Then, for every rational S-map φ from X to Y, there is a unique morphism $f : \mathrm{dom}(\varphi) \to Y$ of S-schemes such that $(f, \mathrm{dom}(\varphi))$ represents φ.*

PROOF. For every point $x \in \mathrm{dom}(\varphi)$, there exists a pair (g_x, U_x) representing φ such that $x \in U_x$. Now, consider the open cover $\mathcal{V} = \{U_x : x \in X\}$; then $(g_x : x \in X)$ will, according to the lemma above, glue together to give a unique morphism $f : \mathrm{dom}(\varphi) \to Y$ such that $f|_{U_x} = g_x$, for all $x \in \mathrm{dom}(\varphi)$. Moreover, if we have any other morphism $f' : \mathrm{dom}(\varphi) \to Y$ such that $(f', \mathrm{dom}(\varphi))$ is equivalent to $(f, \mathrm{dom}(\varphi))$, then there is a dense open subset of $\mathrm{dom}(\varphi)$ on which they agree, and, again, by (2.10.6), they must be equal on $\mathrm{dom}(\varphi)$. □

tional-functions-reduced

COROLLARY 5.2.12. *Let X be a reduced scheme; then, for any dense open subscheme $U \subset X$, the rational functions on X defined at every point of U are in bijection with $\Gamma(U, \mathscr{O}_X)$.*

PROOF. Since $\mathbb{A}^1_X$ is always separated over X, we see that, for a rational function φ on X, $\mathrm{dom}(\varphi)$ contains U if and only if φ corresponds to a section of $\Gamma(U, \mathscr{O}_X)$. Moreover, such a section must be unique, since two sections over U that agree on an open dense subset of U must agree everywhere on U. □

2.3. Birational Morphisms.

2.4. The Sheaf of Rational Functions.

DEFINITION 5.2.13. The *sheaf of rational functions* $\mathscr{K}(X)$ over a scheme X is the sheafification of the presheaf $U \mapsto K(U)$ (this is a presheaf by (5.2.5)). This is clearly a sheaf of $\mathscr{O}_X$-algebras (not necessarily quasi-coherent) over X.

onal-functions-quasi-coh

PROPOSITION 5.2.14. *Let X be a scheme such that every point $x \in X$ has a neighborhood with only finitely many irreducible components (for example, we can take X to be locally Noetherian). Then the $\mathscr{O}_X$-algebra $\mathscr{K}(X)$ is quasi-coherent; moreover, for any open subscheme $U \subset X$ with only finitely many irreducible components, the natural map $R(U) \to \Gamma(U, \mathscr{K}(X))$ is an isomorphism.*

PROOF. We can immediately reduce to the case where X has only finitely many irreducible components $X_1, \ldots, X_n$, and show that $U \mapsto K(U)$ is in fact a sheaf on X. First observe that we have, by (5.2.7), the natural isomorphism

$$K(U) \cong \prod_{U \cap X_i \neq \emptyset} K(X_i).$$

Now, $U \mapsto K(U)$ is clearly separated; so it suffices to show that if we have a weak covering sieve $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of an open subset $U \subset X$, then the map $K(U) \to \mathcal{V}(K)$ is surjective. Let (s_α) be an element of $\mathcal{V}(K)$; then, for each irreducible component X_i with $X_i \cap U \neq \emptyset$, and for all pairs (α, β) such that $X_i \cap V_\alpha \cap V_\beta \neq \emptyset$, s_α and s_β determine the same element of $K(X_i)$. Thus the coherent sequence (s_α) determines a unique element of $\prod_{U \cap X_i \neq \emptyset} K(X_i)$ and thus a unique element of $K(U)$ that restricts to s_α over each V_α. This shows that $U \mapsto K(U)$ is in fact a sheaf.

For the quasi-coherence, observe that if $M = \oplus_{i=1}^{n} \mathscr{O}_{X_i,\xi_i}$, where ξ_i is the generic point of the component X_i, for $1 \leq i \leq n$, then $\widetilde{M}$ is clearly isomorphic to $\mathscr{K}(X)$, as can be checked on stalks. □

3. Normal Schemes and Normalization

3.1. Serre's Criterion.

Note on Notation 4. In this section, all our schemes will be *locally Noetherian.*

Definition 5.3.1. An scheme X is *normal* if, for every $x \in X$, $\mathscr{O}_{X,x}$ is a normal ring.

`algv-normal-equiv-prps`

Proposition 5.3.2. *Let X be an integral scheme. Then the following are equivalent:*

1. *For every open subscheme $U \subset X$, $\Gamma(U, \mathscr{O}_X)$ is a normal domain.*
2. *For every affine open $U \subset X$, $\Gamma(U, \mathscr{O}_X)$ is normal.*
3. *There is an affine open cover $\{U_i : i \in I\}$ of X such that $\Gamma(U_i, \mathscr{O}_X)$ is normal, for all $i \in I$.*
4. *X is normal.*

If, in addition, the underlying topological space of X is Noetherian, then these statements are equivalent to: For every closed point $x \in X$, $\mathscr{O}_{X,x}$ is normal.

Proof. The only non-trivial part is (4) ⇒ (1). So let X be an irreducible normal scheme. Then, we find from (1.6.6) that, for any open subscheme $U \subset X$,

$$\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x}.$$

By hypothesis, for each $x \in X$, the ring $\mathscr{O}_{X,x}$ is integrally closed in its fraction ring $K(X)$. Thus, $\Gamma(U, \mathscr{O}_X)$ is also integrally closed in $K(X)$. □

Definition 5.3.3. A scheme X is *regular in codimension n* or *satisfies condition R_n* if, for all $x \in X$, with $\dim \mathscr{O}_{X,x} \leq n$, the ring $\mathscr{O}_{X,x}$ is a regular local ring.

A scheme X *satisfies condition S_n*, if, for all $x \in X$, we have

$$\operatorname{depth} \mathscr{O}_{X,x} \geq \max\{n, \dim \mathscr{O}_{X,x}\}$$

`scheme-serre-criterion`

Proposition 5.3.4. *A scheme X is reduced if and only if it satisfies conditions R_0 and S_1. It is normal if and only if it satisfies conditions R_1 and S_2.*

4. Flat Morphisms

DEFINITION 5.4.1. A morphism $f : X \to Y$ is *flat* if, for every $x \in X$, the natural map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a flat homomorphism of local rings.

In other words, a morphism $f : X \to Y$ is flat if $\mathscr{O}_X$ is flat over $\mathscr{O}_Y$.

local-flat-equiv-prps

PROPOSITION 5.4.2. *The following are equivalent for a morphism $f : X \to Y$:*

1. *f is flat.*
2. *There is an open cover $\{U_i : i \in I\}$ of Y such that the restriction $f^{-1}(U_i) \to U_i$ is flat.*
3. *For every affine open $U = \operatorname{Spec} R \subset Y$ and every affine open $V = \operatorname{Spec} S \subset f^{-1}(U)$, S is flat over R.*
4. *For every affine open $U = \operatorname{Spec} R \subset Y$, there is an affine open cover $\{V_i = \operatorname{Spec} S_i\}$ of $f^{-1}(U)$ such that S_i is flat over R.*

REMARK 5.4.3. Condition (2) says that flatness is local on the base.

PROOF. The only non-trivial implication is (1) ⇒ (3); but this follows from [CA, 3.1.10]. □

local-flat-prps

COROLLARY 5.4.4. (1) *Open immersions are flat morphisms.*
(2) *Flat morphisms are local on the base and on the domain.*
(3) *Flat morphisms are stable under base change.*
(4) *Flat morphisms are stable under composition.*
(5) *Flat morphisms have the going down property (See [NS, 6]).*
(6) *If $f : X \to Y$ is a flat morphism, with Y irreducible, then, for any non-empty open subscheme $U \subset X$, $f(U)$ is dense in Y. In particular, f is dominant. If, in addition, X has only finitely many irreducible components, then the image of every irreducible component under f is dense in Y.*
(7) *If $f : X \to Y$ is a flat morphism of finite type between Noetherian schemes, then f is open.*

PROOF. (1) This is essentially the fact that, for any ring R and any element $f \in R$, the localization R_f is a flat R-module.
(2) Follows from the Proposition.
(3) It suffices to prove this for the affine case, and there it follows from [CA, 3.1.6].
(4) Follows from [CA, 3.1.6].
(5) Let $x \in X$ and $y \in Y$ be such that y is a generization of $f(x)$. We need to show that there is a generization of x that maps to y. Let $U = \operatorname{Spec} R \subset X$ be an affine neighborhood of $f(x)$; then $y \in U$. Choose any affine neighborhood $V = \operatorname{Spec} S \subset f^{-1}(U)$ of x; then S is flat over R (5.4.2). Now the result follows from [CA, 3.6.8].
(6) We reduce immediately to the affine case, and here it reduces to showing that if $f : R \to S$ is a flat map of rings, with $N = \operatorname{Nil}(R)$ prime, then $\ker f \subset N$. For this it suffices to show that the induced map $R/N \to S/NS$ is injective. But now observe that S/NS is flat over R/N by [CA, 3.1.8], and since R/N is a domain we see that S/NS is torsion free as an R/N-module, which tells us that R/N embeds into S/NS. For the second

assertion, simply note that if there are only finitely many irreducible component, then each component contains a non-empty open set–namely the complement of the union of the rest of the components.

(7) By (2.7.6), f is a constructible map. Now, the result follows from [NS, 5.10]. □

DEFINITION 5.4.5. A morphism $f : X \to Y$ is *faithfully flat* if it is flat and surjective.

The next Proposition follows from (5.4.4) and (2.3.2).

cal-faithfully-flat-prps

PROPOSITION 5.4.6. (1) *Faithfully flat morphisms are local on the base and the domain.*
(2) *Faithfully flat morphisms are stable under base change.*
(3) *Faithfully flat morphisms are stable under composition.*

The next theorem tells us that many morphisms that arise geometrically are generically flat.

local-generic-flatness

THEOREM 5.4.7 (Generic Flatness). *Let $f : X \to Y$ be a dominant morphism of finite type between two integral, locally Noetherian schemes. Then there is an open subscheme $V \subset Y$ such that the restriction $f^{-1}(V) \to V$ is faithfully flat.*

PROOF. There is no harm in assuming that $Y = \operatorname{Spec} R$ is affine, with R some Noetherian domain and thus that X is in fact Noetherian, covered by finitely many affine opens $\{U_1, \dots, U_n\}$, with $U_i = \operatorname{Spec} S_i$, where S_i is a finitely generated R-algebra. Now, by [CA, 8.2.1], there is for each i an open subscheme (and in fact a principal open subscheme) $V_i \subset Y$ such that the morphism $f^{-1}(V_i) \cap U_i \to V_i$ is faithfully flat. We finish the proof by taking $V = \cap_i V_i$. □

5. Tangent Spaces and Regularity

5.1. Regular Schemes.

DEFINITION 5.5.1. A scheme X is *regular* if all its local rings are regular. Equivalently, X is regular if it satisfies condition R_n, for all $n \geq 0$.

6. Cohen-Macaulay Schemes

CHAPTER 6

Dimension

chap:dim

ABSTRACT. We will use extensively the results from [NS, 6].

1. Krull Dimension

DEFINITION 6.1.1. The *Krull dimension* or *dimension* $\dim X$ of a scheme X is the dimension of its underlying topological space.

For a closed subscheme $Z \subset X$, the *codimension* of Z, $\operatorname{codim}(Z, X)$ is its codimension in X as a closed subspace of the topological space underlying X.

A subscheme $Z \subset X$ has *pure codimension* n, for some positive integer $n \in \mathbb{N}$, if $\operatorname{codim}(Z_i, X) = n$, for all irreducible components Z_i of Z.

For a point $x \in X$, the *dimension at* x $\dim_x X$ is the Krull dimension of X at x; i.e. it's the infimum $\inf_{U \ni x} \dim U$, taken over the dimensions of all the open subschemes of X containing x.

im-dimension-sup-affines

REMARK 6.1.2. It follows from [NS, 6.2] that

$$\dim X = \sup_{\substack{U \subset X \\ U \text{ affine open}}} \dim U.$$

dim-affine-case

PROPOSITION 6.1.3. *Let X be a scheme.*

(1) *If $X = \operatorname{Spec} R$ is affine,* $\dim X = \dim R$, *where the latter quantity is the Krull dimension of the ring R.*
(2) *For any ideal $I \subset R$,*

$$\operatorname{codim}(\operatorname{Spec} R/I, X) = \operatorname{ht} I.$$

(3) *For any $x \in X$*

$$\operatorname{codim}(\overline{x}, X) = \dim \mathscr{O}_{X,x}.$$

PROOF. The first two are immediate from the definitions; for the third, note that, by part (5) of [NS, 6.6], we can assume that $X = \operatorname{Spec} R$ is affine. In this case, x corresponds to a prime $P \subset R$, and the result now follows from part (2). □

We now investigate the dimension zero case.

dim-dimension-zero

PROPOSITION 6.1.4. *Let X be a scheme.*

(1) *Suppose the underlying space of X is discrete; then* $\dim X = 0$.
(2) *Suppose X is Noetherian; then* $\dim X = 0$ *if and only if X is discrete and finite, if and only if $X = \operatorname{Spec} A$, where A is an Artinian ring.*

PROOF.
Follows from [NS, 6.7].

We get one equivalence from [NS, 6.7]. If X is finite and discrete, then it's clear that $X = \coprod_{x\in X} \operatorname{Spec} \mathscr{O}_{X,x}$, where each ring $\mathscr{O}_{X,x}$ is local Artinian. Thus, $X = \operatorname{Spec} A$, where A is artinian; the other direction follows trivially. □

Next, the relative dimension zero case.

dim-integral-morphism

PROPOSITION 6.1.5. *Let $f : X \to Y$ be an integral morphism of schemes.*

(1) $\dim X \leq \dim Y$.
(2) *For any closed subspace $Z \subset X$, $f(Z)$ is a closed subset of Y, and we have*

$$\dim Z = \dim f(Z).$$

PROOF. Follows from (2.8.3) and [NS, 6.13]. □

2. Jacobson Schemes

Essentially, Jacobson schemes are schemes to which the Nullstellensatz applies. See [CA, 8] for more algebraic details.

DEFINITION 6.2.1. A subset $Z \subset X$ of a topological space X is *very dense* if Z intersects every non-empty locally closed subset non-trivially. Equivalently, Z is very dense in X if $Z \cap W$ is dense in W for every non-empty closed subset $W \subset X$.

A scheme X is *Jacobson* if the subset of closed points is very dense in X.

cobson-scheme-equiv-prps

PROPOSITION 6.2.2. *Let X be a scheme; then the following are equivalent:*

(1) *X is Jacobson.*
(2) *Every non-empty open subscheme $U \subset X$ is Jacobson.*
(3) *For every affine open* $\operatorname{Spec} R \subset X$, *$R$ is a Jacobson ring.*
(4) *There is an affine open cover $\{U_i : i \in I\}$ of X with $U_i = \operatorname{Spec} S_i$, where S_i is a Jacobson ring.*
(5) *Every locally closed point of X is closed.*
(6) *For every X-scheme $f : Y \to X$ that is locally of finite type, a point $y \in Y$ is closed if and only if $f(y)$ is closed in X.*

PROOF. **(1) ⇔ (2):** This is clear.

(2) ⇔ (3): This is equivalent to showing that $\operatorname{Spec} R$ is Jacobson if and only if R is Jacobson. The closure of the set of closed points in $\operatorname{Spec} R$ is $V(\operatorname{Jac} R)$. The equivalence now follows from characterization (2) in [CA, 8.4.2].

(3) ⇒ (4): Trivial.

(4) ⇒ (5): Let $x \in X$ be a locally closed point, and let $i \in I$ be such that $x \in U_i$. Since U_i is Jacobson (via the equivalence (2) ⇔ (3)), x is in fact a closed point in U_i. It remains to show that x is closed in X. For this it suffices to show that x is closed in U_j for any $j \in I$ such that $x \in U_i \cap U_j$. Let $V = \operatorname{Spec} S \subset U_i \cap U_j$ be an open neighborhood of x that is a principal affine open in both U_i and U_j. Now, x corresponds to a maximal ideal in S, and since the map $S_j \to S$ is of finite type, we see from [CA, 8.4.6] that x corresponds to a maximal ideal also in S_j and hence is closed in U_j.

(5) ⇒ (3): We'll use characterization (3) from [CA, 8.4.2]. Let $P \subset R$ be a non-maximal prime ideal, and let I be the intersection of all prime ideals containing P. Suppose $I \neq P$; then there is $f \in I \setminus P$. Consider the localization R_f: $\operatorname{Spec} R_f$ is an open subscheme of $\operatorname{Spec} R$, and P

corresponds to a closed point in $\operatorname{Spec} R_f$ and thus a locally closed point in $\operatorname{Spec} R$. But then P corresponds in fact to a closed point in $\operatorname{Spec} R$ and is therefore maximal. Contradiction!

(4) $\Rightarrow$ (6): Via the equivalence (4) $\Leftrightarrow$ (5), we can assume that X and Y are affine. In this case our result follows from [CA, 8.4.6].

(6) $\Rightarrow$ (5): Let $x \in X$ be a locally closed point, and let $U \subset X$ be an open subscheme such that x is closed in U. Now, the open immersion $U \hookrightarrow X$ is locally of finite type (2.12.1), and so we see that x must in fact be closed in X.

□

REMARK 6.2.3. Characterization (6) above is the Nullstellensatz.

obson-irred-closed-point

COROLLARY 6.2.4. *Let X be a Jacobson scheme with finitely many irreducible components. Then every irreducible component of X contains a closed point that is not contained in any other component.*

PROOF. Let $X_0 \subset X$ be the subset of closed points, and let $Z_1, \dots, Z_n$ be the irreducible components of X; then, for any $i \in \{1, \dots, n\}$, $X_0 \not\subseteq \bigcup_{j \neq i} Z_j$, since $\bigcup_{j \neq i} Z_j$ is closed and X_0 is dense in X. □

dim-algv-jacobson

COROLLARY 6.2.5. *Let X be a Jacobson scheme; then every X-scheme that's locally of finite type is also Jacobson. This is in particular true for $X = \operatorname{Spec} \mathbb{Z}$ or $X = \operatorname{Spec} k$, where k is a field.*

PROOF. We'll use characterization (6) above. Let $f : Y \to X$ be an X-scheme that's locally of finite type and let $g : Z \to Y$ be a Y-scheme that's locally of finite type; then $f \circ g$ is again an X-scheme that's locally of finite type. Therefore, a point $z \in Z$ is closed if and only if $f(g(z))$ is closed in X if and only if $g(z)$ is closed in Y.

The second statement follows immediately from the fact that $\mathbb{Z}$ and k are both Jacobson rings. □

3. Catenary and Universally Catenary Schemes

DEFINITION 6.3.1. A scheme X is *catenary* (resp. *equidimensional*, *equicodimensional*, *biequidimensional*) if its underlying topological space is catenary (resp. equidimensional, equicodimensional, biequidimensional).

A scheme X is *universally catenary* if every X-scheme that's locally of finite type is also catenary, where an X-scheme $f : Y \to X$ is catenary if the domain Y is catenary.

catenary-local-condition

PROPOSITION 6.3.2. *The following are equivalent for a scheme X:*

(1) *X is catenary.*
(2) *For every triple Z, T, W of closed subschemes of X with $Z \subset T \subset W$, we have*
$$\dim \mathscr{O}_{W,\zeta} = \dim \mathscr{O}_{W,\xi} + \dim \mathscr{O}_{T,\zeta},$$
where ζ is the generic point of Z and ξ is the generic point of T.
(3) *Every affine open subscheme $V \subset X$ is catenary.*
(4) *There is an affine open cover $\{V_i : i \in I\}$ such that V_i is catenary, for each i.*
(5) *For every $x \in X$, the local ring $\mathscr{O}_{X,x}$ is catenary.*

Proof. (1) ⇔ (3) ⇔ (4) follows from [NS, 6.16], and (1) ⇔ (2) follows from (1.5.5).

We will show (3) ⇔ (5). This comes down to showing that a ring R is catenary if and only if R_P is catenary for every prime $P \subset R$. This is immediate. □

catenary-local-condition

Corollary 6.3.3. *The following are equivalent for an affine scheme* $X = \operatorname{Spec} R$*:*

(1) *X is universally catenary.*
(2) *$\mathbb{A}^n_X$ is catenary, for all $n \in \mathbb{N}$.*

Proof. By the Proposition above, it suffices to show that every affine scheme of finite type over X is catenary. But every affine scheme of finite type is a closed subscheme of $\mathbb{A}^n_X$, for some $n \in \mathbb{N}$. Hence the result. □

4. Dimension Theory of Varieties

The dimension theory of algebraic varieties is the foundation of our studies.

Definition 6.4.1. An *algebraic variety*, or simply a *variety* over a field k is a separated k-scheme of finite type.

An *affine variety* over a field k is an affine scheme over k.

A *projective variety* over a field k is a projective scheme over k.

dim-algv-main-thm

Theorem 6.4.2. *Let X be an irreducible algebraic variety over a field k, and let ξ be its generic point. Then $\dim X = \operatorname{tr}\deg_k k(\xi)$, and X is biequidimensional. In particular, $\dim X = \dim U$, for any open subscheme $U \subset X$.*

Proof. Let $U = \operatorname{Spec} R \subset X$ be an affine open; then we find from [CA, 8.5.1] that $\dim U = \operatorname{tr}\deg_k k(\xi)$. The first assertion now follows from remark (6.1.2). Moreover, by the same theorem, if $z \in X$ is any closed point, then $\dim \mathscr{O}_{X,z} = \dim X$, which shows that X is biequidimensional. □

dim-algv-univ-catenary

Corollary 6.4.3. *Every algebraic variety over a field k is universally catenary.*

Proof. Since $\mathbb{A}^n_k$ is biequidimensional by (6.4.2), it's catenary by [NS, 6.18]. Now the result follows from (6.3.3). □

dim-algv-sup-trdeg

Corollary 6.4.4. *Let X be an algebraic variety, and let X_1 be its set of generic points; then we have*

$$\dim X = \sup_{x \in X} \operatorname{tr}\deg_k k(x) = \sup_{x \in X_1} \operatorname{tr}\deg_k k(x).$$

Proof. Simply observe that $\operatorname{tr}\deg_k k(x) = \dim \overline{\{x\}}$. □

dim-algv-dim-at-a-point

Corollary 6.4.5. *Let X be an algebraic variety and let $x \in X$ be a point.*

$$\dim_x X = \dim \operatorname{Spec} X, x + \operatorname{tr}\deg_k k(x).$$

Proof. Let $X_1, \ldots, X_n$ be the irreducible components of X, and let U be a neighborhood of X such that $\dim U = \dim_x X$ and such that every irreducible component of U contains x [NS, 6.4]. Then we have

$$\dim_x X = \dim U = \max X_i \ni x \dim(X_i \cap U) = \max X_i \ni x \dim X_i.$$

Equip X_i with the reduced induced subscheme structure; then, since the X_i containing x correspond bijectively to the irreducible components of Spec $\mathscr{O}_{X,x}$, and so we have

$$\dim \mathscr{O}_{X,x} = \max X_i \ni x \dim \mathscr{O}_{X_i,x}.$$

In sum, it suffices to prove the required identity in the case where X is integral. In this case, since U is biequidimensional, we have

$$\dim_x X = \dim U = \dim \operatorname{Spec} X, x + \dim \overline{\{x\}} = \dim \operatorname{Spec} X, x + \operatorname{tr} \deg_k k(x).$$

So □

Definition 6.4.6. Let $X = \operatorname{Proj} S$ be a projective variety over k. The *Hilbert polynomial* H_X of X is just the Hilbert polynomial $H(S, n)$ of the graded k-algebra S.

lgv-projective-varieties

Proposition 6.4.7. *Let $X = \operatorname{Proj} S$ be a projective variety over k. Then*

$$\dim X = \dim S - 1 = \deg H_X.$$

Proof. The second equality follows from [CA, 6.4.3]; so it suffices to prove the first equality. For this, observe that we can compute the dimension of X as the maximal length of a chain of homogeneous primes of S strictly contained in S^+. By [CA, 6.4.1], this is $\operatorname{ht} S^+ - 1$, which, by [CA, 6.4.2], is $\dim S - 1$. □

Now we consider the behavior of dimension under products.

dim-algv-products

Proposition 6.4.8. *Let X, Y be algebraic varieties. Then*

$$\dim X \times_k Y = \dim X + \dim Y.$$

Proof. We can assume that X and Y are affine. In this case, the Proposition is just a restatement of [CA, 8.5.6]. □

5. Dimension of Fibers: Chevalley's Theorem

Now, given any scheme Y and any Y-scheme $X \to Y$, note that the *fibers* of this Y-scheme are algebraic varieties. Hence we can use the results of the previous section to obtain important results about the dimensions of the fibers.

dim-dimension-of-fibers

Proposition 6.5.1. *Let $f : X \to Y$ be a morphism of locally Noetherian schemes, and let $x \in X$ and $y \in Y$ be such that $f(x) = y$.*

(1)
$$\dim \mathscr{O}_{X,x} \leq \dim \mathscr{O}_{Y,y} + \dim \mathscr{O}_{X_y,x},$$
where we treat x as an element of X_y via the homeomorphism of the latter with $f^{-1}(y)$. Equality holds if f satisfies the going down condition, and in particular if f is flat.

(2) *If X and Y are irreducible schemes with f dominant and locally of finite type, then*
$$\dim \mathscr{O}_{X,x} + \operatorname{tr} \deg_{k(y)} k(x) \leq \dim \mathscr{O}_{Y,y} + \operatorname{tr} \deg_{k(\upsilon)} k(\xi),$$
with equality holding whenever Y is universally catenary. Here, υ is the generic point of X, and ξ, that of X.

(3) *If X and Y are irreducible, Y is universally catenary, and f is dominant and locally of finite type, then*

$$\dim_x X_y \geq \operatorname{tr}\deg_{k(\upsilon)} k(\xi),$$

with equality holding if f satisfies the going down condition.

Proof. (1) We reduce immediately to the affine case, and here this is a consequence of [CA, 6.6.1] and [CA, 6.7.1].
(2) See [CA, 8.6.1].
(3) Put (1) and (2) together and use (6.4.5).

□

The next Theorem of Chevalley combines the niceness of flat morphisms with (5.4.7) for the important result that the dimension of fiber is an upper semicontinuous function on the domain.

icontinuity-of-dimension

Theorem 6.5.2 (Semicontinuity of dimension). *Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes, with Y universally catenary, and let $\dim_{X/Y} : X \to \mathbb{N} \cup \{\infty\}$ be the map $\dim_{X/Y}(x) = \dim_x X_{f(x)}$. Then $\dim_{X/Y}$ is upper semicontinuous on X; that is, for every $n \in \mathbb{N}$, $G_n = \{x \in X : \dim_{X/Y}(x) \geq n\}$ is a closed subset of X.*

Remark 6.5.3. The Theorem is true without the hypothesis that Y be universally catenary (or even Noetherian!), but the proof in such generality requires many technical results that we don't have the patience for here. It may however be found by the diligent reader in part 3 of EGA IV.

Proof of Theorem (6.5.2). We'll use Noetherian induction [NS, 3.6]. Let a closed subset Z of Y be said to satisfy property P if, for every closed subscheme of Y supported on Z and for every irreducible Z-scheme (we're conflating Z with the scheme supported on it) Z' of finite type, the function $\dim_{Z'/Z}$ is upper semicontinuous. We will show that this property satisfies the hypotheses needed for the application of Noetherian induction to work: namely, we'll show that if P is true for every proper closed subset of Y then P is true for Y.

First note that we can assume that Y is irreducible. Indeed, if Y' is the scheme theoretic image of any irreducible Y-scheme of finite type, then Y' is irreducible, and moreover $\dim_{X/Y} = \dim_{X/Y'}$. Hence by induction we may assume that Y is itself irreducible. Hence by part (3) of (6.5.1), we see that

$$\dim_{X/Y} \geq e := \operatorname{tr}\deg_{k(\upsilon)} k(\xi),$$

where υ is the generic point of Y and ξ, that of X.

By (5.4.7), we find an open subscheme $V \subset Y$ such that the restriction $f^{-1}(U) \to U$ is faithfully flat, and so, for $x \in f^{-1}(U)$, we have $\dim_{X/Y}(x) = e$. This tells us that, for $n > e$, $G_n \subset W = X \setminus f^{-1}(U)$. Therefore, by the induction, G_n is a closed subset of X (note that $W = f^{-1}(f(W))$), for $n > e$

Now, let X be any Y-scheme of finite type, and suppose $X_1, \ldots, X_n$ are the irreducible components of X. Then we have

$$\dim_{X/Y}(x) = \max_{1 \leq i \leq n} \dim_{X_i/Y}(x),$$

where $\dim_{X_i/Y_i}(x) = 0$, if $x \notin Y_i$. By what we've shown above, $\dim_{X_i/Y}$ is upper semicontinuous for each i, and so it follows that $\dim_{X/Y}$ is also upper semicontinuous. □

6. Pseudovarieties

DEFINITION 6.6.1. A locally Noetherian scheme S is a *pseudovariety* if it satisfies the following conditions:

(1) S is Jacobson and universally catenary.
(2) Every irreducible component of S is equicodimensional (or, equivalently, biequidimensional). That is, for every irreducible component S' and every closed point $s \in S'$, $\dim \mathscr{O}_{S',s} = \dim S'$.

REMARK 6.6.2. Observe that any algebraic variety is a pseudovariety.

im-pseudovariety-fintype

PROPOSITION 6.6.3. *Let S be a pseudovariety*

(1) *Every S-scheme that's locally of finite type is also a pseudovariety.*
(2) *Let $f : X \to Y$ be a dominant, locally of finite type morphism of irreducible S-schemes, where X and Y are both locally of finite type over S. Let ξ be the generic point of X and υ that of Y. Then, we have*

$$\dim X = \dim Y + \operatorname{tr\,deg}_{k(\upsilon)} k(\xi).$$

REMARK 6.6.4. Part (2) is a relative analogue of the statement that, for an irreducible algebraic variety X, we have $\dim X = \operatorname{tr\,deg}_k k(\xi)$.

PROOF. (1) Every S-scheme that's locally of finite type is Jacobson and universally catenary; so it suffices to show that every irreducible S-scheme $f : X \to S$ that's locally of finite type is equicodimensional. Now, f factors as $X \xrightarrow{j} Z \xrightarrow{i} S$, where $i : Z \to S$ is the scheme theoretic image of f. Since X is irreducible, it follows that Z is also irreducible, and so there is an irreducible component $S' \subset S$ containing Z. Now, since S' is biequidimensional, it follows that Z is also biequidimensional [NS, 6.18]. Hence Z is a pseudovariety. Now, let $x \in X$ be a closed point; then $s = f(x) \in Z$ is also a closed point. Since Z is universally catenary, we obtain from part (2) of (6.5.1) the following identity:

$$\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{Z,s} + \operatorname{tr\,deg}_{k(\zeta)} k(\xi) = \dim Z + \operatorname{tr\,deg}_{k(\zeta)} k(\xi),$$

where ζ is the generic point of Z, and ξ, that of X. This shows that $\dim \mathscr{O}_{X,x}$ is independent of x, and hence that X is equicodimensional.

(2) By (1), Y is a pseudovariety and so the result follows as in the proof of (1).

□

DEFINITION 6.6.5. A morphism $f : X \to Y$ is *equidimensional of dimension r* if the following conditions hold:

(1) f is of finite type.
(2) Every irreducible component of X dominates an irreducible component of Y.
(3) $\dim_{X/Y} \equiv r$ is constant.

ovariety-equidimensional

THEOREM 6.6.6. *Let Y be a Noetherian irreducible pseudovariety, and let $f : X \to Y$ be a flat Y-scheme of finite type. Then f is equidimensional of dimension r if and only if X is equidimensional of dimension* $\dim Y + r$.

Proof. Since f is flat, by (5.4.4), we already know that it satisfies condition (2) for equidimensionality. By hypothesis, it satisfies condition (1); so it remains to show that $\dim_{X/Y} \equiv r$ if and only if X is equidimensional of dimension $\dim Y + r$.

First assume that $\dim_{X/Y} \equiv r$; and let $Z \subset X$ be an irreducible component. By (6.2.2), X is also Jacobson, and so, by (6.2.4), we can find a closed point $z \in Z$ such that $\dim \mathscr{O}_{X,z} = \dim Z$. Given this, we have:

$$\dim Z = \dim \mathscr{O}_{X,z} = \dim \mathscr{O}_{Y,f(z)} + \dim \mathscr{O}_{X_{f(z)},z}.$$

where the second equality follows from (6.5.1) and the flatness of f. But now, since f is equidimensional and z is a closed point of $X_{f(z)}$, we see, using (6.4.5), that

$$\dim \mathscr{O}_{X_{f(z)},z} = \dim_z X_{f(z)} = r.$$

Since Y is equicodimensional, we have $\dim \mathscr{O}_{Y,f(z)} = \dim Y$. From these identities, one implication follows.

For the second implication, choose $x \in X$ and let $y = f(x)$. Then, by part (3) of (6.5.1), we have, for every irreducible component W of X containing x,

$$\dim_x W_y = \operatorname{tr}\deg_{k(\upsilon)} k(\omega) = \dim W - \dim Y = r$$

where υ is the generic point of Y and ω is the generic point of W. The second equality follows from (6.6.3). Now the implication we seek results from part (2) of [NS, 6.4]. □

CHAPTER 7

Algebraic Varieties

chap:algv

1. First Properties

DEFINITION 7.1.1. Let X be an algebraic variety over a field k, and let K/k be a field extension. Then, we denote by X_K the base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. This is an algebraic variety over K since both separated morphisms and morphisms of finite type are stable under base change. We denote by $X(K)$ the *set of K-valued points* $\operatorname{Hom}_{\mathrm{Sch}_k}(\operatorname{Spec} K, X)$.

An algebraic variety Y over K is said to be *defined over* k if $Y = X_K$, for some algebraic variety X over k.

algv-preimage-of-point

PROPOSITION 7.1.2. *Let X be an algebraic variety over k, and let $K = \overline{k}$ be the algebraic closure of k. Fix a point $y \in X$, and let $p : X_K \to X$ be the natural projection.*

1. *The closed points of X are dense in X.*
2. *$k(y)$ is an algebraic extension of k if and only if y is a closed point.*
3. *If y is closed, there is a natural bijection between $p^{-1}(y)$ and the number of k-embeddings of $k(y)$ in K. In particular, $\sharp p^{-1}(y) = [k(y)^{sep} : k]$, where $k(y)^{sep}/k$ is the largest separable sub-extension of $k(y)/k$.*

PROOF. (1) Follows from the fact that X is Jacobson (6.2.5).

(2) By (**??**), y is closed in X if and only if it's closed in some affine neighborhood $U = \operatorname{Spec} R$. So we can assume X is affine. In this case, y corresponds to some prime $P \subset R$, and is closed if and only if P is maximal, if and only if $\dim R/P = 0$, if and only if $\operatorname{tr}\deg_k K(R/P) = 0$.

(3) Suppose we're given a k-embedding of $k(y)$ in K. Then we have a morphism $\operatorname{Spec} K \to \operatorname{Spec} k(y)$ of k-schemes, which in turn gives us a morphism $\operatorname{Spec} K \to X$ of k-schemes, whose image is $\{y\}$. By the universal property of fiber products, this gives a morphism $\operatorname{Spec} K \to X_K$, which gives us a closed point of X_K that maps to y. Conversely, suppose we have a k-embedding $k(y) \hookrightarrow K$. Then, for every $x \in p^{-1}(y)$, we have the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\cong} & k(x) \\ \uparrow & & \uparrow p_x^{\sharp} \\ k & \longrightarrow & k(y) \end{array}$$

Note that the top arrow is an isomorphism by part (1), since K is algebraically closed and closed points pull back to closed points (**??**). Hence, we get a k-embedding of $k(y)$ in K. These two processes are inverse to each other, thus giving us our bijective correspondence.

□

-valued-points-morphisms

PROPOSITION 7.1.3. *Let the notation be as in the Proposition above, and let Y be another variety over k. Assume, in addition, that X_K is reduced, and let $f, g : X \to Y$ be two morphisms of k-schemes. If the induced maps $f(K), g(K) : X(K) \to Y(K)$ agree, then $f = g$.*

PROOF. We'll show that the morphisms $f_K, g_K : X_K \to Y_K$ are equal. That is, we'll show that the locus of agreement of f and g, $h : Z \to X$ is such that $h_K : Z_K \to X_K$ is isomorphic to 1_{X_K} (2.10.5). Given this, we claim that h is itself isomorphic to 1_{X_K}.

To see this, first observe that X is also reduced. We can reduce to the case where $X = \operatorname{Spec} R$, for some k-algebra R, with $R \otimes_k K$ a reduced ring. Let $a \in R$ be a nilpotent element; then a must map to 0 in $R \otimes_k K$. But K is flat over k, and so a must have been 0 in R to begin with. Now, since X is reduced, it suffices to show that the underlying topological space of Z is the whole space X. But for this, just observe that $Z = p(Z_K) = p(X_K) = X$, where $p : X_K \to X$ is the natural projection.

It remains to show that $f_K = g_K$. Now,

$$X(K) = \operatorname{Hom}_{\mathrm{Sch}_k}(\operatorname{Spec} K, X) = \operatorname{Hom}_{\mathrm{Sch}_K}(\operatorname{Spec} K, X_K) = X_K(K).$$

Moreover, the set $X_K(K)$ is in bijective correspondence with the closed points of X_K: every morphism $\operatorname{Spec} K \to X_K$ corresponds to a point $x \in X_K$, and an isomorphism $\mathscr{O}_{X_K,x} \xrightarrow{\cong} K$. Since K is algebraically closed, these correspond precisely the closed points of X. So we might as well replace X and Y with X_K and Y_K, and k with K, and assume that we're working with varieties over an algebraically closed field.

Since f and g induce the same maps from $X(k) \to X(k)$, we see that, for every closed point $x \in X$, $f(x) = g(x) = y$, for some closed point $y \in Y$. In particular, as maps of topological spaces, f and g agree on the dense subset of closed points, and hence agree on all of X. To show that they agree as morphisms of schemes, it suffices to consider the case where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine, with A reduced. Now, since B is finitely generated over k, there is a closed immersion $j : Y \to \mathbb{A}^n_k$, for some $n \in \mathbb{Z}$. Therefore, it's enough to consider the case where we have morphism $f, g : \operatorname{Spec} A \to \mathbb{A}^n_k$. In this case, we see from (4.3.8) that f and g are given by n-tuples of sections $(s_1, \dots, s_n)$ and $(s'_1, \dots, s'_n)$ of A. Moreover, the fact that f and g induce the same maps $X(k) \to \mathbb{A}^n_k(k)$ tells us that $s_i \equiv s'_i \pmod{\mathfrak{m}}$, for all i and all maximal ideals $\mathfrak{m} \subset A$. But then $s_i - s'_i \in \operatorname{Jac}(A)$, for all i. Now, since A is Jacobson and reduced, we have $\operatorname{Jac}(A) = \operatorname{Nil}(A) = 0$, and so $s_i = s'_i$, for all i, which of course means that $f = g$. This finishes our proof. □

2. Normal Varieties

algv-normalization

PROPOSITION 7.2.1. *Let X be an integral scheme, and let $L/K(X)$ be an algebraic extension. Then the normalization $\pi : X' \to X$ exists. Moreover, if X is an integral algebraic variety over a field k, then π is a finite morphism. Equivalently, the category of algebraic varieties over a fixed field k is closed under the process of normalization.*

Proof. Using the Fourfold Way and the universal property of normalization, it suffices to construct it for the case where $X = \operatorname{Spec} A$ is affine. In this case, let $A' \subset L$ be the integral closure of A in L, and let $X' = \operatorname{Spec} A'$. There is a natural integral morphism $\pi' : X' \to X$ induced by the inclusion $A \subset A'$. Let's verify that this satisfies the necessary properties. The first one is verified immediately. For the second, let $f : Y \to X$ be an integral morphism satisfying property (1); then, $Y = \operatorname{Spec} B$ is also affine, and f is induced by an embedding $A \subset B$. Since B is normal and $B \subset L$, we deduce that $A' \subset B$, which gives us a morphism $Y \to X'$ via which f factors through $\pi : X' \to X$.

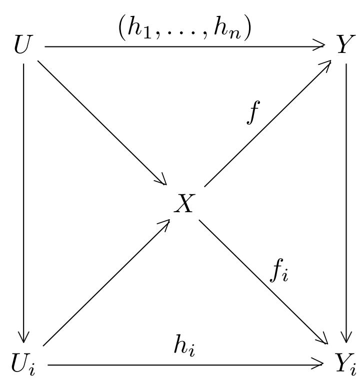
The second assertion follows from [CA, 8.3.1]. □

3. Non-singular Curves

Definition 7.3.1. A *non-singular curve* over a field k is a one dimensional regular variety over k.

-proper-curve-projective Theorem 7.3.2. *Let C be a non-singular proper curve over k; then C is projective over k.*

Proof. Let $\{U_i : 1 \leq i \leq n\}$ be a finite affine open cover for C, and, for $1 \leq i \leq n$, let Y_i be an integral projective variety over k equipped with an open immersion $h_i : U_i \to Y_i$ (to get such an immersion, take any morphism of U_i into a projective variety, and look at its scheme theoretic image). By (**??**), we can extend h_i to a surjective morphism $f_i : X \to Y_i$. Let $f : X \to Y$, where $Y = \times_i Y_i$ be the morphism with co-ordinates f_i (here, the fiber product of the Y_i is taken over k). Let $U = \bigcap_i U_i$; then, for every i, the following diagram commutes; moreover, the outer square is cartesian.



Let $Z \to Y$ be the scheme-theoretic image of f; then, by (2.5.5) it's a reduced and irreducible closed subscheme of Y, and is thus integral. Moreover, the induced morphism $g : X \to Z$ is dominant and is hence surjective, since X is proper. Observe now that the image of Z in Y_i under the natural projection $Y \to Y_i$ is all of Y_i, since the image of X in Y_i is all of Y_i. Therefore, we can decompose the map h_i into a composition of dominant morphisms $U_i \to X \to Z \to Y_i$. So, for $x \in U_i$, by (2.3.6), we have a sequence of injections

$$\mathscr{O}_{Y_i,h_i(x)} \hookrightarrow \mathscr{O}_{Z,g(x)} \hookrightarrow \mathscr{O}_{X,x}.$$

Now the composition here is just the map $\mathscr{O}_{Y_i,h_i(x)} \to \mathscr{O}_{U_i,x}$, which is an isomorphism. Hence the map $\mathscr{O}_{Z,g(x)} \to \mathscr{O}_{X,x}$ is also an isomorphism. Since the U_i cover X, and g is surjective, this implies that Z is also a non-singular curve. But now

$g : X \to Z$ is a birational morphism between regular curves and is therefore an isomorphism (**??**). □

4. Conjugation

DEFINITION 7.4.1. Let $\sigma \in \mathrm{Aut}(K/k)$; then the *conjugation map* $\sigma_X : X_K \to X_K$ is the morphism morphism

$$1_X \times \varphi : X_K \to X_K,$$

where $\varphi : \operatorname{Spec} K \to \operatorname{Spec} K$ is the morphism of k-schemes corresponding to $\sigma^{-1} : K \to K$.

It's clear that if $\sigma, \tau \in \mathrm{Aut}(K/k)$, then $(\sigma\tau)_X = \sigma_X \tau_X$. Hence $\sigma \mapsto \sigma_X$ gives a homomorphism from $\mathrm{Aut}(K/k)$ to $\mathrm{Aut}_{\mathrm{Sch}_k}(X_K)$. In particular, we have a natural action of $\mathrm{Aut}(K/k)$ on X_K. Moreover, if $f : X \to Y$ is a morphism of algebraic varieties over k, then we claim that the following diagram commutes:

$$\begin{array}{ccc} X_K & \xrightarrow{\sigma_X} & X_K \\ {\scriptstyle f_K}\downarrow & & \downarrow{\scriptstyle f_K} \\ Y_K & \xrightarrow{\sigma_Y} & Y_K \end{array}$$

In fact, both routes from X_K to Y_K are the same morphism: $f \times \varphi$, where $\varphi : \operatorname{Spec} K \to \operatorname{Spec} K$ is the morphism induced by the automorphism σ^{-1}.

raic-closure-conjugation

THEOREM 7.4.2. *Let X be an algebraic variety over k, and let $p : X_K \to X$ be the natural projection, where K is algebraic over k. Then*

(1) *p is surjective and closed.*
(2) *Now, suppose in addition that either K is the algebraic closure of k, or that K/k is a finite, Galois extension. For all $x, y \in X_K$, $p(x) = p(y)$ if and only if $x = \sigma_X(y)$, for some $\sigma \in \mathrm{Aut}(K/k)$. That is, for all $z \in X$, $p^{-1}(z)$ is an orbit of $\mathrm{Aut}(K/k)$.*
(3) *With the hypotheses as in the last part, p has finite fibers and is also open.*

PROOF. If $U \subset X$ is an open subscheme, then it's easy to see that $\sigma_X|_{p^{-1}(U)} = \sigma_U$. Therefore, since both things we have to prove are local (for the finite fibers part, recall that X_K is quasi-compact), we can assume that $X = \operatorname{Spec} A$ is affine.

(1) This follows immediately from the fact that the map $\psi : A \to A \otimes_k K$ is injective and integral. See (2.8.3).
(2) Let $P \subset A$ be a prime. Replacing A by $k(P)$, and $A \otimes_k K$ by the fiber

$$(A \otimes_k K) \otimes_A k(P) = k(P) \otimes_k K,$$

we can assume that $A = k'$ is a field. We have to show that $\mathrm{Aut}(K/k)$ acts transitively on $\operatorname{Spec}(k' \otimes_k K)$. Let $Q, Q' \subset k' \otimes_k K$ be two primes. First, suppose K/k is finite; and let $x \in Q'$. Let $\widetilde{x} = \prod_{\sigma \in \mathrm{Gal}(K/k)} \sigma(x)$; then $\widetilde{x}$ is fixed by everything in $\mathrm{Gal}(K/k)$, and so lies in $k' \cap Q' = 0$. In particular, it also lies in Q, and so there is $\sigma \in \mathrm{Gal}(K/k)$ such that $\sigma(x) \in Q$. In sum, we have $Q' \subset \bigcup_{\sigma \in \mathrm{Gal}(K/k)} \sigma(Q)$, and so by prime avoidance, we find that $Q' = \sigma(Q)$, for some $\sigma \in \mathrm{Gal}(K/k)$. Now suppose $K = \overline{k}$, and let Ω be a large extension field of k containing both the fields $(k' \otimes_k K)/Q$

and $(k' \otimes_k K)/Q'$. Then we have two embeddings α, β of k in Ω arising from the two natural maps of k into these two fields. Since both $\alpha(k)$ and $\beta(k)$ are the algebraic closures of k in Ω, it follows that $\alpha(k) = \beta(k)$, and $\alpha = \beta \circ \sigma$, for some $\sigma \in \mathrm{Aut}(\overline{k}/k)$. Therefore, we see that

$$\begin{aligned} \sum_i a_i \otimes b_i \in Q &\Leftrightarrow \sum_i a_i \otimes \alpha(b_i) = 0 \in \Omega \\ &\Leftrightarrow \sum_i a_i \otimes \beta(\sigma(b_i)) = 0 \in \Omega \\ &\Leftrightarrow \sum_i a_i \otimes \sigma(b_i) \in Q', \end{aligned}$$

which shows that $(1 \otimes \sigma)(Q) = Q'$.

(3) Suppose $P \subset A \otimes_k K$ is generated by $f_1, \ldots, f_r$, with $f_i = \sum_j (f'_{ij} \otimes a_{ij})$. Let $L = k[a_{ij}]$; then the subgroup of $\mathrm{Aut}(K/k)$ fixing L has finite index, and is contained in the subgroup fixing P. Hence P has only finitely many conjugates, showing that p has finite fibers.

Now suppose $U \subset X_K$ is an open subset. Then

$$U' = \bigcup_{\sigma \in \mathrm{Aut}(K/k)} \sigma(U),$$

is also open. Moreover, $p(U) = p(U')$; but $U' = p^{-1}(p(U))$, and since $p(X_K \setminus U')$ is closed, we see that $p(U) = p(U)$ is open.

□

k-valued-quotient-galois

Corollary 7.4.3. *With the hypotheses on K as in part (2) of the Proposition, X is the topological quotient of X_K under the action of* $\mathrm{Gal}(K/k)$.

Proof. Follows immediately from parts (1) and (2) of the Proposition. □

Definition 7.4.4. Let X be an algebraic variety over k, and let L/k be an extension of k. Then a closed point $y \in X$ is *L-rational* if $k(y) \cong L$.

algv-k-rational-points

Corollary 7.4.5. *With notation as above, if k is perfect, and $x \in X_K$ is a closed point, then $p(x)$ is k-rational if and only if x is fixed by all conjugations.*

Proof. Note that by Theorem (7.4.2), to say that x is fixed by all conjugations is equivalent to saying that $p^{-1}(p(x)) = \{x\}$. We see Proposition (7.1.2) that this can happen if and only if $[k(p(x))^{sep} : k] = 1$. But since k is perfect, this is equivalent to saying that $[k(p(x)) : k] = 1$, and hence that $k(p(x)) \cong k$. □

iber-product-alg-closure

Example 7.4.6. Let $k = \mathbb{Q}$, and let $K = \overline{\mathbb{Q}}$ (in fact, we can substitute any perfect field for $\mathbb{Q}$). Consider the K-scheme $Z = \operatorname{Spec} K \times_{\operatorname{Spec} k} \operatorname{Spec} K = \operatorname{Spec}(K \otimes_k K)$. The ring $K \otimes_k K$ is the direct limit of subrings of the form $L \otimes_k K$, where L/k is a finite Galois extension. Hence we find that

$$Z = \varprojlim_{L/k \text{ finite Galois}} \operatorname{Spec}(L \otimes_k K).$$

But now $W = \operatorname{Spec} L$ is an algebraic variety over k, and so we can apply the results of the last theorem to W_K. We claim that W_K is a finite, discrete space in one-to-one correspondence with the finite group $\mathrm{Gal}(L/k)$. That it's discrete follows from the fact that $L \otimes_k K$ is finite over K and is thus an artinian ring. We'll first consider the ring $L \otimes_k L$. For every $\sigma \in \mathrm{Gal}(L/k)$, we have a natural map $L \otimes_k L \to L$ given by $1 \otimes \sigma$. Let P_σ denote the kernel of this map. We claim that $\sigma \mapsto P_\sigma$ gives a one-to-one correspondence between $\mathrm{Gal}(L/k)$ and W_L. Indeed, since the action of $\mathrm{Gal}(L/k)$ is transitive by the Theorem, it suffices to show that $P_1 = P_\sigma$ if and only if $\sigma = 1$. Now, for any $\sigma \in \mathrm{Gal}(L/k)$,

$$P_\sigma = \{\sum_i l_i \otimes m_i : \sum_i l_i \sigma(m_i) = 0\}.$$

Observe that for all $l \in L$, $l \otimes l - l^2 \otimes 1 \in P_1$. Hence, if $P_1 = P_\sigma$, then this will also belong to P_σ, for every $l \in L$. But then $l\sigma(l) = \sigma(l)^2$, and hence $\sigma(l) = l$, for every $l \in L$. So, to prove our original claim, we should show that the natural projection $W_K \to W_L$ is a bijection. By the Theorem, it suffices to show that it is an injection. Since all the points in W_L are closed and L-rational, this is accomplished by the next Proposition (7.1.2).

So we find that the underlying topological space of Z is homeomorphic to $\lim_{\longleftarrow} \mathrm{Gal}(L/k)$, where the limit is taken over all finite Galois extensions of k. But this space is precisely the pro-finite group $\mathrm{Gal}(K/k)$! In sum, we've shown that as topological spaces $\mathrm{Spec}\,\overline{\mathbb{Q}} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are homeomorphic. Taking the fiber product of two one-point spaces has given us an infinite space that's not very well understood at all!

ble-connected-components

Corollary 7.4.7. *Let X be an algebraic variety over k, and let K/k be a finite Galois extension or the algebraic closure of k, and let $p : X_K \to X$ be the projection morphism.*

(1) *If X is irreducible with generic point ξ, then $p^{-1}(\xi)$ is the set of generic points of X_K. In particular,* $\mathrm{Aut}(K/k)$ *acts transitively on the generic points of X_K.*
(2) *If X is connected, then* $\mathrm{Aut}(K/k)$ *acts transitively on the connected components of X_K.*

Proof. (1) By the Theorem, it suffices to show that every generic point of X_K maps to the generic point of X. Since any automorphism must preserve generic points, this will do it. First, suppose $X = \mathrm{Spec}\,A$ is affine, where A has a unique minimal prime P. Then, $X_K = \mathrm{Spec}\,B$, where $B = A \otimes_k K$ is integral over A, and contains A. For $Q \subset B$ minimal, B/Q is still integral over $A/(A \cap Q)$, and so

$$\dim A/(A \cap Q) = \dim B/Q = \dim B = \dim A,$$

which shows that $A \cap Q$ is also minimal. This proves the statement for the affine case. For the general case, X has a covering by finitely many affines $U_1, \ldots, U_n$, and every generic point in $f^{-1}(U_i)$ maps to the generic point of U_i, which is of course also the generic point of X.

(2) Since X_K is Noetherian, each of the connected components is both open and closed. Since p is both open and closed as a morphism (the Theorem tells us so), and X is connected, we find that $p(X') = X$, for each connected component $X' \subset X_K$. But now, if X_1 and X_2 are connected components of X_K, and $y \in X_1$, then there is some $y' \in X_2$ such that $p(y) = p(y')$. Hence, by the Theorem, there is some $\sigma \in \mathrm{Aut}(K/k)$ such that $\sigma(y) = y'$; but then $\sigma(X_1) = X_2$. This finishes our proof.

□

Here's one consequence.

eom-connected-k-rational

COROLLARY 7.4.8. *Suppose X is a connected variety over k. If $X(k) \neq \emptyset$, then $X_{\overline{k}}$ is connected.*

PROOF. By the Corollary above, we know that $\mathrm{Aut}(K/k)$ acts transitively on the connected components of X_K, where $K = \overline{k}$. Assume, for the moment, that $X = \operatorname{Spec} A$ is affine, and let $x \in X$ be a k-rational point (whose existence we're guaranteed by hypothesis), and consider the fiber $X_K \times_X \operatorname{Spec} k(x)$: this is just

$$\operatorname{Spec}((A \otimes_k K) \otimes_A k(\mathfrak{m})) = \operatorname{Spec}((A \otimes_A k) \otimes_k K) = \operatorname{Spec}(k \otimes_k K) = \operatorname{Spec} K.$$

Hence, this is one point space, which implies that X_K has only one connected component. Now, in the general case, take any connected component Y of X_K, and an affine open $U \subset Y$, which contains a point in the preimage of x. Let $V = \cup_{\sigma \in \mathrm{Aut}(K/k)} \sigma(U)$: this is also an affine open of X_K, but has only one connected component, by the argument above. Hence X_K also has only one connected component. □

EXAMPLE 7.4.9. This is definitely not true without the hypothesis that $X(k)$ be non-empty. For example, consider any Galois extension L/k, and let $X = \operatorname{Spec} L$. In this case, we see from Example (7.4.6) that X_L is a discrete space (in bijection with $\mathrm{Gal}(L/k)$ in fact), and hence heavily disconnected.

Back to conjugation: note that the morphism $p : X_K \to X$ is a base change of the morphism $\operatorname{Spec} K \to \operatorname{Spec} k$, and so, by (4.3.5), we see that p is X-isomorphic to the X-scheme $\operatorname{Spec} g^*\mathscr{K} \to X$, where $\mathscr{K}$ is the quasi-coherent sheaf on $\operatorname{Spec} k$ induced by K and $g : X \to k$ is the structure morphism. Hence it follows that $p_*\mathscr{O}_{X_K} \cong g^*\mathscr{K}$. So for every $U \subset X$, we have an isomorphism of rings

$$\Gamma(p^{-1}(U), \mathscr{O}_{X_K}) \cong \Gamma(U, g^*\mathscr{K}) = \Gamma(U, \mathscr{O}_X) \otimes_k K,$$

where the last equality follows from the sheaf axiom and the fact that it's true for affine opens $U \subset X$. Thus, for every $U \subset X$, $\Gamma(U, \mathscr{O}_X)$ is a subring of $\Gamma(p^{-1}(U), \mathscr{O}_{X_K})$.

algv-k-topology-sheaf

PROPOSITION 7.4.10. *Let the notation be as in Theorem (7.4.2), with K the algebraic closure of k, and let $U \subset X$ be an open subscheme.*

(1) *There is an isomorphism*

$$\Gamma(U, \mathscr{O}_X) \otimes_k K \to \Gamma(p^{-1}(U), \mathscr{O}_{X_K}).$$

For all closed points $x \in p^{-1}(U)$, all $s \in \Gamma(U, \mathscr{O}_X)$, and every $\sigma \in \mathrm{Aut}(K/k)$, we have

$$(*) \qquad s(\sigma_X(x)) = \sigma(s(x)) \in K,$$

where we're treating s as a section of $\mathscr{O}_{X_K}$ via the isomorphism above.

(2) *Suppose X_K is reduced, and k is perfect; then the elements of the subring $\Gamma(U, \mathscr{O}_X)$ are precisely the ones that satisfy condition $(*)$ given above.*

PROOF. Note that the existence of the isomorphism was proved in the discussion right above. Also, in the equation $(*)$, we're implicitly using the Nullstellensatz to ensure that the residue fields at closed points in X_K is K.

(1) This question is local, so we can assume $X = U = \operatorname{Spec} R$ is affine. Suppose $\mathfrak{m} \subset R \otimes_k K$ is a maximal ideal corresponding to a closed point $x \in X_K$. Then, we observe that

$$\sigma_X(x) = \left[(1 \otimes \sigma^{-1})^{-1}([\mathfrak{m}])\right] = [(1 \otimes \sigma)(\mathfrak{m})].$$

So to say that $s(\sigma_X(x)) = \sigma(s(x))$, for some $s \in R$, is precisely to say that

$$s \pmod{(1 \otimes \sigma)(\mathfrak{m})} = \sigma(s \pmod{\mathfrak{m}}) \in K.$$

Now, suppose $s = a + m$, with $a \in K$(this is actually $1 \otimes a$) and $m \in \mathfrak{m}$; then we have

$$s = (1 \otimes \sigma)(s) = \sigma(a) + (1 \otimes \sigma)(m),$$

and so we have our result.

(2) Now, suppose $R \otimes_k K$ is reduced; then since K is Jacobson, we see that $\operatorname{Jac}(R \otimes_k K) = 0$. Moreover, since k is perfect, K/k is Galois, and so k is the fixed field of $\operatorname{Aut}(K/k) = \operatorname{Gal}(K/k)$. Suppose $(*)$ holds for some element $s \in R \otimes_k K$. By the argument in the previous part, given a maximal ideal $\mathfrak{m} \subset R \otimes_k K$, we have

$$\sigma(s \pmod{\mathfrak{m}}) = (1 \otimes \sigma)(s) \pmod{(1 \otimes \sigma)(\mathfrak{m})}.$$

So if $(*)$ holds, then for all maximal ideals $\mathfrak{m}$, we have (by replacing $\mathfrak{m}$ with $(1 + \sigma)(\mathfrak{m})$) that $s - (1 \otimes \sigma)(s) \in \mathfrak{m}$. Since the Jacobson radical is 0, this implies that $s = (1 \otimes \sigma)(s)$, for all $\sigma \in \operatorname{Gal}(K/k)$. But then $s \in (R \otimes_k K)^{\operatorname{Gal}(K/k)} = R$.

□

Definition 7.4.11. Let K/k be a field extension, and let Y be an algebraic variety over K equipped with an action by $\operatorname{Aut}(K/k)$, which satisfies the constraint that, for every $\sigma \in \operatorname{Aut}(K/k)$, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\widetilde{\sigma}} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\varphi_\sigma} & \operatorname{Spec} K, \end{array}$$

where φ_σ is the morphism induced by the automorphism σ^{-1} of K/k.

The *k-topology* on Y is the set of $\operatorname{Aut}(K/k)$-invariant open subsets of Y.

Suppose now that $K = \overline{k}$. Y is said to have a *k-structure* if there is an action of $\operatorname{Aut}(K/k)$ on Y, and there is a subsheaf $\mathscr{G}$ of $\mathscr{O}_Y$ in the k-topology, such that, for all open sets U in the k-topology, we have

$$\Gamma(U, \mathscr{G}) \otimes_k K \cong \Gamma(U, \mathscr{O}_Y),$$

and $\Gamma(U, \mathscr{G}) \subset \Gamma(U, \mathscr{O}_Y)^{\operatorname{Aut} K/k}$.

overing-from-base-change

Theorem 7.4.12. *Let k be a field, and let K/k be the algebraic closure of k. Suppose Y is an algebraic variety over K.*

(1) *Suppose Y has a k-structure; then there is a variety Y_0 over k such that $Y = (Y_0)_K$, and the induced k-structure agrees with the inherent one.*

(2) *If Y is reduced and k is perfect, then giving a k-structure is equivalent to simply giving an action of* $\mathrm{Gal}(K/k)$ *on Y. In this case, the variety Y_0 is determined uniquely.*
(3) *Let k be perfect, and let Y and Y' be two reduced varieties over K with k-structures. Then, to give a k-morphism $Y_0 \to Y_0'$ is equivalent to giving a* $\mathrm{Gal}(K/k)$*-equivariant K-morphism $Y \to Y'$. In particular, the category of reduced varieties over K with k-structures and* $\mathrm{Gal}(K/k)$*-equivariant morphisms is equivalent to the category of reduced varieties over k.*

Proof. (1) The underlying topological space of Y_0 is the quotient space of Y by the action of $\mathrm{Aut}(K/k)$. We give Y_0 the structure of a ringed space by setting, for each $\mathrm{Aut}(K/k)$-invariant open set $U \subset Y$,

$$\Gamma(p(U), \mathscr{O}_{Y_0}) = \Gamma(U, \mathscr{G}),$$

where $p : Y \to Y_0$ is the quotient map. To see that this actually gives us the structure of a k-scheme on Y_0, it's enough to consider the case where $Y = \operatorname{Spec} S$ is affine. We'll show that in this case $Y_0 = \operatorname{Spec} \Gamma(Y, \mathscr{G})$. First note that the map $\Gamma(Y, \mathscr{G}) \to \Gamma(Y, \mathscr{O}_Y)$ induces a morphism of schemes $p' : Y \to \operatorname{Spec} \Gamma(Y, \mathscr{G})$. Since S is integral over $\Gamma(Y, \mathscr{G})$, this morphism is surjective and closed. To show that Y_0 is homeomorphic to $\operatorname{Spec} \Gamma(Y, \mathscr{G})$, it suffices to show that the fibers of p' are precisely the orbits of $\mathrm{Aut}(K/k)$. Let $X = \operatorname{Spec} \Gamma(Y, \mathscr{G})$; then we find that $Y = X \times_k K$. It's enough to show that the induced action by $\mathrm{Aut}(K/k)$ via this decomposition is the same as the original action; for the result will then follow from Theorem (7.4.2). Let $\sigma \in \mathrm{Aut}(K/k)$, and denote its original action on Y by the automorphism $\widetilde{\sigma} : S \to S$. Since $\widetilde{\sigma}$ fixes $\Gamma(Y, \mathscr{G})$, we obtain immediately that $\widetilde{\sigma} = 1 \otimes \sigma^{-1}$ (recall that the action of $\widetilde{\sigma}$ on K is via σ^{-1}) So it's enough to show that $\mathrm{Aut}(K/k)$ fixes $\Gamma(Y, \mathscr{G})$.
(2) Proposition (7.4.10) tells us that $\Gamma(Y, \mathscr{G})$, and hence the scheme structure on Y_0, is completely determined by the action of $\mathrm{Gal}(K/k)$.
(3) We showed earlier that a morphism $f : X \to X'$ induces a $\mathrm{Gal}(K/k)$-equivariant morphism $f_K : X_K \to X'_K$. Conversely, if we have a $\mathrm{Gal}(K/k)$-equivariant morphism $g : Y \to Y'$, then it clearly induces a continuous map $g_0 : Y_0 \to Y_0'$. It suffices to prove that $g^\sharp(\mathscr{O}_{Y_0'}) \subset g_* \mathscr{O}_{Y_0}$. For this we will take $U \subset Y_0'$ open and show that, for every $s \in \Gamma(U, \mathscr{O}_{Y_0'})$, $s' = g_U^\sharp(s)$ satisfies condition $(*)$. Indeed, let $y \in U$ be a closed point; then so is $g(y) \in U'$, and we have

$$\sigma(s'(y)) = \sigma(s(g(y))) = s(\sigma_{Y_0'}(g(y))) = s(g(\sigma_{Y_0}(y))) = s'(\sigma_{Y_0}(y)).$$

This finishes our proof.

□

Example 7.4.13 (Make more precise). Let $k = \mathbb{R}$, $X = \operatorname{Spec} \mathbb{R}[x,y]/(x^2 + y^2 - 1)$; then $K = \mathbb{C}$ and $X_\mathbb{C}$ is a plane conic with two points at infinity. $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ acts on $X_\mathbb{C}$ via conjugation. Now, $\mathbb{C}[x,y]/(x^2 + y^2 - 1)$ is clearly a domain of dimension 1, and so all its points (except for the generic point) are closed, corresponding to maximal ideals of the form $(x - z, y - w)$, with $z^2 + w^2 = 1$. If $(z, w) \notin \mathbb{R}^2$, consider the line in $\mathbb{A}^2_\mathbb{C}$ containing (z, w) and $(\overline{z}, \overline{w})$: this is cut out by the ideal $(-\Im(w)x + \Im(z)y + (\Im(w)z - \Im(z)w))$. Let $\alpha = -\frac{\Im(w)}{\Im(w)z - \Im(z)w}$, and let

$\beta = \frac{\Im(z)}{\Im(w)z - \Im(z)w}$. Then the $\mathbb{C}$-rational point in X corresponding to this conjugate pair is the maximal ideal $(\alpha x + \beta y - 1)$ (it is certainly maximal, and its extension splits into the maximal ideals of each of the conjugates).

Now, note that $X_{\mathbb{C}}$ is homeomorphic to the punctured complex plane, while X is homeomorphic to the punctured disc, via the map $(\alpha, \beta) \mapsto (x - \alpha, y - \beta)$ if $\alpha^2 + \beta^2 = 1$, and $(\alpha, \beta) \mapsto (\alpha x + \beta y - 1)$ if $\alpha^2 + \beta^2 < 1$.

Going back to $X_{\mathbb{C}}$, note that its projective closure Y is a non-singular projective plane conic, which is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. The argument above can be extended to show that $Y_0 = \operatorname{Proj} \mathbb{R}[x, y, z]/(x^2 + y^2 - z^2)$. Of course, it's also true that if $X' = \mathbb{P}^1_{\mathbb{C}}$, then $X'_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$. This gives us two non-isomorphic $\mathbb{R}$-varieties that extend to the same complex variety! But what happened to the uniqueness promised us in the Theorem? The point is that the the conjugation action on Y, and the conjugation action on $X'_{\mathbb{C}}$ are entirely different. The first one flips the two constituents of the standard affine open cover on $\mathbb{P}^1_{\mathbb{C}}$, while the second one doesn't.

5. Behavior under Base Change

DEFINITION 7.5.1. If P is a property of schemes (say, reducedness or integrality, for example), then X is *geometrically* P if $X_{\overline{k}}$ has property P.

The next technical lemma will be very useful in the study of the behavior of algebraic varieties under base change.

d-subschemes-base-change

LEMMA 7.5.2. *Let X be an algebraic variety over k, and let K be an algebraic extension of k. Suppose $Z \subset X_K$ is a reduced, closed subscheme. Then, there exists a finite extension $K' \supset K$ and a unique reduced closed subscheme $W \subset X_{K'}$ such that $W_K = Z$.*

PROOF. First assume $X = \operatorname{Spec} A$ is an affine algebraic variety, and let $Z \subset X_K$ correspond to some radical ideal $I \subset A \otimes_k K$. Suppose $I = (f_1, \ldots, f_r)$, and let K'/k be the extension of k generated by the coefficients of the f_i (i.e. if $f_i = \sum_j (f'_{ij} \otimes a_{ij})$, then let $K' = k[a_{ij}]$). Let $W \subset X_{K'}$ be the closed subscheme cut out by $I' = (f_1, \ldots, f_r) \subset A \otimes_k K'$. Then it's clear that $W_K = Z$. Moreover, since the underlying topological space of W is the image of Z under the morphism $X_K \to X_{K'}$, the uniqueness of W follows from its reducedness.

For the general case, suppose X has a covering by finitely many affine opens $\{U_i\}$. Let K'/k be a finite extension, such that, for each i, $W_i \subset (U_i)_{K'}$ is the unique reduced closed subscheme such that $(W_i)_K = Z \cap (U_i)_K$. By the uniqueness condition, and the separatedness of X, it follows that we can glue together the W_i to obtain a reduced closed subscheme $W \subset X_{K'}$ such that $W_K = Z$. □

lly-p-finite-extension-p

PROPOSITION 7.5.3. *Let X be an algebraic variety over k. Let P be a property of varieties, where P is one of: reduced, connected, irreducible and integral. Then the following are equivalent:*

(1) *X is universally P.*
(2) *For every finite extension K/k, X_K is P.*

PROOF. (1) ⇒ (2)**:** Connectedness and irreducibility follow immediately since X_K is the image of $X_{\overline{k}}$ under the natural morphism $p : X_{\overline{k}} \to X_K$. It suffices to prove the statement for the case where P is reducedness, since the statement for integrality will follow from this combined with the one for irreducibility. But this follows immediately from the fact, that for every open set $U \subset X_K$, the ring $\Gamma(U, \mathscr{O}_{X_K})$ injects into $\Gamma(p^{-1}(U), \mathscr{O}_{X_{\overline{k}}})$ (7.4.10).

(2) $\Rightarrow$ (1)**:** We'll do reducedness first. For this, we can assume that $X = \operatorname{Spec} A$ is affine. Suppose $A \otimes_k \overline{k}$ has a nilpotent element s; then in fact this nilpotent element will also live in a finite extension of k, by the argument of the Lemma. Hence if X_K is reduced for every finite extension K/k, then it's geometrically reduced.

Now we turn to connectedness: for this, we use the criterion from (1.5.7). According to this criterion, it suffices to show that $\Gamma(X_{\overline{k}}, \mathscr{O}_{X_{\overline{k}}})$ has no idempotents. But if this contains an idempotent, then we can find a finite extension K/k such that the ring of global sections of X_K also contains an idempotent (as always, just adjoin the coefficients of the idempotent).

Finally, we get to irreducibility. Let Z be an irreducible component of $X_{\overline{k}}$ equipped with the reduced induced subscheme structure; then we can find a finite extension K/k, and a unique reduced closed subscheme $W \subset X_K$ such that $W_{\overline{k}} = Z$ (Lemma (7.5.2) makes this possible). By (7.4.7), every generic point of $X_{\overline{k}}$ maps to the generic point of X_K, which is irreducible, by hypothesis. In particular, this means that W, which is the image of Z, contains the generic point of X_K. But then $W = X_K$, and hence $Z = W_{\overline{k}} = X_{\overline{k}}$.

□

nseparable-homeomorphism

PROPOSITION 7.5.4. *Let X be an algebraic variety over k.*

(1) *If K/k is a purely inseparable extension, then the projection $p : X_K \to X$ is a homeomorphism.*
(2) *Let k^{sep} be the separable closure of k in $\overline{k}$. Then X is geometrically connected (resp. irreducible) if and only if $X_{k^{sep}}$ is connected (resp. irreducible).*

PROOF. (1) First, suppose $K = k[t]/(t^q - a)$, for some $a \notin k^q$, where $q = \operatorname{char} k$. We can assume that $X = \operatorname{Spec} A$ is affine; let $P \subset A$ be a prime. We want to show that $K \otimes_k k(P)$ has only one prime. Now, this tensor product looks like $k(P)[t]/(t^q - a)$; so it suffices to show that, for any field L over k, the ring $L[t]/(t^q - a)$ has a solitary prime. Indeed, let $\lambda \in \overline{L}$ be such that $\lambda^q = a$. Then

$$L[t]/(t^q - a) \hookrightarrow \overline{L}[t]/(t - \lambda)^q$$

is an integral inclusion and thus induces a surjective map of ring spectra. But the ring on the right quite clearly has only one prime ideal (it's the prime $(t - \lambda)$).

Now, if K/k is a finite purely inseparable extension, then by repeating the above step finitely many times, we find that $p : X_K \to X$ is injective.

Let K/k be any purely inseparable extension; then $K \otimes_k k(P)$ is the direct limit of the rings $L \otimes_k k(P)$, where $L \subset K$ ranges over the finite subextensions of K. This tells us that, topologically, $\operatorname{Spec}(K \otimes_k k(P))$ is the inverse limit of the one point spaces $\operatorname{Spec}(L \otimes_k k(P))$, and is thus itself a one point space, which is what we wanted to show: the fact that p is a homeomorphism follows immediately from this via Theorem (7.4.2).

(2) Suppose $X_{k^{sep}}$ is connected (resp. irreducible), then, by the previous part, $X_{\overline{k}}$ is homeomorphic to $X_{k^{sep}}$ and is thus also connected (resp. irreducible). Conversely, if X is geometrically connected (resp. irreducible), then the same argument as in the proof of the first implication of the last Proposition gives us that $X_{k^{sep}}$ is connected (resp. irreducible).

□

EXAMPLE 7.5.5 (Reduced, yet not geometrically reduced). Taking inspiration from the proof above, consider the field $L = \mathbb{F}_q(t)[u]/(u^q - t)$: $X = \operatorname{Spec} L$ is a reduced variety over $\mathbb{F}_q(t)$, but

$$X_{\mathbb{F}_q(t^{1/q})} = \operatorname{Spec}(\mathbb{F}_q(t^{1/q})[u]/(u - t^{1/q})^q)$$

is evidently not reduced. This is actually the simplest possible example of this phenomenon; it can't happen if the field k is perfect, as we'll show very soon.

More generally, let K/k be any purely inseparable extension. Then $X = \operatorname{Spec} K$ is reduced over k, but X_K is not. To see this, simply observe that $K \otimes_k K$ is not a domain, but has a solitary prime ideal, since its spectrum is homeomorphic to $\operatorname{Spec} K$, according to the last Proposition. Again, we needed k to be imperfect for this to work.

EXAMPLE 7.5.6 (Irreducible, but not geometrically irreducible). Let k be any field of characteristic different from 2 such that $k^2 \neq k$. Let $a \in k \setminus k^2$, and consider the projective variety

$$X = \operatorname{Proj} k[x, y]/(x^2 - ay^2).$$

We claim that this is integral. It is connected (its ring of global sections is k) and has only finitely many irreducible components; so it's enough to show that its stalk at every point is integral (1.6.5). Over the open subscheme $X_{(x)}$, the ring of global sections is $k[t]/(t^2 - a^{-1})$, and over the open subscheme $X_{(y)}$, it's $k[u]/(u^2 - a)$. Since a is not a square, both of these are degree 2 extension fields of k, and so we see that X is integral. But now, if $K = k[\sqrt{a}]$, then

$$X_K = \operatorname{Proj} K[x, y]/(x^2 - ay^2) = \operatorname{Proj}(K[x, y]/(x - \sqrt{a}y)(x + \sqrt{a}y)).$$

This is evidently not irreducible, since, for example,

$$(X_K)_{(y)} = \operatorname{Spec}(K[u]/(u - \sqrt{a}) \times K[w]/(w + \sqrt{a}))$$

is not connected and thus is not irreducible. Incidentally, if we consider $Y = \operatorname{Spec} k[u]/(u^2 - a)$, then this is an example of a connected variety that is not geometrically connected.

algv-separable-reduced

PROPOSITION 7.5.7. *Let X be a reduced variety over k.*

(1) *Let K/k be a separable extension; then X_K is also reduced.*
(2) *If k is perfect, then X is reduced if and only if it is geometrically reduced.*
(3) *If $\operatorname{char} k = q > 0$, and $K = k^{-q^\infty}$ is the perfect closure of k, then X is geometrically reduced if and only if X_K is reduced.*

PROOF. These are local questions; so we can assume $X = \operatorname{Spec} A$ is affine, for some reduced finitely generated k-algebra A.

(1) Let $\{P_1, \ldots, P_n\}$ be the minimal primes of A; then the natural map $A \to \prod_i A/P_i$ is an injection. Hence, sine K is flat over k, we see that $A_K \to \prod_i (A/P_i)_K$ is also an injection. So we can reduce to the case where A is a domain, and since A_K embeds in $K(A)_K$, we can in fact assume that

$A = L$ is a field. Now, if $L \otimes_k k'$ is reduced, for all finite subextensions k'/k of K/k, then it's clear that $L \otimes_k K$ will be reduced (use Lemma (7.5.2)). So we can assume that K/k, in which case $K = k[t]/(f(t))$, for some separable polynomial $f(t) \in k[t]$. Then, we find

$$L \otimes_k K = L[t]/(f(t))$$

is still reduced, since $f(t) \in L[t]$ is still separable. For this, just observe that $L[t]/(f(t)) \hookrightarrow \overline{L}[t]/(f(t))$, and the latter is clearly a reduced ring.

(2) If k is perfect, $\overline{k}$ is a separable extension of k, and we're done by the last part, and Proposition (7.5.3).

(3) X is geometrically reduced if and only if $X_{\overline{k}}$ is reduced if and only if X_K is reduced. This follows from the last part, and the fact that K is perfect. □

Integral varieties are easier to handle.

al-varieties-base-change

PROPOSITION 7.5.8. *Let X be an integral variety over a field k.*

(1) *For any extension K/k, X_K is reduced (resp. integral) if and only if $K(X) \otimes_k K$ is reduced (resp. integral).*

CHAPTER 8

Vector Bundles

chap:vect

vect-vector-bundles

1. Vector Bundles and Locally Free Sheaves

1.1. The Sheaf of Local Sections. Let X be a scheme. Recall from [NOS, 6.2] the notion of a bundle section. We showed in [NOS, 6.3] that for every map $V \to X$ we can form the associated sheaf of sections Γ_V over X. If $V \to X$ is an X-scheme, we impose the additional restriction that our sections must be morphisms of schemes, we still get a sheaf, since being a morphism of schemes is again a local property. We'll call this $\mathscr{S}_{V/X}$, the associated sheaf of scheme-theoretic sections.

Definition 8.1.1. If $\mathscr{A}$ and $\mathscr{B}$ are two $\mathscr{O}_X$-algebras, then we define the presheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_X\text{-alg}}(\mathscr{A}, \mathscr{B})$ to be given by the assignment

$$U \mapsto \mathrm{Hom}_{\mathscr{O}_U\text{-alg}}(\mathscr{A}|_U, \mathscr{B}|_U).$$

undle-sections-sheaf-hom

Lemma 8.1.2. *Let $f : V \to X$ be a scheme over X. Then we have a natural isomorphism of sheaves of sets*

$$\mathscr{S}_{V/X} \cong \underline{\mathrm{Hom}}_{\mathscr{O}_X\text{-}alg}(f_*\mathscr{O}_V, \mathscr{O}_X).$$

In particular, the presheaf on the right is in fact a sheaf.

Proof. Observe that, by definition, for an open set $U \subset X$, $\mathscr{S}_{V/X}(U)$ is the set $\mathrm{Hom}_{\mathrm{Sch}_U}(U, f^{-1}(U))$. By (4.3.7), we have natural bijections

$$\begin{aligned} \mathscr{S}_{V/X}(U) &\cong \mathrm{Hom}_{\mathrm{Sch}_U}(U, f^{-1}(U)) \\ &\cong \mathrm{Hom}_{\mathscr{O}_U\text{-alg}}(f_*(\mathscr{O}_V|_{f^{-1}(U)}), \mathscr{O}_U) \\ &\cong \mathrm{Hom}_{\mathscr{O}_U\text{-alg}}((f_*\mathscr{O}_V)|_U, \mathscr{O}_U). \end{aligned}$$

This shows the isomorphism that we sought. Observe that the second bijection we showed above follows from (4.3.7). □

We will now investigate the behavior of the sheaf of sections under base change. Let $g : V \to X$ and $f : Y \to X$ be two X-schemes. Via base change, we obtain a Y-scheme $g_Y : V_Y = V \times_X Y \to Y$. How is $\mathscr{S}_{V_Y/Y}$ related to $\mathscr{S}_{V/X}$?

cal-sections-base-change

Proposition 8.1.3. *With the notation as in the discussion above, we have a natural morphism*

$$\mathscr{S}_{V/X} \longrightarrow f_*\mathscr{S}_{V_Y/Y}.$$

PROOF. For, given a local section $s : U \to g^{-1}(U)$, we get a local section $\widetilde{s} : f^{-1}(U) \to g^{-1}(U) \times_U f^{-1}(U)$ via the following base change diagram

$$\begin{array}{ccccc} f^{-1}(U) & \xrightarrow{\widetilde{s}} & g^{-1}(U) \times_U f^{-1}(U) & \longrightarrow & f^{-1}(U) \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{s} & g^{-1}(U) & \xrightarrow{g} & U \end{array}$$

Here we used the isomorphism

$$(f^{-1}(U) \times_U g^{-1}(U)) \times_{g^{-1}(U)} U \cong f^{-1}(U) \times_U U \cong f^{-1}(U).$$

The composition of the bottom row is the identity on U, and so the composition of the top row will be the identity on $f^{-1}(U)$. We still have to check that this defines a morphism of sheaves. For this, suppose $V \subset U$ is another open set, and let $s|_V$ be the restriction of s to V. Then it follows from the following fiber diagram that $\widetilde{s}|_{f^{-1}(V)} = \widetilde{s|_V}$.

$$\begin{array}{ccccc} f^{-1}(V) & \longrightarrow & f^{-1}(U) & \xrightarrow{\widetilde{s}} & g{-}1(U) \times_U f^{-1}(U) \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U & \xrightarrow{s} & g^{-1}(U) \end{array}$$

□

vect-V-construction

1.2. The $\mathbb{V}$-construction.

DEFINITION 8.1.4. For any $\mathscr{O}_X$-module $\mathscr{E}$, we set $\mathbb{V}(\mathscr{E}) \to X$ to be the X-scheme

$$\operatorname{Spec}(\operatorname{Sym}(\mathscr{E})) \to X.$$

epresentability-glob-sym

LEMMA 8.1.5. *For any $\mathscr{O}_X$-module $\mathscr{E}$, and any X-scheme Y, we have a natural bijection*

$$\operatorname{Hom}_{\mathscr{O}_Y}(f^*\mathscr{E}, \mathscr{O}_Y) \cong \operatorname{Hom}_{\mathrm{Sch}_X}(Y, \mathbb{V}(\mathscr{E})).$$

In other words, the functor

$$\mathrm{Sch}_X^{op} \to \mathrm{Ab}$$
$$(f : Y \to X) \mapsto \operatorname{Hom}_{\mathscr{O}_Y}(f^*\mathscr{E}, \mathscr{O}_Y) \cong \Gamma(Y, f^*\check{\mathscr{E}})$$

is represented by the X-scheme $\mathbb{V}(\mathscr{E}) \to X$.

PROOF. We have the following sequence of natural bijections

$$\begin{aligned} \operatorname{Hom}_{\mathscr{O}_Y}(f^*\mathscr{E}, \mathscr{O}_Y) &\cong \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E}, f_*\mathscr{O}_Y) \\ &\cong \operatorname{Hom}_{\mathscr{O}_X\text{-alg}}(\operatorname{Sym}(\mathscr{E}), f_*\mathscr{O}_Y) \\ &\cong \operatorname{Hom}_{\mathrm{Sch}_X}(Y, \mathbb{V}(\mathscr{E})), \end{aligned}$$

where the second bijection follows from [RS, **??**], and the third, from (4.3.7). □

isms-into-affine-n-space

EXAMPLE 8.1.6. Take $\mathscr{E} = \mathscr{O}_X^n$ in the above lemma. Then we find that $\operatorname{Hom}_{\mathrm{Sch}_X}(Y, \mathbb{A}_X^n)$ is in natural bijection with $\Gamma(Y, \mathscr{O}_Y^n)$. More descriptively, given an n-tuple of global sections $(s_1, \ldots, s_n)$ over Y, we get a natural map $\Gamma(X, \mathscr{O}_X)^n \to \Gamma(Y, \mathscr{O}_Y)$, which gives a morphism $\mathscr{O}_X^n \to f_*\mathscr{O}_Y$, and thus induces a morphism $Y \to \mathbb{A}_X^n$.

1.3. Vector Bundles.

DEFINITION 8.1.7. A *vector bundle* over a scheme X is an X-scheme $f : V \to X$ satisfying the following conditions:

(1) There is an open cover $\{U_i\}$ of X equipped with isomorphisms of X-schemes
$$\rho_i : f^{-1}(U_i) \to \mathbb{A}_{U_i}^n = \mathbb{A}_{\mathbb{Z}}^n \times_{\mathbb{Z}} U_i.$$
We call the U_i *trivializing opens*, and the collection $\{U_i, \rho_i\}$ a *trivializing collection*. Observe that the n need not be the same over all trivializing opens.

(2) For any pair (i, j) of indices and any affine open $\operatorname{Spec} R \subset U_i \cap U_j$, the horizontal map in this diagram
$$\begin{array}{ccc} \operatorname{Spec} R[x_1, \ldots, x_n] & \longrightarrow & \operatorname{Spec} R[x_1, \ldots, x_n] \\ & {\scriptstyle \rho_i^{-1}} \searrow \quad \swarrow {\scriptstyle \rho_j^{-1}} & \\ & f^{-1}(U_i \cap U_j) & \end{array}$$
is induced by a linear isomorphism. That is, it's induced by the assignment
$$x_i \mapsto \sum_j a_{ij} x_j,$$
for some $a_{ij} \in R$, so that $A = (a_{ij})$ is an invertible matrix over R. Observe that we've used the isomorphism
$$\mathbb{A}_{\mathbb{Z}}^1 \times_{\mathbb{Z}} \operatorname{Spec} R = \operatorname{Spec}(\mathbb{Z}[x_1, \ldots, x_n] \otimes R) = \operatorname{Spec} R[x_1, \ldots, x_n].$$

A morphism between two vector bundles $f : V \to X$ and $g : W \to X$ is a just a morphism of X-schemes $h : V \to W$.

-sheaf-of-local-sections

PROPOSITION 8.1.8. *If $f : V \to X$ is a vector bundle over X, the associated sheaf of scheme-theoretic sections $\mathscr{S}_{V/X}(U)$ has a natural $\mathscr{O}_X$-module structure, and is in fact a locally free sheaf.*

PROOF. Let $U = \operatorname{Spec} R \subset X$ be contained in a trivializing open set, and let $\rho_U : f^{-1}(U) \to \mathbb{A}_R^n$, be the trivialization of V on U. Using the Lemma above, we get the isomorphisms:
$$\begin{aligned} \mathscr{S}_{V/X}(U) &\cong \operatorname{Hom}_{\mathscr{O}_U\text{-alg}}((f_*\mathscr{O}_V)|_U, \mathscr{O}_U) \\ &\cong \operatorname{Hom}_{\mathscr{O}_{\operatorname{Spec} R}\text{-alg}}(f_*\mathscr{O}_{\mathbb{A}_R^n}, \mathscr{O}_U) \\ &\cong \operatorname{Hom}_{R\text{-alg}}(R[x_1, \ldots, x_n], R) \\ &\cong \operatorname{Hom}_R(R^n, R) \\ &\cong R^n \end{aligned}$$
which allows us to define a natural R-module structure on $\mathscr{S}_{V/X}(U)$. Suppose U is also contained in a different trivializing open, giving another trivialization ρ'_U of V

over U. We want to show that the two R-module structures induced on $\mathscr{S}_{V/X}(U)$ agree. For this, we must investigate how we got the isomorphism

$$\mathrm{Hom}_{R\text{-alg}}(R[x_1,\dots,x_n],R) \cong \mathrm{Hom}_R(R^n,R),$$

in the first place. We can identify $R[x_1,\dots,x_n]$ with the symmetric algebra $\mathrm{Sym}(R^n)$, and we have a natural isomorphism

$$\mathrm{Hom}_{R\text{-alg}}(\mathrm{Sym}(R^n),R) \cong \mathrm{Hom}_R(R^n,R),$$

which we get from the universal property of the symmetric algebra. In particular, this gives the set on the left a natural R-module structure that's *independent* of the choice of basis for R^n. But, by the first condition in the definition of a vector bundle, the two 'different' R-module structures on $\mathscr{S}_{V/X}(U)$ differ only by a linear isomorphism ${\rho^\sharp_U}^{-1} \circ \rho'^\sharp_U$ of $\mathrm{Sym}(R^n)$, which amounts only to a change of basis for R^n. Thus, we see that the two R-module structures are in fact the same: there is a *natural* R-module structure on $\mathscr{S}_{V/X}(U)$ independent of choice of trivialization.

Since we can do this for every trivializing affine open in X, and since such affine opens give us a basis for the topology on X, we can extend this to a $\mathscr{O}_X$-module structure on $\mathscr{S}_{V/X}$ in standard fashion. That this makes $\mathscr{S}_{V/X}$ a locally free sheaf follows from the proof above (over a trivializing $\mathrm{Spec}\,R$ it's isomorphic to the sheaf $\widetilde{R^n}$). □

Given a locally free sheaf $\mathscr{E}$ of rank n on X, we can consider the corresponding affine morphism $\mathbb{V}(\mathscr{E}) \to X$. Observe that on any open set U over which $\mathscr{E}$ is free, this is just the morphism $\mathrm{Spec}\,\mathrm{Sym}(\mathscr{O}_U^n) \to U$. As we saw in Example (4.3.6), this is just the morphism $\mathbb{A}^n_U \to U$, which depends only on the choice of isomorphism $\phi_U : \mathscr{E}|_U \to \mathscr{O}_U^n$. So on the intersection of two trivializing opens $U \cap V$, we have two morphisms $\mathbb{A}^n_{U\cap V} \to U \cap V$, and an isomorphism $\mathbb{A}^n_{U\cap V} \xrightarrow{\cong} \mathbb{A}^n_{U\cap V}$ induced by the linear isomorphism $\phi_U^{-1} \circ \phi_V$. So we see $\mathbb{V}(\mathscr{E}) \to X$ is indeed a vector bundle.

tions-vect-dual-isomorph

Lemma 8.1.9. *If $\mathscr{E}$ is a locally free sheaf of finite rank on X, then we have a natural isomorphism of $\mathscr{O}_X$-modules*

$$\check{\mathscr{E}} \cong \mathscr{S}_{\mathbb{V}(\mathscr{E})/X}.$$

Proof. For, by the (proof of) Lemma (8.1.5) above, and by Lemma (8.1.2), we have natural isomorphisms of sheaves

$$\begin{aligned}\check{\mathscr{E}} \cong \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E},\mathscr{O}_X) &\cong \underline{\mathrm{Hom}}_{\mathscr{O}_X\text{-alg}}(\mathrm{Sym}(\mathscr{E}),\mathscr{O}_X)\\ &= \underline{\mathrm{Hom}}_{\mathscr{O}_X\text{-alg}}(f_*\mathscr{O}_{\mathbb{V}(\mathscr{E})},\mathscr{O}_X)\\ &\cong \mathscr{S}_{\mathbb{V}(\mathscr{E})/X}\end{aligned}$$

Now we will get the isomorphism we want, as long as we've shown that the composition of morphisms of sheaves above actually gives us a morphism of $\mathscr{O}_X$-modules. But this we can do locally over an affine open $U = \mathrm{Spec}\,R$, over which $\mathscr{E}$ is free, and which, therefore, is a trivializing open for $\mathbb{V}(\mathscr{E})$. Here, by the construction of the module structure on $\mathscr{S}_{\mathbb{V}(\mathscr{E})/X}$, we have isomorphisms of $\mathscr{O}_U$-modules

$$\begin{aligned}\mathscr{S}_{\mathbb{V}(\mathscr{E})/X}|_U &\cong \underline{\mathrm{Hom}}_{\mathscr{O}_U\text{-alg}}(f_*\mathscr{O}_{\mathbb{V}(\mathscr{E})}|_U,\mathscr{O}_U)\\ &\cong \underline{\mathrm{Hom}}_{\mathscr{O}_U\text{-alg}}(\mathrm{Sym}(\mathscr{O}_U^n),\mathscr{O}_U)\\ &\cong \underline{\mathrm{Hom}}_{\mathscr{O}_U}(\mathscr{O}_U^n,\mathscr{O}_U)\\ &\cong \underline{\mathrm{Hom}}_{\mathscr{O}_U}(\mathscr{E}|_U,\mathscr{O}_U).\end{aligned}$$

This finishes the proof. □

vect-bundles-base-change

LEMMA 8.1.10. *Vector bundles are stable under base change.*

PROOF. Follows from the fact that $\mathbb{A}^n_U \times_U f^{-1}(U) \cong \mathbb{A}^n_{f^{-1}(U)}$, for any X-scheme $f : V \to X$, and any open subscheme $U \subset X$. □

les-locally-free-sheaves

THEOREM 8.1.11. *The assignment $F : \mathscr{E} \mapsto \mathbb{V}(\check{\mathscr{E}})$ gives an equivalence from the category of locally free sheaves of finite rank over X to the category of vector bundles of finite rank over X. The assignment $G : (V \to X) \mapsto \mathscr{S}_{V/X}$ provides the inverse functor.*

PROOF. It's easy to see that both assignments are functorial. From Lemma (8.1.9) above, we see that

$$GF\mathscr{E} \cong \check{\check{\mathscr{E}}} \cong \mathscr{E},$$

by [RS, 3.6].

We also have

$$FG(V \to X) \cong (\mathbb{V}(\check{\mathscr{S}}_{V/X}) \to X)$$

Now, the functor $(f : Y \to X) \mapsto \Gamma(Y, f^*\mathscr{S}_{V/X})$ is represented by $\mathbb{V}(\check{\mathscr{S}}_{V/X})$, where we've used (8.1.5) and the isomorphism

$$(f^*\check{\mathscr{S}}_{V/X})\check{} \cong f^*\check{\check{\mathscr{S}}}_{V/X} \cong f^*\mathscr{S}_{V/X},$$

which we obtain from [RS, 3.6] and [RS, 3.5]. To show that $g : V \to X$ is isomorphic to $\mathbb{V}(\check{\mathscr{S}}_{V/X}) \to X$, by Yoneda's Lemma, we only have to show that V also represents the same functor. For this, observe that for any X-scheme $f : Y \to X$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sch}\,X}(Y, V) &\cong \mathrm{Hom}_{\mathrm{Sch}\,Y}(Y, V \times_X Y) \\ &\cong \Gamma(Y, \mathscr{S}_{V_Y/Y}), \end{aligned}$$

where $V_Y \to Y$ is the base change of $V \to X$ along $Y \to X$. So to finish our proof, it suffices to show that

$$f^*\mathscr{S}_{V/X} \cong \mathscr{S}_{V_Y/Y}.$$

Now, we have a natural morphism (see (8.1.3))

$$\mathscr{S}_{V/X} \to f_*\mathscr{S}_{V_Y/Y}.$$

This gives us a natural morphism

$$f^*\mathscr{S}_{V/X} \to \mathscr{S}_{V_Y/Y}.$$

Locally, this is an isomorphism, since both sides are locally free $\mathscr{O}_Y$-modules of the same local rank (note that $V_Y \to Y$ is also a vector bundle by the last Lemma). This finishes our proof. □

DEFINITION 8.1.12. A vector bundle $V \to X$ is *trivial* if it's isomorphic to the bundle $\mathbb{A}^n_X \to X$.

ct-trivial-vector-bundle

COROLLARY 8.1.13. *A vector bundle $V \to X$ is trivial if and only if $\mathscr{S}_{V/X} \cong \mathscr{O}^n_X$, for some n.*

PROOF. Follows immediately from the Theorem, by observing that $\mathscr{S}_{\mathbb{A}^n_X/X} \cong \mathscr{O}^n_X$. To get this isomorphism, it's enough to show that $\mathscr{S}_{\mathbb{A}^n_{\mathbb{Z}}/\operatorname{Spec}\mathbb{Z}} \cong \mathbb{Z}^n$, since $\mathbb{A}^n_X \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X$. Now, since $\operatorname{Spec}\mathbb{Z}$ is Noetherian and $\mathscr{S}_{\mathbb{A}^n_{\mathbb{Z}}/\operatorname{Spec}\mathbb{Z}}$ is locally free of finite rank, it's in particular quasi-coherent, and so is completely determined by its module of global sections. But this consists precisely of the ring homomorphisms

$$\mathbb{Z}[x_1, \dots, x_n] \longrightarrow \mathbb{Z},$$

which, as we saw earlier, are in one-to-one correspondence with the group $\mathbb{Z}^n$. □

1.4. Examples of Vector Bundles. After all this heavy categorical theory, one can still legitimately ask: What exactly does a vector bundle look like? Observe that over any Noetherian affine scheme $\operatorname{Spec} R$, a locally free sheaf, being coherent (see 4.1.6), is of the form $\widetilde{M}$ for some locally free (and hence projective) R-module M. If R is a PID, then every finitely generated projective module is free, and so every vector bundle over $\operatorname{Spec} R$ is trivial.

1.4.1. *A Baby Example.* Let's do a baby example of the case where R isn't a PID. Take $R = \mathbb{Z}/6\mathbb{Z}$; then $\operatorname{Spec} R$ is a discrete set with 2 points, (2) and (3). Now, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are both PIDs (fields, in fact), and so every vector bundle over them is trivial. Since $\{\{(2)\}, \{(3)\}\}$ is an open cover for $\operatorname{Spec} R$ with no common intersection, we see that every vector bundle over $\operatorname{Spec} R$ corresponds to a map of rings

$$\mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n] \times \mathbb{Z}/3\mathbb{Z}[y_1, \dots, y_m].$$

This corresponds to the projective module $(\mathbb{Z}/2\mathbb{Z})^n \oplus (\mathbb{Z}/3\mathbb{Z})^m$. Note that if $n = m$, then we get the trivial bundle $\mathbb{A}^n_R \to \operatorname{Spec} R$, which corresponds to the fact that the sheaf of local sections is now R^n. Also observe that the rank of this vector bundle need not be constant (in fact, if it is constant, then it's trivial!). This is because our scheme is very disconnected.

1.4.2. *The Tangent Bundle of the Sphere.* Now, let $R = k[x_1, \dots, x_n]/(\sum_i x_i^2 - 1)$; when $k = \mathbb{R}$, this is the co-ordinate ring of the $n-1$-sphere. Now, we have a short exact sequence

$$0 \longrightarrow T \longrightarrow R^n \xrightarrow{\phi} R \longrightarrow 0,$$

where $\phi : R^n \to R$ is the map $(a_1, \dots, a_n) \mapsto \sum_i x_i a_i$. The map is surjective, because $\phi(x_1, \dots, x_n) = 1$. Now, since R is free, this sequence splits, and we can give the splitting map explicitly by sending $b \in R$ to $(x_1 b, \dots, x_n b)$. In particular, $T \oplus R \cong R^{n+1}$, and so T is a projective R-module.

As an aside, this process is quite general: whenever we have a set of elements $\{x_1, \dots, x_n\} \subset R$ (where R is any ring), generating the unit ideal, we have an exact sequence of the above form, and the kernel will be a projective R-module, which will become free when localized at any of the x_i.

1.4.3. *Vector Bundles over $\mathbb{P}^1_{\mathbb{C}}$.* See the definition of the $\mathbb{P}^1_{\mathbb{C}}$ in 3.2.8. We get this by gluing together two copies of $\mathbb{A}^1_{\mathbb{C}}$ along the open set $\operatorname{Spec}\mathbb{C}[z, z^{-1}]$ via the isomorphism $z \mapsto z^{-1}$. Since $\mathbb{C}[z]$ is a PID, any vector bundle over $\mathbb{A}^1_{\mathbb{C}}$ is trivial. So any vector bundle over $\mathbb{P}^1_{\mathbb{C}}$ is obtained by gluing together two copies of $\mathbb{A}^n_{\mathbb{C}}$ (one over each copy of the affine line) along a suitable isomorphism. What this means is that we take a free sheaf $\widetilde{\mathbb{C}[z]^n}$ over each affine line and glue together the two copies along some linear automorphism of $\mathbb{C}[z, z^{-1}]^n$.

CHAPTER 9

Quasi-coherent Cohomology over Schemes

chap:cohom

In this chapter we investigate the cohomology of quasi-coherent sheaves over affine schemes. For details on the derived functor approach see [HA, 7].

1. Cohomology of Sheaves over a Scheme

Definition 9.1.1. For a scheme X, a closed subscheme $Z \subset X$, and an $\mathscr{O}_X$-module $\mathscr{M}$, the n^{th} *cohomology of X with local support Z and coefficients in $\mathscr{M}$* is the group $H^n_Z(X, \mathscr{M})$, where we are thinking of X as a ringed space.

If $Z = X$, then we denote $H^n_X(X, \mathscr{M})$ simply as $H^n(X, \mathscr{M})$ and call it the n^{th} *cohomology of X with coefficients in $\mathscr{M}$*; or more colloquially the n^{th} *cohomology of $\mathscr{M}$*.

Observe that $H^\bullet_Z(X, \_\_) : \mathscr{O}_X\text{-mod} \to \mathrm{Ab}$ is a universal δ-functor, since it is the right derived functor of the sections with local support functor $\Gamma_Z(X, \_\_)$.

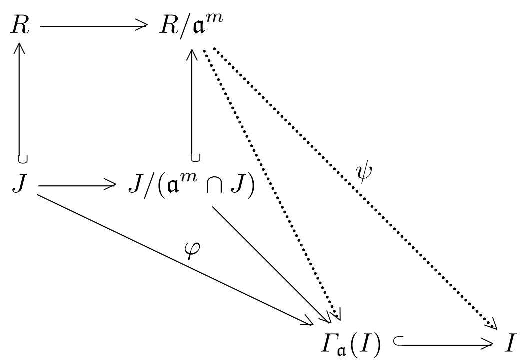
Remark 9.1.2. If X is a scheme over an affine scheme $\operatorname{Spec} R$, then the cohomology groups defined above are in fact R-modules.

The most crucial step in the actual computation of the cohomology of quasi-coherent sheaves over schemes is the fact that it is trivial over affine schemes. This, using [HA, 7.5.4], lets us compute cohomology using Čech complexes. For now, we'll only prove this vanishing statement for *Noetherian* affine schemes, though it is valid for all affine schemes. What we will show is that the functor $M \mapsto \widetilde{M}$ from R-mod to $\mathscr{O}_X$-mod takes injective modules to flabby (and hence $\Gamma_Z(X, \_\_)$-acyclic) sheaves. Before that we need two preliminary lemmas.

ctive-loc-supp-injective

Lemma 9.1.3. *Let R be a Noetherian ring, let I be an injective R-module, and let $\mathfrak{a} \subset R$ be an ideal; then $\Gamma_{\mathfrak{a}}(I)$ is also injective.*

Proof. Let $J \subset R$ be an ideal, and let $\varphi : J \to \Gamma_{\mathfrak{a}}(I)$ be a homomorphism. Since J is finitely generated, there exists $n \geq 0$ such that $\mathfrak{a}^n \varphi(J) = 0$; therefore φ factors through $J/\mathfrak{a}^n J$, for n large enough. By Artin-Rees [CA, 2.2.7], $\mathfrak{a}^m \cap J \subset \mathfrak{a}^n J$, for m large enough. Therefore, φ in fact factors through $J/(\mathfrak{a}^m \cap J)$. Hence we have the following picture:



where the dotted map ψ is an extension of the map from $J/(\mathfrak{a}^n \cap J)$ to I. It's clear now that $\operatorname{im}\psi$ lies in $\Gamma_{\mathfrak{a}}(I)$, and hence that the composition of ψ with the natural map $A \to A/\mathfrak{a}^n$ is a lifting of φ. Thus $\Gamma_{\mathfrak{a}}(I)$ is injective. □

ive-principal-loc-flabby

LEMMA 9.1.4. *Let R be a Noetherian ring, and let I be an injective R-module; then, for every $f \in R$, the natural map $I \to I_f$ is surjective.*

PROOF. Let $\mathfrak{a}_n = \operatorname{ann}(f^n)$; then since R is Noetherian, we see that $\mathfrak{a}_n = \mathfrak{a}_m$, for all n, m large enough. Pick $a/f^r \in I_f$, where $a \in I$, and $r \geq 0$; we want to show that that $a/f^r = b/1$, for some $b \in I$. Equivalently, we want to show that there exist $b \in I$ and $m \in \mathbb{N}$ such that $f^m a = f^{m+r} b$. For this, choose m so large that $\mathfrak{a}_m = \mathfrak{a}_{m+s}$, for all $s \geq 0$, and define a map $(f^{m+r}) \to I$ by $f^{m+r} \mapsto f^m a$. Observe that $\operatorname{ann}(f^{m+r}) = \mathfrak{a}_{m+r} = \mathfrak{a}_m$ acts trivially on $f^m a$. Hence this map is well-defined. Since I is injective, it extends to a map $\psi : R \to I$ such that $f^{m+r}\psi(1) = f^m a$; now let $b = \psi(1)$, to finish the proof. □

m-injective-tilde-flabby

PROPOSITION 9.1.5. *Let $X = \operatorname{Spec} R$ be a Noetherian affine scheme, and let I be an injective R-module; then $\widetilde{I}$ is a flabby sheaf.*

PROOF. We use Noetherian induction; so suppose $Z \subset X$ is a closed subset, and suppose that, for all injective sheaves $\mathscr{J}$ over X with $\operatorname{Supp}\mathscr{J}$ contained in some proper closed subset of Z, $\mathscr{J}$ is flasque. We wish to show that every injective sheaf $\mathscr{I}$ with support Z is flasque. Let $U \subset X$ be an open subscheme; we want to show that $\Gamma(X,\mathscr{I}) \to \Gamma(U,\mathscr{I})$ is a surjective map. If $Z \cap U = \emptyset$, then there is nothing to prove; so assume that $Z \cap U \neq \emptyset$.

Now, if $Y = X \setminus X_f$, where $X_f \subset U$ and $X_f \cap Z \neq \emptyset$, then $\operatorname{Supp}\underline{H}^0_Y(\mathscr{I}) = Y \cap Z$ is a proper closed subset of Z. We see from (4.3.15) that $\underline{H}^0_Y(\mathscr{I}) \cong \widetilde{\Gamma_{(f)}(I)}$, where I is the injective R-module satisfying $\widetilde{I} \cong \mathscr{I}$. Thus, by (9.1.3), $\underline{H}^0_Y(\mathscr{I})$ is an injective sheaf over X, and is therefore flabby, by our induction hypothesis.

We have the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma_Y(X,\mathscr{I}) & \longrightarrow & \Gamma(X,\mathscr{I}) & \longrightarrow & \Gamma(X_f,\mathscr{I}) \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \Gamma_{Y\cap U}(U,\mathscr{I}) & \longrightarrow & \Gamma(U,\mathscr{I}) & \longrightarrow & \Gamma(X_f,\mathscr{I})
\end{array}$$

The vertical map on the left is surjective since $\underline{H}^0_Y(\mathscr{I})$ is flabby, and the map from $\Gamma(X,\mathscr{I})$ to $\Gamma(X_f,\mathscr{I})$ is surjective, by (9.1.4). Thus, by a diagram chase, we see that the vertical arrow in the middle is also surjective, which is exactly what we wanted. □

In algebraic geometry, we are mostly concerned with quasi-coherent sheaves and their cohomology. The next Proposition will tell us that we need not go outside the category of quasi-coherent $\mathscr{O}_X$-sheaves for our cohomological computations.

uasi-coherent-injectives

PROPOSITION 9.1.6. *Let X be a Noetherian scheme.*

1. *$\mathscr{O}_X$-qcoh has enough injective.*
2. *For every injective object $\mathscr{I}\in\mathscr{O}_X$-qcoh and every open subscheme $U\subset X$, $\mathscr{I}|_U$ is injective in $\mathscr{O}_U$-qcoh.*
3. *Every injective sheaf in $\mathscr{O}_X$-qcoh is flabby.*

PROOF. For assertion (1), first observe that, for any Noetherian affine scheme $Y=\operatorname{Spec}R$ and any injective R-module I, $\widetilde{I}$ is an injective object in $\mathscr{O}_Y$-qcoh. This follows trivially from the fact that $\sim$ induces an isomorphism of categories between R-mod and $\mathscr{O}_Y$-qcoh. Now, if X is any Noetherian scheme, then we can cover X by finitely many affine opens $U_1,\dots,U_n$. Pick $\mathscr{F}\in X$-qcoh, and, for each j, let $\mathscr{I}_j$ be an injective object in $\mathscr{O}_{U_j}$-qcoh such that there is a monomorphism $\mathscr{F}|_{U_j}\to\mathscr{I}_j$. Let $f_j:U_j\to X$ be the inclusion map and set

$$\mathscr{I}=\bigoplus_{j=1}^n (f_j)_*\mathscr{I}_j.$$

Then there is an induced monomorphism $\mathscr{F}\to\mathscr{I}$. We claim that $\mathscr{I}$ is injective in $\mathscr{O}_X$-qcoh. Indeed, for any $\mathscr{O}_X$-module $\mathscr{M}$, we have a natural isomorphism

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{M},\mathscr{I})\cong\bigoplus_{j=1}^n\operatorname{Hom}_{\mathscr{O}_{U_j}}(\mathscr{M}|_{U_j},\mathscr{I}_j)$$

The functor on the right hand side is clearly exact, which shows that $\mathscr{I}$ is in fact injective in $\mathscr{O}_X$-qcoh.

Now, we move on to (2). Here it suffices to show that every morphism from a coherent ideal sheaf $\mathscr{J}\subset\mathscr{O}_U$ to $\mathscr{I}|_U$ extends to one from $\mathscr{O}_U$ to $\mathscr{I}|_U$. For this, we use (4.2.20) to find a coherent ideal sheaf $\mathscr{K}\subset\mathscr{O}_X$ such that $\mathscr{K}|_U\cong\mathscr{J}$.

Note that assertion (3) is true in the affine case, by (9.1.5). In the general case, it follows at once from (2) and [HA, 7.1.8]. □

uasi-coherent-cohomology

PROPOSITION 9.1.7. *Let X be a Noetherian scheme, and let $Z\subset X$ be a closed subscheme. Consider the functor*

$$\Gamma_Z(X,\_\_):\mathscr{O}_X\text{-}qcoh\to\mathrm{Ab}\,.$$

The right derived functors of $\Gamma_Z(X,\_\_)$ are naturally equivalent to the cohomology functors on X. That is, for every quasi-coherent sheaf $\mathscr{F}\in\mathscr{O}_X$-qcoh, and every $n\geq 0$, we have a natural isomorphism

$$R^n(\Gamma_Z(X,\_\_))(\mathscr{F})\cong H^n_Z(X,\mathscr{F}).$$

PROOF. Follows immediately from (9.1.6), since injectives in $\mathscr{O}_X$-qcoh are flabby in $\operatorname{Shf}(X,\mathrm{Ab})$. □

m-quasi-coherent-acyclic

Corollary 9.1.8. *Let $X = \operatorname{Spec} R$ be a Noetherian affine scheme. Then, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, and all $n > 0$, we have*

$$H^n(X, \mathscr{M}) = 0.$$

Proof. This follows immediately from the Proposition above, and the fact that $\Gamma(X, \_\_)$ is an exact functor from $\mathscr{O}_X$-qcoh to Ab (4.1.5). □

ohom-quasi-coherent-cech

Corollary 9.1.9. *Let X be a separated Noetherian scheme, and let $\mathcal{V}$ be any affine open cover of X; then, for every quasi-coherent sheaf $\mathscr{F}$ over X, we have a natural isomorphism:*

$$H^\bullet(X, \mathscr{F}) \cong \check{H}^\bullet(\mathcal{V}, \mathscr{F}),$$

where $\check{H}^\bullet(\mathcal{V}, \mathscr{F})$ denotes the Čech cohomology of $\mathscr{F}$.

Proof. Follows from (9.1.8) and [HA, 7.5.4]. □

2. Serre's Criterion for Affineness

We are now ready to present the long awaited cohomological criterion for affineness. After that we'll give some immediate applications of the criterion.

cal-criterion-affineness

Theorem 9.2.1 (Cohomological Affineness Criterion). *The following are equivalent for a Noetherian scheme X:*

1. *X is affine.*
2. *Every quasi-coherent $\mathscr{O}_X$-module is $\Gamma(X, \_\_)$-acyclic.*
3. *For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have*
$$H^1(X, \mathscr{F}) = 0.$$
4. *For every quasi-coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$, we have*
$$H^1(X, \mathscr{I}) = 0.$$

Proof. Observe that (2) ⇒ (3) ⇒ (4) follows trivially, and that (1) ⇒ (2) is (9.1.8). So it suffices to show (4) ⇒ (1). For this we'll use our old criterion for affineness from (1.4.3). To apply this criterion, we need to find a finite affine open cover $\{X_{f_i} : f_i \in \Gamma(X, \mathscr{O}_X)\}$ such that $(f_1, \dots, f_n)$ is the unit ideal of $A := \Gamma(X, \mathscr{O}_X)$.

First let $x \in X$ be a closed point, and let Y be a closed subscheme of X associated to the closed subspace $X \setminus U$, where U is an affine neighborhood of x. Let $\mathscr{I}_x, \mathscr{I}_Y$ be the ideal sheaves corresponding to x and Y, respectively. Consider the exact sequence

$$0 \to \mathscr{I}_x \cap \mathscr{I}_Y \to \mathscr{I}_Y \to \mathscr{I}_Y / \big(\mathscr{I}_x \cap \mathscr{I}_Y\big) \to 0.$$

Since $x \notin Y$, we find that $\mathscr{I}_Y / \big(\mathscr{I}_x \cap \mathscr{I}_Y\big)$ is just the skyscraper sheaf $\operatorname{Sky}_x(k(x))$ (this is local, so we can assume that everything in sight is affine; in this case, the statement is obvious). If we now look at the long exact sequence of cohomology associated to this sequence, we see that we have a surjection $\Gamma(X, \mathscr{I}_Y) \to k(x)$. Hence we can find $f_x \in \Gamma(X, \mathscr{I}_Y)$ such that its image in $k(x)$ is the identity. Now, since f_x vanishes on Y, we see that $X_{f_x} \subset U$. Since it contains x, we see that we have found an affine neighborhood of x of the form X_{f_x}, for some $f_x \in A$ (we can identify $\Gamma(X, \mathscr{I}_Y)$ with an ideal in A).

Now, pick any point $y \in X$; then since X is Zariski $\overline{\{y\}}$ contains a closed point x. Now, any open neighborhood of x will contain y, and so we see that the affine open cover $\{X_{f_x} : x \in X \text{ closed}\}$ in fact covers X. Since X is quasi-compact, we can pick finitely many neighborhoods $X_{f_1}, \ldots, X_{f_n}$ to cover X. It now remains to show that the unit ideal in A is generated by the f_i.

Consider the morphism $\varphi : \mathscr{O}_X^n \to \mathscr{O}_X$ given by the n-tuple $(f_1, \ldots, f_n)$. By construction this is an epimorphism. Let $\mathscr{F} = \ker \varphi$; we claim that $H^1(X, \mathscr{F}) = 0$. Indeed, consider a filtration on $\mathscr{O}_X^n$ given by

$$\mathscr{O}_X^n \supset \mathscr{O}_X^{n-1} \supset \ldots \supset \mathscr{O}_X \supset 0,$$

for some appropriate choice of free subsheaves. This induces a filtration

$$\mathscr{F} \cap \mathscr{O}_X^n \supset \mathscr{F} \cap \mathscr{O}_X^{n-1} \supset \ldots \supset \mathscr{F} \cap \mathscr{O}_X \supset 0$$

on $\mathscr{F}$. Repeatedly using long exact sequences of cohomology and the fact that $\mathscr{F} \cap \mathscr{O}_X^r / \mathscr{F} \cap \mathscr{O}_X^{r-1}$ is isomorphic to an ideal sheaf of $\mathscr{O}_X$, for $1 \leq r \leq n$, we see that $H^1(X, \mathscr{F})$ is indeed 0. Given this, we use the long exact sequence associated to the exact sequence

$$0 \to \mathscr{F} \to \mathscr{O}_X^n \to \mathscr{O}_X \to 0$$

we conclude that the induced map $\Gamma(X, \mathscr{O}_X^n) \to A$ is a surjection, and thus the f_i do actually generate the unit ideal of A. $\square$

ine-iff-reduction-affine

Corollary 9.2.2. *A Noetherian scheme X is affine if and only if each of its irreducible components is affine when equipped with the reduced induced sub-scheme structure. In particular, X is affine if and only if X_{red} is affine.*

Proof. Only one direction requires proof.

We first reduce to the case where X is reduced; so suppose X_{red} is affine; we'll show that $\mathscr{I}$ is $\Gamma(X, \_\_)$-acyclic, for all coherent ideal sheaves $\mathscr{I}$ over X. By the Theorem above, this says that X is affine. Indeed, let $\mathscr{N}_X$ be the nilradical of X; then, since X is Noetherian, there is $s \in \mathbb{N}$ such that $\mathscr{N}_X^s = 0$. Now, for any coherent ideal sheaf $\mathscr{I}$, we have a finite filtration

$$\mathscr{I} \supset \mathscr{N}_X \mathscr{I} \supset \ldots \supset \mathscr{N}_X^{s-1} \mathscr{I} \supset 0$$

where each quotient $\mathscr{N}_X^i \mathscr{I} / \mathscr{N}_X^{i+1} \mathscr{I}$ is $\Gamma(X, \_\_)$-acyclic, since it's a sheaf over X_{red} (Observe that X_{red} and X have the same underlying topological space, and that sheaf cohomology is a purely topological invariant). Now, we conclude from [HA, **??**] that $\mathscr{I}$ must also be $\Gamma(X, \_\_)$-acyclic.

Assume now that X is reduced and that each of its irreducible components, call them $X_1, \ldots, X_n$, is affine. For $k = 1, \ldots, n$, let $\mathscr{I}_k \subset \mathscr{O}_X$ be the prime ideal sheaf associated to the component X_i. Since X is reduced, we have $\prod_k \mathscr{I}_k = 0$. Now, let $\mathscr{F} \in \mathscr{O}_X$-coh be any coherent sheaf, and consider the filtration:

$$\mathscr{F} \supset \mathscr{I}_1 \mathscr{F} \supset \mathscr{I}_1 \mathscr{I}_2 \mathscr{F} \supset \ldots (\prod_{k=1}^{n-1} \mathscr{I}_k) \mathscr{F} \supset 0$$

Now, for any $\mathscr{O}_X$-module $\mathscr{G}$ and any $1 \leq k \leq n$, $\mathscr{G} / \mathscr{I}_k \mathscr{G}$ can be considered as a sheaf over X_k, since its support is contained entirely inside X_k. Since X_k is affine, for all k, we see that the components of the graded sheaf associated to the filtration on $\mathscr{F}$ are all $\Gamma(X, \_\_)$-acyclic. So, by [HA, **??**] again, we find that $\mathscr{F}$ is $\Gamma(X, \_\_)$-acyclic; therefore, X is affine by Serre's criterion. $\square$

3. Higher Direct Images and Local, Global Ext

3.1. Higher Direct Images.

er-dirimg-quasi-coherent

PROPOSITION 9.3.1. *Let $f : X \to Y$ be a quasi-compact, separated morphism of locally Noetherian schemes, and consider the direct image functor $f_* : X$-qcoh $\to$ Y-qcoh (observe that the direct image of a quasi-coherent sheaf under f is quasi-coherent by (4.2.11)).*

(1) *The derived functors of f_* agree with the higher direct images $R^\bullet f_*$ (the notation is confusing, but the meaning should be clear; the higher direct images are the derived functors of $f_* : \mathrm{Shf}(X, \mathrm{Ab}) \to \mathrm{Shf}(Y, \mathrm{Ab})$).*

(2) *For every quasi-coherent sheaf $\mathscr{F} \in X$-qcoh, and for all $i \geq 0$, $R^i f_* \mathscr{F}$ is also quasi-coherent over Y, and for $U \subset X$ affine, we have*

$$(R^i f_* \mathscr{F})|_U \cong H^i(\widetilde{f^{-1}(U)}, \mathscr{F}).$$

PROOF. Statement (1) follows immediately from the fact that an injective resolution in X-qcoh is flabby in $\mathrm{Shf}(X, \mathrm{Ab})$ (9.1.6). For (2), $R^i f_* \mathscr{F}$ is tautologically quasi-coherent; so it suffices to find $\Gamma(U, R^i f_* \mathscr{F})$. For this, we use the Leray spectral sequence [HA, 7.7.4], which gives us a natural monomorphism

$$H^i(f^{-1}(U), \mathscr{F}) \to H^0(U, R^i(f|_{f^{-1}(U)})^* \mathscr{F}).$$

Since U is affine, the spectral sequence collapses to the vertical axis on the second page. Thus the natural monomorphism is in fact an isomorphism. Now to finish the proof, it suffices to show that we have a natural isomorphism:

$$(R^i f_* \mathscr{F})|_U \cong R^i(f|_{f^{-1}(U)})_* \mathscr{F}.$$

Consider the following commutative (up to natural isomorphism) diagram of functors:

$$\begin{array}{ccc} \mathscr{O}_X\text{-qcoh} & \xrightarrow{|_{f^{-1}(U)}} & \mathscr{O}_{f^{-1}(U)}\text{-qcoh} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f|_{f^{-1}(U)})_*} \\ \mathscr{O}_Y\text{-qcoh} & \xrightarrow{|_U} & \mathscr{O}_U\text{-qcoh} \end{array}$$

Restriction to an open sub-scheme is an exact functor and takes injectives to injectives according to (9.1.6). Now the natural isomorphism that we need follows from [HA, 3.6.7]. □

r-dirimg-cech-cohomology

PROPOSITION 9.3.2. *Let $f : X \to Y$ be a quasi-compact, separated morphism between locally Noetherian schemes, and let $\mathscr{F}$ be a quasi-coherent sheaf over X. Let $\mathcal{V} = \{V_i : i \in I\}$ be an affine open cover for X, and let $\underline{C}^\bullet(\mathcal{V}, \mathscr{F})$ be the Čech resolution of $\mathscr{F}$. Then, for each $i \geq 0$, we have a natural isomorphism*

$$R^i f_* \mathscr{F} \cong H^i(f_* \underline{C}^\bullet(\mathcal{V}, \mathscr{F})).$$

PROOF. If $U \subset Y$ is open, then we have

$$(R^i f_* \mathscr{F})|_U \cong R^i(f_*|_{f^{-1}(U)})(\mathscr{F}|_{f^{-1}(U)}),$$

and

$$H^i(f_* \underline{C}^\bullet(\mathcal{V}, \mathscr{F}))|_U \cong H^i((f|_{f^{-1}(U)})_*(\underline{C}^\bullet(\mathcal{V} \cap f^{-1}(U), \mathscr{F}|_{f^{-1}U})).$$

If we show that $\mathcal{V} \cap U$ is an affine cover of U □

3.2. A Global Affineness Criterion. The next result is a generalization of Serre's criterion for affineness.

ffine-morphism-criterion

PROPOSITION 9.3.3. *Let $f : X \to Y$ be a quasi-compact, separated morphism of locally Noetherian schemes. Then the following statements are equivalent:*

(1) *f is affine.*
(2) *$f_* : X$-qcoh $\to Y$-qcoh is an exact functor.*
(3) *Every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is f_*-acyclic.*
(4) *Every coherent $\mathscr{O}_X$-module $\mathscr{F}$ is f_*-acyclic.*
(5) *Every coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$ is f_*-acyclic.*

PROOF. The implication (1) $\Rightarrow$ (2) is from (4.3.11). The only implication that now needs proof is (5) $\Rightarrow$ (1). Let $U \subset Y$ an affine open; we'd like to show that $f^{-1}(U)$ is also affine. For this, it suffices to show that $H^i(f^{-1}(U), \mathscr{I}) = 0$, for every coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_{f^{-1}(U)}$, and every $i > 0$. By (4.2.20), we can find an ideal sheaf $\mathscr{J} \subset \mathscr{O}_X$ such that $\mathscr{J}|_{f^{-1}(U)} \cong \mathscr{I}$. By part (2) of (9.3.1), since $\mathscr{J}$ is f_*-acyclic, we find that

$$H^i(f^{-1}(U), \mathscr{I}) \cong H^i(f^{-1}(U), \mathscr{J}) = 0,$$

which is precisely what we wanted. □

fine-morphism-cohomology

COROLLARY 9.3.4. *Let $f : X \to Y$ be an affine morphism of Noetherian schemes, and let $Z \subset Y$ be a closed subscheme; then, for every sheaf $\mathscr{F} \in \mathscr{O}_X$-qcoh, we have natural isomorphisms*

$$H^n_{f^{-1}(Z)}(X, \mathscr{F}) \cong H^n_Z(Y, f_*\mathscr{F}).$$

PROOF. Immediate from the fact that f_* is an exact functor; see [HA, 7.7.5]. □

3.3. Cohomology of Fibers. The next result will help us investigate the cohomology of fibers in terms of the fibers of the cohomology.

ism-pullback-pushforward

THEOREM 9.3.5. *Let $f : X \to S$ and $g : Y \to S$ be two separated, quasi-compact S-schemes over a locally Noetherian scheme S; let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the two natural projections, so that we have the following picture:*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p} & X \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & S \end{array}$$

Then, for every quasi-coherent sheaf $\mathscr{F}$ over X, we have natural maps:

$$g^*(R^\bullet f_*\mathscr{F}) \to (R^\bullet q_*)(p^*\mathscr{F}).$$

If g is flat, then this map is in fact an isomorphism.

PROOF. Let $U \subset S$ be an affine open. We have natural maps:

$$H^i(f^{-1}(U), \mathscr{F}) \to H^i((p \circ f)^{-1}(U), p^*\mathscr{F}) \to H^0(g^{-1}(U), R^i q_* p^* \mathscr{F}),$$

where the second map is obtained from the Leray spectral sequence [HA, 7.7.4], and the fact that $p \circ f = q \circ g$.

Observe that when $i = 0$, this is the natural map

$$H^0(f^{-1}(U), \mathscr{F}) \to H^0((q \circ g)^{-1}(U), p^*\mathscr{F}) = H^0(g^{-1}(U), q_* p^* \mathscr{F})$$

Using (9.3.1), we see that this gives us a natural morphism:

$$R^i f_* \mathscr{F} \to g_*(R^i q_* p^* \mathscr{F}),$$

and thus a natural morphism

$$g^*(R^i f_* \mathscr{F}) \to R^i q_*(p^* \mathscr{F})$$

as claimed.

Now suppose g is flat. To show that the map is an isomorphism, we can assume that $S = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A'$ are affine. Indeed, if $V = \operatorname{Spec} A \subset S$ and $U = \operatorname{Spec} A' \subset g^{-1}(V)$ are affine opens, then A' is flat over A, and we have

$$\Gamma(U, g^*(R^i f_* \mathscr{F})) = H^i(f^{-1}(V), \mathscr{F}) \otimes_A A'.$$
$$\Gamma(U, R^i q_*(p^* \mathscr{F})) = H^i(q^{-1}(U), p^* \mathscr{F}).$$

So, reducing to the case where S and Y are affine, we must show that the following natural map is an isomorphism, for all i:

$$H^i(X, \mathscr{F}) \otimes_A A' \longrightarrow H^i(X \times_S Y, p^* \mathscr{F}).$$

Consider the functor

$$T : \mathscr{O}_X\text{-qcoh} \to A'\text{-mod}$$
$$X \longrightarrow \Gamma(X, \mathscr{F}) \otimes_A A'.$$

Since A' is flat over A, T is a left exact functor. Moreover, we also have

$$R^i T(\mathscr{F}) = H^i(X, \mathscr{F}) \otimes_A A',$$

and so $H^\bullet(X, \mathscr{F}) \otimes_A A'$ is a universal δ-functor, and so there is a unique morphism of δ-functors from $H^\bullet(X, \mathscr{F}) \otimes_A A'$ to $H^\bullet(X \times_S Y, p^* \mathscr{F})$ (the latter is a δ-functor, since p^* is an exact functor) for a given morphism $\Gamma(X, \_\_) \otimes_A A' \to \Gamma(X \times_S Y, p^* \_\_)$.

Now suppose that $U = \operatorname{Spec} B \subset X$ is an affine open. In this case, if $\mathscr{F}|_U = \widetilde{M}$, for some B-module M, the natural map

$$H^0(U, \mathscr{F}) \otimes_A A' \to H^0(U \times_S Y, p^* \mathscr{F})$$

is an isomorphism, since we have

$$M \otimes_A A' \cong M \otimes_B (B \otimes_A A').$$

Let $\mathcal{V} = \{U_i : i \in I\}$ be a finite affine open cover for X, and let $\check{C}^\bullet(\mathcal{V}, \mathscr{F})$ be the Čech complex for $\mathscr{F}$ over $\mathcal{V}$. Then $\mathcal{V}' = \{U_i \times_S Y : i \in I\}$ is an affine open cover for $X \times_S Y$ and the above computation tells us that $\check{C}^\bullet(\mathcal{V}, \mathscr{F}) \otimes_A A'$ is isomorphic to the Čech complex for $p^* \mathscr{F}$ over $\mathcal{V}'$. Since A' is flat, it commutes with cohomology, and so we see that the map

$$H^i(X, \mathscr{F}) \otimes_A A' \longrightarrow H^i(X \times_S Y, p^* \mathscr{F})$$

induced by the Čech complex (9.1.9) is an isomorphism. We'll be done if we show that this is the original map that we had. Using the fact that $H^i(X, \_\_) \otimes_A A'$ is a universal δ-functor it suffices to show that the map induced on the global sections remains the same. But this follows immediately from the fact that it is just the map between inverse limits

$$\varprojlim H^0(U_i, \mathscr{F}) \otimes_A A' \to \varprojlim H^0(U_i \times_S Y, p^*\mathscr{F}),$$

induced by the original natural map on each U_i (Note the importance of the finiteness of the cover: tensoring with A' would not commute with the inverse limit if the cover were infinite). $\square$

REMARK 9.3.6. It is possible, as in Hartshorne, to use the Čech complex to *define* the map, but to do this rigorously seems painful. There are still some details to fill in in the proof above, but at least the line of argument is fleshed out fully.

`cohom-fibers`

COROLLARY 9.3.7. *Let $f : X \to Y$ be a quasi-compact, separated morphism between locally Noetherian schemes, with $Y = \operatorname{Spec} A$ affine. For $y \in Y$, and any quasi-coherent sheaf $\mathscr{F}$ over X, we have natural isomorphisms:*

$$H^i(X_y, \mathscr{F}_y) \cong H^i(X, \mathscr{F} \otimes_A k(y)),$$

where $\mathscr{F}_y$ is the pullback of $\mathscr{F}$ to X_y.

PROOF. Let Z be the scheme-theoretic image of $\operatorname{Spec} k(y) \to Y$; then $\mathscr{F} \otimes_A k(y)$ is supported in $f^{-1}(Z)$, and so, using (9.3.4), we can replace Y with Z, X with $f^{-1}(Z)$, and assume that y is the generic point of Y. Moreover, since $\mathscr{F}_y$ is also the pullback of $\mathscr{F} \otimes_A k(y)$ to X_y, we can also replace $\mathscr{F}$ with $\mathscr{F} \otimes_A k(y)$, and assume that $\mathscr{F}$ is a sheaf of $k(y)$-vector spaces.

In this case the map $\operatorname{Spec} k(y) \to Y$ is flat, and so by the Theorem above, we have isomorphisms

$$H^i(X, \mathscr{F}) \otimes_A k(y) \cong H^i(X_y, \mathscr{F}_y).$$

But $H^i(X, \mathscr{F})$ is already a $k(y)$-vector space, and so $H^i(X, \mathscr{F}) \otimes_A k(y) \cong H^i(X, \mathscr{F})$. $\square$

`cohom-acyclic-fibers`

COROLLARY 9.3.8. *We keep the notation of the corollary above. If $\mathscr{F}_y$ is in addition a $\Gamma(X_y, \_\_)$-acyclic sheaf, then we have isomorphisms*

$$H^0(X, \mathscr{F}) \otimes_A k(y) \cong H^0(X_y, \mathscr{F}_y).$$

PROOF. Base-changing along the flat morphism $\operatorname{Spec} \mathscr{O}_{Y,y} \to Y$, and using (9.3.5), we get isomorphisms

$$H^0(X, \mathscr{F}) \otimes_A \operatorname{Spec} \mathscr{O}_{Y,y} \cong H^0(X \times_Y \operatorname{Spec} \mathscr{O}_{Y,y}, \mathscr{F} \otimes_A \mathscr{O}_{Y,y}).$$

So, to prove our statement, we can replace Y with $\operatorname{Spec} \mathscr{O}_{Y,y}$ and X with $X \times_Y \operatorname{Spec} \mathscr{O}_{Y,y}$, and assume that y is a closed point of Y. In this case, choose we have a free presentation

$$A^r \to A \to k(y) \to 0$$

of $k(y)$. Tensoring this with $\mathscr{F}$ gives an exact sequence

$$\mathscr{F}^r \to \mathscr{F} \to \mathscr{F} \otimes_A k(y) \to 0,$$

and taking the long exact sequence of cohomology $\square$

3.4. Ext Sheaves.

om-ext-sheaves-coherence

PROPOSITION 9.3.9. *Let X be a Noetherian scheme, and let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent sheaves over X.*

(1) *Suppose in addition that $\mathscr{M}$ is in fact coherent; then, for all $n \geq 0$, $\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is also quasi-coherent over X.*
(2) *If $\mathscr{N}$ is also coherent, then $\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is in fact coherent.*
(3) *If $X = \operatorname{Spec} R$ is affine, and $\mathscr{M} = \widetilde{M}$, $N = \widetilde{N}$, where M and N are R-modules, with M finitely generated over R, then we have natural isomorphisms:*

$$\mathrm{Ext}^n_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N} \cong \mathrm{Ext}^n_R(M, N),$$
$$\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Ext}^n_R(M, N)}.$$

PROOF. (1) and (2) follow immediately from [HA, 7.6.6]. For the first isomorphism, consider the following diagram:

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{\sim} & \mathscr{O}_X\text{-qcoh} \\ {\scriptstyle \mathrm{Hom}_R(M, \_\_)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \_\_)} \\ R\text{-mod} & \xrightarrow{\sim} & \mathscr{O}_X\text{-qcoh} \end{array}$$

According to (4.1.7), this commutes up to natural isomorphism. Moreover, $\sim$ takes injectives to injectives, since it's an equivalence of categories. So we get the isomorphism immediately from [HA, 3.6.7].

For the second isomorphism, since $\underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N})$ is quasi-coherent, it suffices to prove that we have a natural isomorphism

$$\Gamma(X, \underline{\mathrm{Ext}}^n_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N})) \cong \mathrm{Ext}^n_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}).$$

But this follows immediately from the Ext spectral sequence [HA, 7.6.3], and the fact that higher cohomology groups vanish over the affine scheme X. □

4. Local Cohomology

m-noetherian-local-cohom

PROPOSITION 9.4.1. *Let $X = \operatorname{Spec} R$ be a Noetherian affine scheme, and let $Z \subset X$ be a closed subscheme cut out by an ideal $\mathfrak{a} \subset R$. Then, for every R-module M, we have an isomorphism of δ-functors*

$$H^\bullet_Z(X, \widetilde{M}) \cong \widetilde{H^\bullet_{\mathfrak{a}}(M)},$$

from R-mod to R-mod, where $H^\bullet_{\mathfrak{a}}(M)$ is the local cohomology of M [HA, 5].

PROOF. Consider the following diagram:

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{M \mapsto \widetilde{M}} & \mathscr{O}_X\text{-mod} \\ {\scriptstyle \Gamma_I(\_\_)}\downarrow & & \downarrow{\scriptstyle \Gamma_Z(X, \_\_)} \\ R\text{-mod} & \xrightarrow{M \mapsto \widetilde{M}} & \mathscr{O}_X\text{-mod}. \end{array}$$

We know by (4.3.15) that this commutes up to natural equivalence. Now the result follows from (9.1.5) and [HA, 3.6.7]. □

We can now finally understand, at least in the Noetherian case, what the mysterious modules of sections over arbitrary open subschemes of an affine scheme look like.

ohom-sections-over-opens

PROPOSITION 9.4.2. *Let $X = \operatorname{Spec} R$ be a Noetherian, affine scheme, and let $\mathfrak{a} \subset R$ be an ideal. Set $U = X - V(\mathfrak{a})$; then, for any R-module M, we have*

$$\Gamma(U, \widetilde{M}) = \varinjlim \operatorname{Hom}_R(\mathfrak{a}^n, M).$$

PROOF. □

CHAPTER 10

Sheaves of Modules over Projective Schemes

chap:qsps

1. The Tilde Functor

Suppose we have a graded R-module M, we can define a quasi-coherent sheaf $\widetilde{M}$ on $X = \operatorname{Proj} R$ by setting $\widetilde{M}|_{X_{(f)}} \cong \widetilde{M_{(f)}}$ (this is the usual tilde construction for affine schemes; see Section 1.1 of Chapter 4), for any homogeneous element $f \in R$. This, of course, defines a presheaf on the open base $\{X_{(f)} : f \in R\}$. That this in fact satisfies the condition for being a sheaf [NOS, 11.1] follows from part (4) of (3.1.4). Moreover, part (5) of the same Proposition tells us that $\widetilde{M}_P \cong M_{(P)_0}$, for any homogeneous prime $P \in \operatorname{spc}(X)$. That this is quasi-coherent follows from (4.2.1).

The assignment that sends M to $\widetilde{M}$ actually defines an exact functor from $R^{\mathbb{Z}}$-mod to $\mathscr{O}_X$-qcoh. To show this it suffices to show that for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

and any homogeneous element $f \in R$, the sequence of morphisms of $\mathscr{O}_{X_{(f)}}$-modules

$$0 \to \widetilde{M'_{(f)}} \to \widetilde{M_{(f)}} \to \widetilde{M''_{(f)}} \to 0$$

is exact. But this follows from the fact that both the tilde functor for affine schemes and the functor that sends M to $M_{(f)}$ are exact.

Now, suppose N is another graded R-module. Consider the graded R-module ${}^*\operatorname{Hom}_R(M,N)$, and the induced quasi-coherent sheaf ${}^*\widetilde{\operatorname{Hom}_R(M,N)}$. For every homogeneous element $f \in R$, we have a natural homomorphism (see [CA, 1.6.3])

$${}^*\operatorname{Hom}_R(M,N)_{(f)} \to \operatorname{Hom}_{R_{(f)}}(M_{(f)}, N_{(f)}).$$

By the process described in Section 1.1 of Chapter 4, this gives us a natural morphism of $\mathscr{O}_{X_{(f)}}$-modules

$${}^*\widetilde{\operatorname{Hom}_R(M,N)}|_{X_{(f)}} \to \underline{\operatorname{Hom}}_{\mathscr{O}_{X_{(f)}}}(\widetilde{M}|_{X_{(f)}}, \widetilde{N}|_{X_{(f)}}).$$

We can glue these together to get a natural morphism of $\mathscr{O}_X$-modules

$${}^*\widetilde{\operatorname{Hom}_R(M,N)} \longrightarrow \underline{\operatorname{Hom}}_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}).$$

We consider tensor products now. Again by [CA, 1.6.3], we have a natural monomorphism

$$M_{(f)} \otimes_{R_{(f)}} N_{(f)} \longrightarrow (M \otimes_R N)_{(f)}$$

This gives us a natural monomorphism of $\mathscr{O}_{X_{(f)}}$-modules

$$\widetilde{M}|_{R_{(f)}} \otimes_{\mathscr{O}_{X_{(f)}}} \widetilde{N}|_{R_{(f)}} \longrightarrow \left(\widetilde{M \otimes_R N}\right)|_{X_{(f)}}.$$

We glue together these isomorphisms to get a global monomorphism of $\mathscr{O}_X$-modules

$$\widetilde{M}\otimes_{\mathscr{O}_X}\widetilde{N}\longrightarrow \widetilde{M\otimes_R N}.$$

qsps-twisting-sheaf

DEFINITION 10.1.1. If $X=\operatorname{Proj} R$, and $n\in\mathbb{Z}$, we define $\mathscr{O}_X(n)=\widetilde{R(n)}$. These are the *twisting sheaves of Serre*. For any $\mathscr{O}_X$-module $\mathscr{F}$, we set $\mathscr{F}(n)=\mathscr{F}\otimes\mathscr{O}_X(n)$.

Let's record all we've done in the next Proposition.

qsps-tilde-qc-proj

PROPOSITION 10.1.2. *For any graded R-module M, we have a quasi-coherent sheaf $\widetilde{M}$ on $X=\operatorname{Proj} R$ such that*

(1) $\widetilde{M}|_{X_{(f)}}\cong\widetilde{M_{(f)}}$.
(2) *The assignment $M\mapsto\widetilde{M}$ gives us an exact functor from $R^{\mathbb{Z}}$-mod to $\mathscr{O}_X$-qcoh.*
(3) $\widetilde{M}_P\cong M_{(P)_0}$.
(4) *If N is another graded R-module, then we have a natural morphism of $\mathscr{O}_X$-modules*
$$\widetilde{M}\otimes_{\mathscr{O}_X}\widetilde{N}\longrightarrow\widetilde{M\otimes_R N}.$$
(5) *For any $n\in\mathbb{Z}$, we have a natural morphism*
$$\widetilde{M}(n)\longrightarrow\widetilde{M(n)}.$$
In particular, we have a natural morphism
$$\mathscr{O}_X(n)\otimes_{\mathscr{O}_X}\mathscr{O}_X(m)\longrightarrow\mathscr{O}_X(n+m).$$
(6) *If N is another graded R-module, we have a natural morphism of $\mathscr{O}_X$-modules*
$$\widetilde{{}^*\operatorname{Hom}_R(M,N)}\longrightarrow\underline{\operatorname{Hom}}_{\mathscr{O}_X}(\widetilde{M},\widetilde{N}).$$
(7) *Let $\{M_i:i\in I\}$ be a filtered system of graded R-modules. Then* $\operatorname{colim}_i\widetilde{M}_i=\widetilde{\operatorname{colim}_i M_i}$.
(8) *Suppose R is a positively graded ring; then, for any integer $d\in\mathbb{Z}$, $\widetilde{M^{\geq d}}\cong\widetilde{M}$.*
(9) *Suppose R is a positively graded Noetherian ring, and suppose also that M is* quasifinitely generated *over R, in the sense that $M^{\geq d}$ is finitely generated over R, for some $d\in\mathbb{Z}$. Then $\widetilde{M}$ is a coherent sheaf over X.*

PROOF. The only things that remain to be proved are parts (5), (7) and 8. (5) follows immediately from (4), and the second part follows from the observation

$$R(n)\otimes_R R(m)\cong R(n+m).$$

(7) follows immediately from part (5) of [CA, 1.6.3], and (8) follows from part (6) of the same Proposition.

For (9), it's enough to note that $\widetilde{M}\cong\widetilde{M^{\geq d}}$, and so $\widetilde{M}|_{X_{(f)}}$ is of finite type over each affine open $X_{(f)}$. Now, since X is Noetherian (see the last part of (3.1.6), $\widetilde{M}$ must be coherent (4.2.6). □

REMARK 10.1.3. Observe that $\widetilde{R}\cong\mathscr{O}_X$.

We now specialize to the case that's most common in real life.

sps-tilde-qc-deg-one-gen

PROPOSITION 10.1.4. *Let R be a graded ring, and let I_1 be the ideal generated by R_1, the degree one component of R. Assume that $R^+ \subset \operatorname{rad}(I_1)$ (this is true, for example, when R_1 generates R as an R_0-algebra). Let $X = \operatorname{Proj} R$.*

(1) *For any $n \in \mathbb{Z}$, $\mathscr{O}_X(n)$ is locally free of rank 1.*
(2) *If N is another graded R-module, then we have a natural isomorphism of $\mathscr{O}_X$-modules:*
$$\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \longrightarrow \widetilde{M \otimes_R N}.$$
(3) *For any $n \in \mathbb{Z}$, $\widetilde{M}(n) \cong \widetilde{M(n)}$. In particular, $\mathscr{O}_X(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(m) \cong \mathscr{O}_X(n+m)$.*
(4) *For any graded R-modules M and N, with M finitely presented, we have an isomorphism of $\mathscr{O}_X$-modules*
$$\widetilde{{}^*\operatorname{Hom}_R(M,N)} \cong \underline{\operatorname{Hom}}_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}).$$
In particular,
$$\underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{O}_X(n), \widetilde{N}) \cong \widetilde{N}(-n).$$

PROOF. Recall from part (8) of 3.1.5 that with the given hypothesis $\{X_{(f)} : f \in R_1\}$ is an open cover for X.

(1) Follows from part (3) of [CA, 1.6.3]; for we have for every $f \in R_1$
$$\mathscr{O}_X(n)|_{X_{(f)}} \cong \widetilde{R(n)_{(f)}} \cong \widetilde{R_{(f)}}.$$
(2) Part (4) gives us a natural map
$$\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \to \widetilde{M \otimes_R N}.$$
Now we use part (2) of [CA, 1.6.3] to conclude that this is indeed an isomorphism.
(3) Follows immediately from the previous part.
(4) Follows from part (4) of [CA, 1.6.3] and 4.1.7; for we have for every $f \in R_1$
$$\begin{aligned}\widetilde{{}^*\operatorname{Hom}_R(M,N)}|_{X_{(f)}} &\cong \widetilde{{}^*\operatorname{Hom}_R(M,N)_{(f)}}\\ &\cong \widetilde{\operatorname{Hom}_{R_{(f)}}(M_{(f)}, N_{(f)})}\\ &\cong \underline{\operatorname{Hom}}_{\mathscr{O}_{X_{(f)}}}(\widetilde{M}|_{X_{(f)}}, \widetilde{N}|_{X_{(f)}}).\end{aligned}$$
We've used the elementary fact that if M is a finitely presented graded R-module, then $M_{(f)}$ is a finitely presented $R_{(f)}$-module, which follows from the exactness of the functor $M \mapsto M_{(f)}$. The second statement follows from the isomorphism
$${}^*\operatorname{Hom}_R(R(n), N) \cong N(-n).$$

□

ps-veronese-make-deg-one

REMARK 10.1.5. Although this hypothesis seems a little restrictive, it's not really so. Suppose R is positively graded and finitely generated over R_0. By part (3) of [CA, 1.6.5], we can choose some $d \in \mathbb{N}$, such that $R^{(d)}$ is generated by its degree 1 component over R_0. Since, by (3.2.4), $\operatorname{Proj} R^{(d)} \cong \operatorname{Proj} R$, we're back in our nice situation.

We now study the behavior of the tilde functor under the taking of direct and inverse images. For the next Proposition, we use notation from (3.2.1).

qsps-dirimg-invimg-proj

PROPOSITION 10.1.6. *Let R and S be graded rings and let $\phi : R \to S$ be a homomorphism of rings, such that for all $n \in \mathbb{Z}$, $\phi(R_n) \subset S_{en}$, for some fixed integer e. Let $U = G(\phi)$, and let $\operatorname{Proj}(\phi) : U \to X := \operatorname{Proj} R$ be the induced morphism.*

(1) *If M is a graded S-module, then $\operatorname{Proj}(\phi)_*(\widetilde{M}|_U) \cong \widetilde{{}_RM}$. In particular,*

$$\operatorname{Proj}(\phi)_*(\mathscr{O}_Y(n)|_U) \cong (\operatorname{Proj}(\phi)_*(\mathscr{O}_Y|_U))(n).$$

(2) *If N is a graded R-module, $e = 1$, and R satisfies the hypotheses of 10.1.4, then $\operatorname{Proj}(\phi)^*\widetilde{N} \cong \widetilde{M \otimes_R S}|_U$. In particular,*

$$\operatorname{Proj}(\phi)^*(\mathscr{O}_X(n)) = \mathscr{O}_Y(n)|_U$$

PROOF. Let $f \in R$ be a homogeneous element; then $(\phi^*)^{-1}(X_{(f)}) = Y_{(\phi(f))}$, where $Y = \operatorname{Spec} S$. So $\operatorname{Proj}(\phi)$ induces a morphism

$$\psi_{(f)} : Y_{(\phi(f))} \longrightarrow X_{(f)}$$

via restriction.

(1) Observe that, by 4.1.11

$$\psi_{(f)*}(\widetilde{M}|_{Y_{(\phi(f))}}) = \psi_{(f)*}(\widetilde{M_{(\phi(f))}}) = \widetilde{{}_{R_{(f)}}M_{(\phi(f))}} = \widetilde{({}_RM)_{(f)}}.$$

Note also that

$$\widetilde{{}_RM}|_{X_{(f)}} \cong \widetilde{({}_RM)_{(f)}}.$$

It's easy to see these isomorphisms glue together to give us our result. For the second part, observe that

$$\widetilde{{}_RS(n)} = {}_RS\,\widetilde{\otimes_R R(n)} \cong \widetilde{{}_RS}(n).$$

(2) We need $e = 1$, because we want S to have a natural graded R-module structure. The other hypothesis is needed to ensure that we can find an open cover of X by subschemes of the form $X_{(f)}$ where f has degree 1. Here, by 4.1.11 and part (2) of [CA, 1.6.3], we have

$$\begin{aligned}\psi^*_{(f)}(\widetilde{N}|_{X_{(f)}}) &= \psi^*_{(f)}(\widetilde{N_{(f)}})\\ &= N_{(f)}\,\widetilde{\otimes_{R_{(f)}} S_{(\phi(f))}}\\ &\cong (N\,\widetilde{\otimes_R S)_{(\phi(f))}}\\ &\cong \widetilde{N \otimes_R S}|_{Y_{(\phi(f))}}.\end{aligned}$$

Gluing these isomorphisms together, we get our result. For the second statement, observe that

$$(\operatorname{Proj}(\phi))^*(\mathscr{O}_X(n)) = \widetilde{R(n) \otimes_R S}|_U \cong \widetilde{S(n)}|_U = \mathscr{O}_Y(n)|_U$$

□

2. Global Sections of the Twisted Sheaves

The next definition will lead us to the nice place where everything that we care about is tilde of something.

DEFINITION 10.2.1. Let R be a graded ring generated by R_1 over R_0, and let $X = \operatorname{Proj} R$. For an $\mathscr{O}_X$-module $\mathscr{F}$, we define $\Gamma_*(\mathscr{F})$, the *graded R-module associated to $\mathscr{F}$*, to be the direct sum of R_0-modules $\oplus_{n\in\mathbb{Z}}\Gamma(X, \mathscr{F}(n))$.

For every $\mathscr{O}_X$-module $\mathscr{F}$, we have natural maps

$$\Gamma(X, \mathscr{O}_X(n)) \otimes_{\Gamma(X,\mathscr{O}_X)} \Gamma(X, \mathscr{F}(m)) \to \Gamma(X, \mathscr{F}(n+m)),$$

given essentially by the sheafification map. This makes $\Gamma_*(\mathscr{O}_X)$ into a graded ring and $\Gamma_*(\mathscr{F})$ into a graded $\Gamma_*(\mathscr{O}_X)$-module, for every $\mathscr{O}_X$-module $\mathscr{F}$. But of course there's a natural morphism of graded rings from R to $\Gamma_*(\mathscr{O}_X)$ that sends an element $s \in R_n$ to its corresponding section of $\mathscr{O}_X(n)$ over X obtained by gluing together the sections $s/1$ over the principal opens $X_{(f)}$. Thus $\Gamma_*(\mathscr{F})$ is in fact a graded R-module, for every $\mathscr{O}_X$-module $\mathscr{F}$.

We will investigate the relationship between R and $\Gamma_*(\operatorname{Proj} R)$ in this next Theorem, which will then allow us to recover important information about the

qsps-globstar-ring

THEOREM 10.2.2. *Let R be a graded ring finitely generated over R_0 by R_1, and let $X = \operatorname{Proj} R$.*

(1)
$$\Gamma_*(\mathscr{O}_X) = \varprojlim_{i_1<\ldots<i_k} R_{f_{i_1}\cdots f_{i_k}}.$$

(2) *Suppose R is a domain; then the natural map of graded rings*
$$R \longrightarrow \Gamma_*(\mathscr{O}_X)$$
is an injection, and $\Gamma_*(\mathscr{O}_X) \subset K(R)$.

(3) *If each f_i is prime in R, then the natural map above is an isomorphism.*

(4) *If R is a Noetherian domain, then $R' = \oplus_{n\geq 0}\Gamma(X, \mathscr{O}_X(n))$ is integral over R.*

PROOF. (1) Now suppose R is generated over R_0 by elements $f_1, \ldots, f_t \in R_1$. To give a global section s of $\mathscr{O}_X(n)$ then is equivalent to giving sections $s_i \in \Gamma(f_i, \mathscr{O}_X(n)) = R(n)_{(f_i)}$ such that

$$s_i|_{X_{(f_i f_j)}} = s_j|_{X_{(f_i f_j)}},$$

for all pairs i, j. Hence $\Gamma_*(\mathscr{O}_X)$ corresponds to t-tuples of elements $(s_1, \ldots, s_t)$, with $s_i \in R_{f_i}$, and s_i and s_j map to the same element in $R_{f_i f_j}$. Now, $R_{f_i} \subset K(R)$, for each i, and we see that each element in $\Gamma_*(\mathscr{O}_X)$ corresponds uniquely to an element of the inverse limit

$$\varprojlim_{i_1<\ldots<i_k} R_{f_{i_1}\cdots f_{i_k}}.$$

(2) From the definition of the natural map, we see that an element $r \in R$ goes to 0 if and only if $r/1 = 0 \in R_f$, for all $f \in R_1$. Since R is a domain, this means that $r = 0 \in R$. Hence the map is injective. So we see that

$$\Gamma_*(\mathscr{O}_X) = \bigcap_{i_1<\ldots<i_k} R_{f_{i_1}\cdots f_{i_k}} = \cap_i R_{f_i}.$$

In fact the argument shows that

$$\Gamma_*(\mathscr{O}_X) = \cap_i R_{f_i} \subset K(R).$$

(3) Now suppose each of the f_i is prime. Let $s \in \Gamma_*(\mathscr{O}_X)$. Then $s \in R_{f_i} \cap R_{f_j}$, and so $s = \frac{a_i}{f_i^k} = \frac{a_j}{f_j^l}$, with $f_j^l a_i = f_i^k a_j$. Since f_i is prime, we must have either $f_j \in (f_i)$, in which case $(f_j) = (f_i)$; or $a_i \in (f_i)$. Suppose the second case is true: then $a_i = f_i a_i'$, and so

$$f_j^l a_i' = f_i^{k-1} a_j.$$

Proceeding in this way, we will find $b_i \in R$ such that $f_j^l b_i = a_j$. But then $\frac{a_j}{f_j^l} = b_i \in R$, and so we find that $s \in R$. Now, suppose the first case holds. In this case, either f_i does not generate R over R_0, in which case, we can find another f_k such that $(f_k) \neq (f_i)$ and run the same argument as above; or, f_i generates R over R_0, in which case R is a quotient of $R_0[t]$ by a homogeneous ideal. If $R \neq R_0[t]$, then f_i is nilpotent in R, and so $X = \emptyset$; there is nothing to prove in this case. If $R = R_0[t]$ (see 3.2.7), then $X = \operatorname{Spec} R_0$, and in this case it's easy to conclude that $\mathscr{O}_X(n) \cong \mathscr{O}_X$, for all n. From this it follows that $R = \Gamma_*(\mathscr{O}_X)$.

(4) Let $\alpha : R \to \Gamma_*(\mathscr{O}_X)$ be the natural map. Suppose $s \in R'$ is homogeneous of non-negative degree; then we can find $n \in \mathbb{N}$ such that $\alpha(f_i^n)s \in \alpha(R)$, for all i. Moreover, since R_m is generated by monomials in f_i of degree m, for m large enough (say $m \geq rn$), $\alpha(R_m)s \subset \alpha(R)$. Let $R^{\geq rn} = \oplus_{m \geq rn} R_m$; then we see that $\alpha(R^{\geq rn})s \subset \alpha(R^{\geq rn})$. Observe that $R^{\geq rn}$ is an ideal of R and is thus finitely generated, since R is Noetherian. Now, apply the Cayley-Hamilton theorem [CA, 4.1.1] to see that s satisfies a monic equation over R, and is hence integral over R

□

ections-projective-space

COROLLARY 10.2.3. *Let $R = S[t_0, \dots, t_n]$ be the polynomial ring in $n+1$-variables over a ring R, so that $X = \operatorname{Proj} R = \mathbb{P}^n_S$.*

$$\Gamma(X, \mathscr{O}_X(d)) = \begin{cases} 0, \text{ if } d < 0 \\ R[t_0, \dots, t_n]_d, \text{ if } d \geq 0. \end{cases}$$

In particular, we have

$$\dim \Gamma(X, \mathscr{O}_X(d)) = \begin{cases} 0, \text{ if } d < 0 \\ \binom{n+d}{d}, \text{ if } d \geq 0. \end{cases}$$

PROOF. Follows immediately from part (3) of the Theorem above, since the x_i are all prime. □

3. Going the Other Way

What we would really like is an analogue of (4.1.3) that would tell us that every quasi-coherent sheaf over $\operatorname{Proj} R$ looks like $\widetilde{M}$ for some graded R-module M. This is not true in general, but is true under the hypotheses of Proposition (10.1.4) above, with some additional finiteness conditions thrown in. Before we show that, we'll need a technical lemma that's a close cousin of (4.1.2).

l-opens-invertible-twist

Lemma 10.3.1. *Let X be any scheme, $\mathscr{M}$ a quasi-coherent $\mathscr{O}_X$-module, and $\mathscr{L}$ a locally free $\mathscr{O}_X$-module of rank 1 (in other words, it's an invertible sheaf). For $s \in \Gamma(X, \mathscr{L})$, set $X_{\mathscr{L},s} = \{x \in X : \mathscr{O}_{X,x} s_x = \mathscr{L}_x\}$. By [RS, 4.3], $X_{\mathscr{L},s}$ is open. For $f \in \Gamma(U, \mathscr{L})$, we denote $f^{\otimes n} \in \Gamma(U, \mathscr{L}^{\otimes n})$ by f^n.*

1. *Suppose X is quasi-compact, and let $a \in \Gamma(X, \mathscr{M})$ be such that $\operatorname{res}_{X,X_{\mathscr{L},s}} = 0$. Then, there exists $n \in \mathbb{N}$ such that $s^n \otimes a = 0 \in \Gamma(X, \mathscr{L}^{\otimes n} \otimes \mathscr{M})$.*
2. *Suppose in addition that X is quasi-separated; then for any section $b \in \Gamma(X_{\mathscr{L},s}, \mathscr{M})$, there is $n \in \mathbb{N}$ such that $s^n \otimes b \in \Gamma(X_{\mathscr{L},s}, \mathscr{L}^{\otimes n} \otimes \mathscr{M})$ is the restriction of a global section of $\mathscr{L}^{\otimes n} \otimes \mathscr{M}$.*

Proof. We'll prove this by reducing to the easier case mentioned above. First, observe that we can find an open cover of X by affine opens U_i such that $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$. Therefore $U_i \cap X_{\mathscr{L},s}$ is also affine (it's just $(U_i)_s$), and if X is quasi-compact, this shows that $X_{\mathscr{L},s}$ is also quasi-compact, since we can find finitely many U_i to do the job.

Now, Let $\mathscr{A} = T(\mathscr{L})$, be the tensor algebra of $\mathscr{L}$, and consider the morphism $f : Y := \operatorname{Spec} \mathscr{A} \to X$. This is affine, and is in particular both quasi-compact and quasi-separated. Since both these properties are stable under composition, we see that Y is quasi-compact (resp. quasi-separated) if X is quasi-compact (resp. quasi-separated). We see from (4.3.10) that

$$f^*\mathscr{M} \cong \widetilde{\mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{M}}.$$

Therefore, for any open subscheme $U \subset X$,

$$(f^*\mathscr{M})(f^{-1}(U) = (\mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{M})(U) = \oplus_{n \geq 0} (\mathscr{L}^n \otimes_{\mathscr{O}_X} \mathscr{M})(U).$$

This equality follows from the definition of the tilde construction. Now, we can look at s as a section in $\Gamma(Y, \mathscr{O}_Y)$, and consider the open subscheme Y_s of Y. By definition

$$Y_s = \{y \in Y : s_y \mathscr{O}_{Y,y} = \mathscr{O}_{Y,y}\}.$$

We will show that $Y_s = f^{-1}(X_{\mathscr{L},s})$. For this we can assume that $X = \operatorname{Spec} R$ is affine and that $\mathscr{L} = \mathscr{O}_X$. In this case, $Y = \operatorname{Spec} R[t]$, and $Y_s = \operatorname{Spec} R[t]_{st}$, and it is clear that $Y_s \subset f^{-1}(X_{\mathscr{L},s}) = f^{-1}(\operatorname{Spec} R_s)$. After all this work, the two statements will now follow easily from (4.1.2), and the fact that its hypotheses hold for every quasi-compact scheme (something that turned out to be a *consequence* of that Lemma).

1. Y is also quasi-compact as shown above. Treat a as a section of $f^*\mathscr{M}$ over Y. Since $\operatorname{res}_{Y,Y_s}(a) = 0$ (since $Y_s \subset f^{-1}(X_{\mathscr{L},s})$), we see that there is $n \in \mathbb{N}$ such that $s^n a = 0 \in \Gamma(Y, \mathscr{O}_Y)$. This of course gives us our result.
2. Again, Y is also quasi-separated. Now, treat b as a section of $f^*\mathscr{M}$ over $f^{-1}(X_{\mathscr{L},s})$. Then, we can find $r \in \mathbb{N}$ such that $(s^r b)|_{Y_s} \in \Gamma(Y_s, f^*\mathscr{M})$ is the restriction of a global section of $\mathscr{O}_Y$, where of course by s we mean $\operatorname{res}_{Y,f^{-1}(X_{\mathscr{L},s})}(s)$. This shows that there is a global section $\widetilde{b} \in \Gamma(Y, f^*\mathscr{M})$ such that
$$\widetilde{b}|_{f^{-1}(X_{\mathscr{L},s})} - (s^r b) =: b_1$$
restricts to 0 over Y_s. Since $f^{-1}(X_{\mathscr{L},s})$ is quasi-compact, we can find $m \in \mathbb{N}$ such that $s^m b_1 = 0$. But then if $n = r + m$ we see see that $s^m \widetilde{b}$ restricts to $s^n b$ over $f^{-1}(X_{\mathscr{L},s})$. This, translated suitably, is exactly what we wanted to prove.

□

Now, let R be a graded ring finitely generated over R_0 by R_1, and let $\mathscr{F}$ be a quasi-coherent sheaf over $X = \operatorname{Proj} R$. We will now construct a natural map β from $\widetilde{\Gamma_*(\mathscr{F})}$ to $\mathscr{F}$ in the following fashion. Choose $f \in R_1$ and consider the map

$$\begin{aligned} \beta_{(f)} : \Gamma_*(\mathscr{F})_{(f)} &\to \mathscr{F}|_{X_{(f)}} \\ \frac{s}{f^n} &\mapsto s|_{X_{(f)}} \otimes f^{-n}. \end{aligned}$$

Here, $s \in \Gamma(X, \mathscr{F}(n))$, and $s|_{X_{(f)}} \otimes f^{-n}$ is evaluated in $\Gamma(X_{(f)}, \mathscr{F})$ by the map

$$\Gamma(X_{(f)}, \mathscr{F}(n)) \otimes \Gamma(X_{(f)}, \mathscr{O}_X(-n)) \to \Gamma(X_{(f)}, \mathscr{F})$$

induced by the isomorphism $\mathscr{F}(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-n) \cong \mathscr{F}$.

t-globstar-tilde-inverse

Theorem 10.3.2. *Let R be a graded ring finitely generated over R_0 by R_1, and let $X = \operatorname{Proj} R$. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{M}$, the natural morphism*

$$\beta : \widetilde{\Gamma_*(\mathscr{M})} \to \mathscr{M}$$

is an isomorphism.

Proof. We'll show that $\beta_{(f)}$ is an isomorphism for every f with $\deg f = 1$. We know from (3.1.6) that X is separated. Under the given hypotheses, we also know that it's of finite type over $\operatorname{Spec} R_0$, from which it follows that it is quasi-compact. In particular, X satisfies the hypotheses of Lemma (10.3.1). We take $\mathscr{O}_X(1)$ to be the $\mathscr{L}$ in the notation of that Lemma. Observe then that $X_{\mathscr{L},f} = X_{(f)}$, and so for every section $s \in \mathscr{M}(X_{(f)})$ we can find $n \in \mathbb{N}$ such that $f^n \otimes s \in \mathscr{M}(n)(X_{(f)})$ is the restriction of a global section $\widetilde{s}$ of $\mathscr{M}(n)$. In this case, $\frac{\widetilde{s}}{f^n}$ maps to s under β_f, and β_f is surjective. Since $\mathscr{O}_X(1)|_{X_{(f)}} \cong \mathscr{O}_{X_{(f)}}$, $\beta_f(\frac{s}{f^r}) = 0$ if and only if $s|_{X_{(f)}} = 0$. In this case, we see by the Lemma that there is $n \in \mathbb{N}$ such that $f^n \otimes s = 0 \in \Gamma(X, \mathscr{F}(n))$, which means that $\frac{s}{f^r} = 0 \in \Gamma_*(\mathscr{M})_{(f)}$. Hence $\beta_{(f)}$ is also injective, which finishes our proof. □

The next corollary describes all closed subschemes of projective n-space over an affine scheme.

-ideals-algebras-deg-one

Corollary 10.3.3. *With the hypotheses and notation as in the Theorem above:*

(1) *Every quasi-coherent $\mathscr{O}_X$-algebra is of the form $\widetilde{S}$ for some graded R-algebra S.*
(2) *If, in addition, R is generated over R_0 by prime elements of degree 1, then every closed subscheme of X is isomorphic to $\operatorname{Proj} R/I$, for some homogeneous ideal $I \subset R$. In particular, every closed subscheme of Proj^n_S is of the form $\operatorname{Proj} S[t_1, \dots, t_n]/I$, for some homogeneous ideal $I \subset S[t_1, \dots, t_n]$.*

Proof. Most of this is immediate from the Proposition.

(1) From the Theorem, we see that any quasi-coherent $\mathscr{O}_X$-algebra is of the form $\widetilde{S}$ for some graded R-module S.
(2) We already know that $\operatorname{Proj} R/I$ is a closed subscheme of X. If R is generated over R_0 by prime elements, then from part (3) of (10.2.2), we know that $\Gamma_*(\mathscr{O}_X) = R$. Suppose we have a quasi-coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$; then $\mathscr{I} = \widetilde{\Gamma_*(\mathscr{I})}$, by part (1) of the Proposition. But $\Gamma_*(\mathscr{I})$

injects into $\Gamma_*(\mathscr{O}_X) = R$, and so $\mathscr{I} = \widetilde{I}$ for some homogeneous ideal $I \subset R$. By the bijective correspondence between quasi-coherent ideal sheaves and closed subschemes (4.3.14), and the exactness of the tilde functor, we see that every closed subscheme of X is of the claimed form. □

qsps-qc-of-finite-type

Corollary 10.3.4. *Let $X = \operatorname{Proj} R$, where R is finitely generated by R_1 over R_0. If $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_X$-module of finite type, then there exists a finitely generated graded R-module M such that $\mathscr{M} = \widetilde{M}$. In particular, if R_0 is Noetherian, then every coherent sheaf over X is of the form $\widetilde{M}$ for some finitely generated graded R-module M.*

Proof. Since $\mathscr{M}$ is quasi-coherent, we can find some R-module N such that $\mathscr{M} = \widetilde{N}$. Now, N is the direct limit of its finitely generated graded R-submodules, and hence, by (10.1.2), $\mathscr{M}$ is the direct limit of $\mathscr{O}_X$-modules of the form $\widetilde{M_i}$, where M_i is finitely generated. Now, we can cover X by finitely many affine opens of the form $X_{(f)}$, and thus, by an argument analogous to that in (4.2.19), we conclude that there is some finitely generated submodule $M \subset N$ such that $\mathscr{M} = \widetilde{M}$. □

isting-global-generation

Corollary 10.3.5. *Let $X = \operatorname{Proj} R$, where R is a graded ring finitely generated over R_0 by R_1, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists $n \in \mathbb{N}$ such that $\mathscr{F}(n)$ is generated by global sections. In particular, there exists a surjection onto $\mathscr{F}$ from a locally free sheaf.*

Proof. By the previous corollary, $\mathscr{F} = \widetilde{M}$, where M is finitely generated. Hence M is the quotient of a finite direct sum of modules of the type $R(m)$, for some $m \in \mathbb{Z}$. So it suffices to prove this for $\mathscr{F} = \mathscr{O}_X(m)$, for some $n \in \mathbb{Z}$. Consider the graded R-module $R(m)$. The zeroth degree component $R(m)_0 = R_m$ is finitely generated over R_0 by certain elements $r_1, \ldots, r_k$. Consider the map $\alpha : R^k \to R(m)$ that takes the basis of R^k to $(r_1, \ldots, r_k)$. Since $R_t = (R_1)^t$, for all $t \in \mathbb{N}$, we see that $\operatorname{im} \alpha = \bigoplus_{t \geq 0} R(m)_t$. This means that we have a short exact sequence

$$R^k \to R(m) \to \bigoplus_{-m \leq t < 0} R(m)_t \to 0.$$

Applying the tilde functor to this sequence, and observing that the cokernel in the sequence above has finite length and thus induces the zero $\mathscr{O}_X$-module, we conclude that we have a surjection $\mathscr{O}_X^k \to \mathscr{O}_X(m)$.

We get the second assertion by twisting $\mathscr{F}(n)$ back to $\mathscr{F}$. □

CHAPTER 11

Coherent Cohomology over Projective Schemes

chap:cohproj

1. Cohomology of Projective Space

The fundamental computation in the cohomology of schemes is that of the cohomology of the twisting sheaves over the projective space $\mathbb{P}^n_A$, for some ring A. Here, we will do this in the Noetherian case.

hproj-projective-r-space

THEOREM 11.1.1 (Cohomology of Twisting Sheaves). *Let $X = \mathbb{P}^r_R$ be projective r-space over a Noetherian ring A, and let $S = R[T_0, \ldots, T_r]$ be the graded polynomial ring in $r+1$-variables over R, so that $X = \operatorname{Proj} S$.*

(1)

$$H^0(X, \mathscr{O}_X(n)) = \begin{cases} S_n & \text{if } n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *For $r > 0$, we have*

$$H^r(X, \mathscr{O}_X(n)) \cong \begin{cases} R & \text{if } n = -r-1, \\ 0 & \text{if } n > -r-1. \end{cases}$$

(3) *For $n \geq 0$, there is a perfect pairing of R-modules:*

$$H^0(X, \mathscr{O}_X(n)) \times H^r(X, \mathscr{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathscr{O}_X(-r-1) \cong R.$$

(4) *For $0 < i < r$ and all $n \in \mathbb{Z}$, we have*

$$H^i(X, \mathscr{O}_X(n)) = 0.$$

PROOF. Let $\mathscr{F} = \oplus_{n \in \mathbb{Z}} \mathscr{O}_X(n)$, and let $C^\bullet$ be the Čech complex $C^\bullet(\mathcal{V}, \mathscr{F})$ associated to the open cover $\mathcal{V} = \{U_0, \ldots, U_r\}$, where $U_i = X_{(T_i)}$. Since each of the U_i is affine, we have by (9.1.9), natural isomorphisms

$$H^n(X, \mathscr{F}) \cong \check{H}^n(\mathcal{V}, \mathscr{F}),$$

for all $n \in \mathbb{Z}$. But observe that the Čech complex looks like the following complex of graded S-modules:

$$\prod_i S_{T_i} \xrightarrow{d^0} \prod_{i,j} S_{T_i T_j} \xrightarrow{d_1} \ldots \xrightarrow{d_{r-2}} \prod_i S_{T_0 \ldots \hat{T}_i \ldots T_r} \xrightarrow{d_{r-1}} S_{T_0 T_1 \ldots T_r}.$$

So we have isomorphisms of graded S-modules:

$$\begin{aligned} H^0(X, \mathscr{F}) &= \ker d^0 \\ &= (s_i) : \; s_i \in S_{T_i}, \; s_i = s_j \in S_{T_i T_j} \\ &\cong S. \end{aligned}$$

This proves the first part of the Theorem.

For the second part, note that $S_{T_0 \dots T_r}$ is a free graded S-module, spanned by monomials of the form $T_0^{j_0} \dots T_r^{j_r}$, for some $r+1$-tuple $(j_0, \dots, j_r) \in \mathbb{Z}^{r+1}$. On the other hand, the image of d^{r-1} is spanned by such monomials with $j_i \geq 0$, for at least one i. Hence we find

$$\begin{aligned} H^r(X, \mathscr{F}) &= \operatorname{coker} d^{r-1} \\ &\cong (T_0^{j_0} \dots T_r^{j_r} : j_i < 0, \text{ for all } i) \subset S_{T_0 \dots T_r}. \end{aligned}$$

Therefore,

$$\begin{aligned} H^r(X, \mathscr{O}_X(n)) &= H^r(X, \mathscr{F})_n \\ &\cong (T_0^{j_0} \dots T_r^{j_r} : j_i < 0 \sum_i j_i = n). \end{aligned}$$

From this the second part of the Theorem follows.

For the third, we define a pairing:

$$H^0(X, \mathscr{O}_X(n)) \times H^r(X, \mathscr{O}_X(-n-r-1)) \to H^r(X, \mathscr{O}_X(-r-1))$$

$$(T_0^{i_0} \dots T_r^{i_r}, T_0^{j_0} \dots T_r^{j_r}) \mapsto \begin{cases} (T_0^{i_0+j_0} \dots T_r^{i_r+j_r}) & \text{if } i_k + j_k < 0, \text{ for all } k \\ 0 & \text{otherwise} \end{cases}$$

Here, $(i_0, \dots, i_r)$ is an $r+1$-tuple of non-negative integers such that $\sum_k i_k = n$ and $(j_0, \dots, j_r)$ is an $r+1$-tuple of negative integers such that $\sum_k j_k = -n-r-1$. The result is either a monomial of degree $-r-1$ or 0. To see that this is a perfect pairing, it suffices to show that for every $r+1$-tuple $(i_0, \dots, i_r)$, with $\sum_k i_k = r$, one can find an $r+1$-tuple $(j_0, \dots, j_r)$ of negative integers with $\sum_k j_k = -n-r-1$ and $i_k + j_k < 0$, for all k. This is easy: simply take $j_k = -i_k - 1$, for all k.

The proof of the last assertion is the most involved. For this, we will use induction on r. If $r = 1$, then there is nothing to prove; so assume $r > 1$, and let H be the the hyperplane in X determined by the ideal (T_r). We have an exact sequence of S-modules:

$$(*) \qquad 0 \to S(-1) \xrightarrow{T_r} S \to S/(T_r) \to 0.$$

Sheafifying this, we have:

$$0 \to \mathscr{O}_X(-1) \xrightarrow{T_r} \mathscr{O}_X \to i_* \mathscr{O}_H \to 0,$$

where $i : H \to X$ is the natural closed immersion. Taking the direct sum of all the twists of this sequence, we get

$$(**) \qquad 0 \to \mathscr{F}(-1) \xrightarrow{T_r} \mathscr{F} \to i_* \mathscr{F}_H \to 0,$$

where $\mathscr{F}_H = \bigoplus_{n \in \mathbb{Z}} \mathscr{O}_H(n)$.

Now, by induction, we have, for $0 < i < r-1$, and all $n \in \mathbb{Z}$:

$$H^i(X, i_* \mathscr{O}_H(n)) \cong H^i(H, \mathscr{O}_H(n)) = 0.$$

So taking the long exact sequence of cohomology associated to the short exact sequence (**) above, we get isomorphisms

$$H^i(X, \mathscr{F}(-1)) \xrightarrow[\cong]{T_r} H^i(X, \mathscr{F}),$$

for $1 < i < r-1$. We claim that these maps are isomorphisms even for $i = 1, r-1$. Indeed, for $i = 0$, we have an exact sequence:

$$0 \to H^0(X, \mathscr{F}(-1)) \xrightarrow{T_1} H^0(X, \mathscr{F}) \xrightarrow{H}{}^{0}\ (X, i_*\mathscr{F}_H) \to 0,$$

since this is just the sequence (*) above written differently.

Now, we claim that the sequence

$$0 \to H^{r-1}(X, i_*\mathscr{F}_H) \xrightarrow{\delta} H^r(X, \mathscr{F}(-1)) \xrightarrow{T_r} H^r(X, \mathscr{F})$$

is exact. The kernel of T_r is generated by all monomials $T_0^{j_0} \dots T_r^{-1}$, with $j_k < 0$, for all k; so it suffices to show the connecting map δ is just multiplication by T_r^{-1}. For this we will have to go back to the Čech complexes $C^\bullet$ and $C^\bullet(-1)$ of graded S-modules corresponding to $\mathscr{F}$ and $\mathscr{F}(-1)$ respectively. Let $S' = S/(T_r)$; then we have the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_i S_{T_0 \dots \hat{T}_i \dots T_r}(-1) & \xrightarrow{T_r} & \prod_i S_{T_0 \dots \hat{T}_i \dots T_r} & \longrightarrow & S'_{T_0 \dots T_{r-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_{T_0 \dots T_r}(-1) & \xrightarrow{T_r} & S_{T_0 \dots T_r} & \longrightarrow & 0 & &
\end{array}$$

If $T_0^{i_0} \dots T_{r-1}^{i_{r-1}}$ is a monomial in $H^{r-1}(H, \mathscr{F}_H)$, with $i_k < 0$, for all k, then it comes from an $(r+1)$-tuple in $\prod_i S_{T_0 \dots \hat{T}_i \dots T_r}$, which maps to $\pm T_0^{i_0} \dots T_{r-1}^{i_{r-1}}$ in $S_{T_0 \dots T_r}$, which is in turn mapped onto by the monomial $T_0^{i_0} \dots T_{r-1}^{i_{r-1}} T_r^{-1}$ in $S_{T_0 \dots T_r}(-1)$. So $\delta(T_0^{i_0} \dots T_{r-1}^{i_{r-1}})$ is represented by the monomial $T_0^{i_0} \dots T_{r-1}^{i_{r-1}} T_r^{-1}$ in $H^r(X, \mathscr{F}(-1))$.

So we have shown that multiplication by T_r is an isomorphism on the i^{th} cohomology $H^i(X, \mathscr{F})$ (forgetting the grading), for $0 < i < r$. Now, we know by (9.1.8), that $H^i(U_r, \mathscr{F}|_{U_r}) = 0$, for all $i \in \mathbb{N}$. We will show

$$H^\bullet(X, \mathscr{F})_{T_r} \cong H^\bullet(U_r, \mathscr{F}|_{U_r}) = 0,$$

Since multiplication by T_r is an isomorphism on $H^i(X, \mathscr{F})$, for $0 < i < r$, this will show that these R-modules must in fact be zero. But the claimed isomorphism follows simply by observing that the Čech complex $C^\bullet(\mathcal{V} \cap U_r, \mathscr{F}|_{U_r})$ is simply the localization of $C^\bullet$ at T_r, and that localization is an exact functor, and so preserves cohomology. □

2. Some Important Finiteness Results

The importance of the next Theorem is inestimable.

hproj-projective-schemes

THEOREM 11.2.1 (Serre). *Let $X = \operatorname{Proj} S$ be a projective scheme over* $\operatorname{Spec} R$ *(where, again, R is Noetherian).*

1. *For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, and every $i \geq 0$, $H^i(X, \mathscr{F})$ is a finitely generated R-module.*
2. *For every coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists $n_0 \in \mathbb{Z}$ such that, for $n \geq n_0$, $\mathscr{F}(n)$ is $\Gamma(X, \_\_)$-acyclic.*

PROOF. Since S is finitely generated by S_1 over R, we have a closed embedding $i : X \to \mathbb{P}_R^r$, for some suitable $r \geq 1$. We have $H^i(X, \mathscr{F}) = H^i(\mathbb{P}_R^r, i_*\mathscr{F})$

[HA, 7.8.4], and we also have $i_*(\mathscr{F}(n)) \cong (i_*\mathscr{F})(n)$ (10.1.6). Hence it suffices to prove the Theorem for the case where $X = \mathbb{P}^r_R$, for some $r \in \mathbb{N}$.

In this case, (1) holds for the twisting sheaves $\mathscr{O}_X(n)$, by (11.1.1), and so it's true for all finite direct sums of such sheaves. We'll prove it in general by descending induction on i, starting from $i = r+1$. For the base case, note that, since $\dim X = r$, $H^k(X, \mathscr{F}) = 0$, for all sheaves $\mathscr{F}$ [HA, 7.8.6], and all $k \geq r+1$. Now, suppose $i < r+1$ and that $H^{i+1}(X, \mathscr{F})$ is finitely generated, for every coherent sheaf $\mathscr{F}$. By (10.3.5), for any coherent sheaf $\mathscr{F}$, we have an exact sequence:

$$(*) \qquad 0 \to \mathscr{I} \to \mathscr{E} \to \mathscr{F} \to 0,$$

where $\mathscr{E} \cong \oplus_{i=1}^r \mathscr{O}_X(q_i)$, for some suitable integers q_i. Now, from the long exact sequence of cohomology associated to this sequence, we have an exact sequence:

$$H^i(X, \mathscr{E}) \to H^i(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{I}),$$

where $H^i(X, \mathscr{E})$ is finitely generated, since $\mathscr{E}$ is a direct sum of twisting sheaves, and $H^{i+1}(X, \mathscr{I})$ is finitely generated by the induction hypothesis. Since R is Noetherian, this implies $H^i(X, \mathscr{F})$ is also finitely generated, thus finishing our proof of (1).

Now observe that again (2) holds for twisting sheaves and hence for finite direct sums of twisting sheaves by (11.1.1); more specifically, we see that $\mathscr{O}_X(n)$ is $\Gamma(X, \_\_)$-acyclic, for $n \geq 0$. We again argue by descending induction on i starting from $i = r+1$. The base step is the same; so, assuming that $i < r+1$, and that for all coherent sheaves $\mathscr{F}$ over X, there exists $n_0 \in \mathbb{Z}$ such that, for $n \geq n_0$, $H^k(X, \mathscr{F}) = 0$, for $k \geq i+1$. Now, twisting the short exact sequence (*) by n, and then taking the long exact sequence of cohomology, we get an exact sequence:

$$H^i(X, \mathscr{E}(n)) \to H^i(X, \mathscr{F}(n)) \to H^{i+1}(X, \mathscr{I}(n)).$$

By induction, there exists an n_0 such that both $H^i(X, \mathscr{E}(n))$ and $H^{i+1}(X, \mathscr{I}(n))$ are 0, for $n \geq n_0$. From this (2) follows immediately. □

j-globsec-exact-twisting

Corollary 11.2.2. *Let X be a projective scheme over a Noetherian ring R, and let $\mathscr{F}^1 \to \ldots \to \mathscr{F}^n$ be an exact sequence of coherent sheaves over X. Then, there exists $n_0 \in \mathbb{Z}$ such that, for all $r \geq n_0$, the sequence*

$$\Gamma(X, \mathscr{F}^1(r)) \to \ldots \to \Gamma(X, \mathscr{F}^(r))$$

is also exact.

Proof. By splitting the exact sequence into short exact sequences, it suffices to prove the statement for a short exact sequence

$$0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \mathscr{F}^3 \to 0.$$

For this, it suffices to find $n_0 \in \mathbb{Z}$ such that $\mathscr{F}^1(r)$ is $\Gamma(X, \_\_)$-acyclic, for all $r \geq n_0$, and this we can always do, by the Theorem. □

oj-ext-twisting-commutes

Lemma 11.2.3. *Let X be a projective scheme over a Noetherian ring R, and let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves over X. Then, for any $n \in \mathbb{Z}$, we have:*

$$\begin{aligned} \operatorname{Ext}^\bullet_{\mathscr{O}_X}(\mathscr{F}(-n), \mathscr{G}) &\cong \operatorname{Ext}^\bullet_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n)) \\ \underline{\operatorname{Ext}}^\bullet_{\mathscr{O}_X}(\mathscr{F}(-n), \mathscr{G}) &\cong \underline{\operatorname{Ext}}^\bullet_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n)) \\ &\cong \underline{\operatorname{Ext}}^\bullet_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})(n). \end{aligned}$$

Proof. Follows from [HA, 7.6.4]. □

twisting-global-sections

COROLLARY 11.2.4. *Let X be a projective scheme over a Noetherian ring R, and let $\mathscr{F}, \mathscr{G}$ be coherent sheaves over X. Then, for every $i \geq 0$, there is an integer $n_0 \in \mathbb{Z}$ such that, for all $n \geq n_0$, we have*

$$\operatorname{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n)) \cong \Gamma(X, \underline{\operatorname{Ext}}^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n))).$$

PROOF. By the last lemma and the Theorem above, there exists $n_0 \in \mathbb{Z}$ such that $\underline{\operatorname{Ext}}^{i-1} \bullet_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n))$ is $\Gamma(X, \_\_)$-acyclic, for $n \geq n_0$ (note that the $\underline{\operatorname{Ext}}$ sheaf is coherent by (9.3.9)). Now, from the Ext spectral sequence [HA, 7.6.3], we see that there is a monomorphism

$$\operatorname{Ext}^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n)) \hookrightarrow \Gamma(X, \underline{\operatorname{Ext}}^i_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n)))$$

whose cokernel is contained in $H^2(X, \underline{\operatorname{Ext}}^{i-1}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}(n)))$. But this last R-module is 0 by our choice of n_0, and hence the monomorphism is in fact an isomorphism, thus finishing our proof. □

3. The Category of Coherent Sheaves

DEFINITION 11.3.1. Let S be a graded ring. The *quasi-equivalence category of graded S-modules*, denoted S-qec, is defined in the following fashion: its objects are just graded S-modules, and, for two graded S-modules M and N, we define:

$$\operatorname{Hom}_{S\text{-qec}}(M, N) = \varinjlim_{m \in \mathbb{Z}} \operatorname{Hom}_{S^{\mathbb{Z}}\text{-mod}}(M^{\geq m}, N^{\geq n}).$$

Composition is defined in the obvious fashion: given $\varphi : M^{\geq m} \to N^{\geq m}$ and $\psi : N^{\geq r} \to P^{\geq r}$, we get $\psi\varphi : M^{\geq s} \to N^{\geq s}$, where $s = \max\{m, r\}$. It's easy to see that this is $\mathbb{Z}$-linear.

Observe that there is a functor from $S^{\mathbb{Z}}$-mod to S-qec induced by the natural map

$$\operatorname{Hom}_{S^{\mathbb{Z}}\text{-mod}}(M, N) \to \operatorname{Hom}_{S\text{-qec}}(M, N).$$

asi-equivalence-category

PROPOSITION 11.3.2. *For any graded ring S, the category S-qec is abelian.*

PROOF. It's clear that S-qec is additive: the 0 object is just the trivial S-module, and the direct sum is just the direct sum of S-modules. Indeed, since direct limit is an additive functor, we have, for graded S-modules M, N, P:

$$\begin{aligned}\operatorname{Hom}_{S\text{-qec}}(M \oplus N, P) &= \varinjlim \operatorname{Hom}_{S^{\mathbb{Z}}\text{-mod}}(M^{\geq n} \oplus N^{\geq n}, P^{\geq n}) \\ &= \varinjlim \operatorname{Hom}_{S^{\mathbb{Z}}\text{-mod}}(M^{\geq n}, P^{\geq n}) \times \varinjlim \operatorname{Hom}_{S^{\mathbb{Z}}\text{-mod}}(N^{\geq n}, P^{\geq n}) \\ &= \operatorname{Hom}_{S\text{-qec}}(M, P) \times \operatorname{Hom}_{S\text{-qec}}(N, P).\end{aligned}$$

Kernels and cokernels are inherited from $S^{\mathbb{Z}}$-mod: suppose $\varphi \in \operatorname{Hom}_{S\text{-qec}}(M, N)$, then there is $n \in \mathbb{Z}$ such that $\varphi : M^{\geq n} \to N^{\geq n}$ in $S^{\mathbb{Z}}$-mod. Let $K = \ker \varphi$, where the kernel is being taken in $S^{\mathbb{Z}}$-mod, and let $\psi \in \operatorname{Hom}_{S\text{-qec}}(P, M)$ be a morphism such that $\varphi\psi = 0$. This implies that we can find $m \geq n$ such that $0 = \varphi\psi : P^{\geq m} \to N^{\geq m}$, and so ψ must factor uniquely through $K^{\geq m}$, which shows that K is also the kernel in S-qec. A formally dual argument works to show the existence of cokernels.

Now it remains to show that, for a monomorphism φ, we have $\varphi = \ker(\operatorname{coker} \varphi)$ and that, for an epimorphism ψ, we have $\psi = \operatorname{coker}(\ker \psi)$. Observe that φ is a monomorphism from M to N in S-qec if and only if its kernel in $S^{\mathbb{Z}}$-mod vanishes in high enough degree. Thus, for $n >> 0$, φ is an honest monomorphism from $M^{\geq n}$

to $N^{\geq n}$ in $S^{\mathbb{Z}}$-mod, and thus is the kernel of its cokernel in $S^{\mathbb{Z}}$-mod, which shows that the same is true in S-qec. □

DEFINITION 11.3.3. We say that a morphism $\varphi : M \to N$ in $S^{\mathbb{Z}}$-mod is *essentially surjective (resp. a quasi-equivalence)* if it is an epimorphism (resp. an isomorphism) in S-qec.

A graded S-module M is *quasi-finitely generated* if there is an essentially surjective morphism $S^n \to M$, for some $n \in \mathbb{N}$.

ngen-finite-globsections

THEOREM 11.3.4. *Let $X = \operatorname{Proj} S$ be a projective scheme over a Noetherian ring R,*

(1) *For every coherent sheaf $\mathscr{M}$ over X, $\Gamma_*(\mathscr{M})$ is quasi-finitely generated over S.*
(2) *For every quasi-finitely generated S-module M, the natural map*

$$M \to \Gamma_*(\widetilde{M})$$

is a quasi-equivalence.

(3) *The functors $M \mapsto \widetilde{M}$ and $\mathscr{M} \mapsto \Gamma_*(\mathscr{M})$ give us an equivalence of categories between X-coh and the full sub-category of quasi-finitely generated S-modules in S-qec.*

PROOF. Suppose $\mathscr{M}$ is a coherent sheaf over X. We can find $n_0 \in \mathbb{Z}$ such that $\mathscr{M}(n)$ is generated by global sections, for $n \geq n_0$ (10.3.5). In this case, we can find $r \geq 0$, and a surjection $\mathscr{O}_X^r \to \mathscr{M}(n_0)$, and thence a surjection

$$\Gamma(X, \mathscr{O}_X(n))^r \twoheadrightarrow \Gamma(X, \mathscr{M}(n)),$$

for every $n >> 0$ (11.2.2). So it suffices to show that $\Gamma_*(\mathscr{O}_X)$ is quasi-finitely generated over S.

For this, embed X in some projective space $\mathbb{P}_R^r = \mathbb{P}$ over R. Let $T = R[T_0, \ldots, T_r]$; then the surjection $T \twoheadrightarrow S$ translates into a surjection of sheaves $\mathscr{O}_{\mathbb{P}} \twoheadrightarrow i_*\mathscr{O}_X$, where $i : X \to \mathbb{P}$ is the embedding. For n large enough, this, by (11.2.2), gives us surjections

$$\Gamma(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(n)) \twoheadrightarrow \Gamma(X, \mathscr{O}_X(n)),$$

Now, by (11.1.1), $\Gamma(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(n)) = T_n$, and so its image in $\Gamma(X, \mathscr{O}_X(n))$ is precisely the image of S_n in $\Gamma(X, \mathscr{O}_X(n))$, under the natural map $\alpha : S \to \Gamma_*(\mathscr{O}_X)$. So we see that $\Gamma_*(\mathscr{O}_X)$ is quasi-equivalent to S, and is in particular quasi-finitely generated over S, thus proving (1).

For (2), there is no harm in assuming that M is finitely generated over S (since $M^{\geq n}$ is finitely generated, for n large enough, and $\widetilde{M^{\geq n}} \cong \widetilde{M}$, by (10.1.2)). By [CA, 1.4.3], we have a finite filtration by graded submodules

$$M = M_n \supset M_{n-1} \supset \ldots \supset M_0 = 0,$$

such that for all i, $M_{i+1}/M_i \cong (S/P_i)(n)$, for some homogeneous prime $P_i \subset R$, and for some $n \in \mathbb{Z}$. Now, the tilde functor is exact, and the global sections functor is left exact. Hence, for every i, we have the following exact sequence:

$$0 \to \Gamma(X, \widetilde{M_i}) \to \Gamma(X, \widetilde{M_{i+1}}) \to \Gamma(X, \widetilde{(S/P_i)}(n)).$$

Considering the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_i & \longrightarrow & M_{i+1} & \longrightarrow & (R/P_i)(n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_*(\widetilde{M_i}) & \longrightarrow & \Gamma_*(\widetilde{M_{i+1}}) & \longrightarrow & \Gamma_*(\widetilde{(R/P_i)(n)}) & &
\end{array}$$

and arguing inductively, we reduce to showing that, for any projective R-scheme $Y = \operatorname{Proj} S'$, the S'-modules S' and $\Gamma_*(\mathscr{O}_Y)$ are quasi-equivalent. But this is precisely what we showed in the paragraph above.

Now, (3) follows from (2) and (10.3.2). □

4. The Hilbert Polynomial

In this section, we will show that the Hilbert polynomial of a coherent sheaf over a projective variety can be defined using its cohomology. This will let us show that for a flat, projective scheme over any affine scheme, the Hilbert polynomial of its fibers stays constant.

Definition 11.4.1. For a projective scheme $X = \operatorname{Proj} S$ over an Artinian ring R, and a coherent sheaf $\mathscr{F}$ over X, the *Hilbert function* of $\mathscr{F}$ is the function $P(\mathscr{F}, n) = \chi(\mathscr{F}(n)) = \sum_i (-1)^i l(H^i(X, \mathscr{F}(n)))$. Note that the A-modules $H^i(X, \mathscr{F}(n))$ have finite length by (11.2.1).

hproj-hilbert-polynomial Theorem 11.4.2. *Let $X = \operatorname{Proj} S$ be a projective scheme over an Artinian ring R, and let $\mathscr{F}$ be a coherent sheaf over X.*

(1) *$P(\mathscr{F}, n)$ is a polynomial function.*
(2) *For n large enough, $P(\mathscr{F}, n) = H^0(X, \mathscr{F}(n))$.*
(3) *If $M = \Gamma_*(\mathscr{F})$, then $P(\mathscr{F}, n)$ is equal to the Hilbert polynomial $H(M, n)$.*

Proof. Note that (2) is an immediate consequence of part (2) of (11.4.2), and that (3) follows from (2) and (1).

For (1), we can, as we have done often before, assume that $X = \mathbb{P}^r_A$, for some $r > 0$. We will prove the statement by induction on $\dim \operatorname{Supp} \mathscr{F}$. If $\dim \operatorname{Supp} \mathscr{F} = 0$, then $\mathscr{F}$ is supported on a finite union of closed points. Since $\mathscr{F}_x \cong \mathscr{F}(n)_x$, for all points $x \in X$, we see immediately that $P(\mathscr{F}, n)$ must be a constant function. Now, suppose $\dim \operatorname{Supp} \mathscr{F} > 0$; in this case, consider the short exact sequence

$$0 \to \mathscr{O}_X(-1) \xrightarrow{T_k} \mathscr{O}_X \to \mathscr{O}_{H_k} \to 0,$$

where H_k is the hyperplane in X cut out by T_k, for $0 \leq k \leq r$(where $X = \operatorname{Proj} A[T_0, \ldots, T_r]$). Tensoring this sequence with $\mathscr{F}$, we obtain a four term exact sequence

$$0 \to \mathscr{K} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{C} \to 0,$$

where $\mathscr{K}$ and $\mathscr{C}$ are supported in $\operatorname{Supp} \mathscr{F} \cap H_k$. Since $\dim \operatorname{Supp} \mathscr{F} > 0$, we can find k such that $\operatorname{Supp} \mathscr{F} \nsubseteq H_k$, and so we can assume that $\mathscr{K}$ and $\mathscr{C}$ are supported in subschemes of dimension strictly lower than that of $\operatorname{Supp} \mathscr{F}$. So, by our inductive hypothesis $P(\mathscr{K}, n)$ and $P(\mathscr{C}, n)$ are polynomial functions. By the additivity of the Euler characteristic, we thus see that $\Delta P(\mathscr{F}, n) = P(\mathscr{F}, n) - P(\mathscr{F}, n-1)$ is an integer valued polynomial function. From this it follows that $P(\mathscr{F}, n)$ is also an integer valued polynomial function. □

j-hilbert-poly-dimension

COROLLARY 11.4.3. *Let* $X = \operatorname{Proj} S$ *be a projective variety over a field* k*. Then* $\deg P(\mathscr{O}_X, n) = \dim X$.

PROOF. This follows from the Theorem above and (6.4.7). □

DEFINITION 11.4.4. The *arithmetic genus* of a projective variety X over a field k is the integer $1 - P(\mathscr{O}_X, 0)$.

cohproj-acyclic-fibers

LEMMA 11.4.5. *Suppose* X *is a projective scheme over an affine scheme* $Y = \operatorname{Spec} A$*, and let* $\mathscr{F}$ *be a coherent sheaf over* X*. For* $y \in Y$*, and* m *large enough, we have isomorphisms*

$$H^0(X, \mathscr{F}(m)) \otimes_A k(y) \cong H^0(X_y, \mathscr{F}_y(m)).$$

PROOF. Base-changing along the flat morphism $\operatorname{Spec} \mathscr{O}_{Y,y} \to Y$, and using (9.3.5), we get isomorphisms

$$H^0(X, \mathscr{F}(m)) \otimes_A \operatorname{Spec} \mathscr{O}_{Y,y} \cong H^0(X \times_Y \operatorname{Spec} \mathscr{O}_{Y,y}, \mathscr{F}(m) \otimes_A \mathscr{O}_{Y,y}).$$

So, to prove our statement, we can replace Y with $\operatorname{Spec} \mathscr{O}_{Y,y}$ and X with $X \times_Y \operatorname{Spec} \mathscr{O}_{Y,y}$, and assume that y is a closed point of Y. In this case, X_y is a closed sub-scheme of X, and we have an isomorphism (9.3.4):

$$H^0(X, \mathscr{F}(m) \otimes_A k(y)) \cong H^0(X_y, \mathscr{F}_y(m))$$

Choose a free presentation

$$A^r \to A \to k(y) \to 0$$

for $k(y)$ over A. Tensoring this with $\mathscr{F}(m)$ gives an exact sequence

$$\mathscr{F}^r(m) \to \mathscr{F}(m) \to \mathscr{F}(m) \otimes_A k(y) \to 0,$$

and taking the long exact sequence of cohomology associated to this sequence, we obtain a sequence

$$H^0(X, \mathscr{F}^r(m)) \to H^0(X, \mathscr{F}^r(m)) \qquad \to H^0(X, \mathscr{F}_y(m)) \to 0,$$

which is exact for m large enough (11.2.2). But if we tensor the free presentation with $H^0(X, \mathscr{F}(m))$, we obtain the exact sequence

$$H^0(X, \mathscr{F}^r(m)) \to H^0(X, \mathscr{F}(m)) \qquad \to H^0(X, \mathscr{F}(m)) \otimes_A k(y) \to 0.$$

This gives us our isomorphism for m large enough. □

j-flat-family-local-ring

PROPOSITION 11.4.6. *Let* $X = \mathbb{P}^n_A$ *be projective* n*-space over a Noetherian local domain* A*, and let* $\mathscr{F}$ *be a coherent sheaf over* X*. Then the following are equivalent:*

(1) $\mathscr{F}$ *is flat over* A.
(2) *For all* m *large enough,* $H^0(X, \mathscr{F}(m))$ *is a free* A*-module of finite rank.*
(3) *The Hilbert polynomial* $P_y = P(\mathscr{F}_y, n)$ *of* $\mathscr{F}_y$*, the pullback of* $\mathscr{F}$ *to* X_y*, for* $y \in Y$*, is independent of* y*.*

PROOF. We begin with (1) ⇒ (2): Let $\mathcal{V}$ be the standard affine open cover for X, and let $\check{C}(\mathcal{V}, \mathscr{F}(m))$ be the corresponding Čech complex for $\mathscr{F}(m)$. For m large enough, $\mathscr{F}(m)$ is $\Gamma(X, \_\_)$-acyclic (11.2.1), and so $\check{C}(\mathcal{V}, \mathscr{F}(m))$ is an acyclic complex (9.1.9). Since $\check{C}^i(\mathcal{V}, \mathscr{F}(m)) = 0$ for $i > n$, and $\check{C}^i(\mathcal{V}, \mathscr{F}(m))$ is a flat A-module, for $0 \leq i \leq n$, we find that $H^0(X, \mathscr{F}(m))$ is a flat syzygy for $\check{C}^n(\mathcal{V}, \mathscr{F}(m))$. Since the latter is a flat A-module, we find that $H^0(X, \mathscr{F}(m))$ must itself be flat over A and hence free, since it's finitely generated over A.

Now on to (2) $\Rightarrow$ (1): If $H^0(X, \mathscr{F}(m))$ is free over A, for m large enough, then $\Gamma_*(\mathscr{F})$ is quasi-equivalent to a free A-module. Since $\mathscr{F} \cong \widetilde{\Gamma_*(\mathscr{F})}$ (11.3.4), we can therefore assume that $\mathscr{F} = \widetilde{M}$, where M is free over A. Since $A[T_0, \dots, T_n]_{(T_i)}$ is flat over A, for each i, this shows that $\widetilde{M}$ must also be flat over A.

Next we show (2) $\Rightarrow$ (3): This is immediate from the lemma above, since, for all $y \in Y$, and all m large enough, we have

$$(*) \qquad \begin{aligned} P_y(m) &= \dim_{k(y)} H^0(X_y, \mathscr{F}_y(m)) \\ &= \dim_{k(y)}(H^0(X, \mathscr{F}(m)) \otimes_A k(y)), \end{aligned}$$

where the first equality follows from (11.4.2) and the second from (11.4.5). But, since $H^0(X, \mathscr{F}(m))$ is a free A-module for m large enough, we see that

$$P_y(m) = \operatorname{rk} H^0(X, \mathscr{F}(m)),$$

for m large enough, and the second quantity is of course independent of y.

Now for (3) $\Rightarrow$ (2): From (4.2.10), it suffices to show that the generic fiber and the special fiber of $H^0(X, \mathscr{F}(m))$ both have the same dimension, for m large enough. But, if P_y is independent of y, this is an immediate consequence of (*) above. $\square$

5. The Theorem on Formal Functions

6. The Semicontinuity Theorem

6.1. Some Homological Nonsense. We set up some notation. Let R be a Noetherian ring, and let $L^\bullet \in \mathrm{Ch}^{\geq 0}\, R$-mod be a bounded complex of finite free R-modules. We define functors

$$\begin{aligned} T^i : R\text{-mod} &\to R\text{-mod} \\ M &\mapsto H^i(L^\bullet \otimes_R M). \end{aligned}$$

Since $L^\bullet$ is a flat complex over R, we see immediately that the T^i give us a δ-functor from R-mod to R-mod. We set $W^i = \operatorname{coker}(L^{i-1} \to L^i)$. Since tensoring preserves cokernels, for every R-module M, we find short exact sequences

$$(**) \qquad 0 \to T^i M \to W^i \otimes_R M \to L^{i+1} \otimes_R M.$$

Now we investigate the exactness properties of these functors T^i.

tfunctors-left-exactness

Proposition 11.6.1. *The following are equivalent:*

1. *T^i is a left exact functor.*
2. *W^i is a projective R-module.*
3. *T^i is co-representable by a finitely generated R-module Q; that is $T^i = \operatorname{Hom}_R(Q, \_\_)$.*

Proof. Let $u : M' \to M$ be a monomorphism of R-modules. Then it follows from (**) above, and the Snake Lemma, that $\ker T^i u = \ker W^i \otimes u$. Hence T^i is left exact if and only if W^i is flat, if and only if W^i is projective. The last equivalence follows since W^i is finitely generated and R is Noetherian. This shows (1) $\Leftrightarrow$ (2).

Clearly (3) implies (1); for (2) $\Rightarrow$ (3), observe that, for any projective R-module P, we have $P \otimes_R M \cong \operatorname{Hom}_R(P^\vee, M)$, where $P^\vee$ is the dual of P. Hence, (**) tells us that T^i is the kernel of two co-representable functors and is hence itself co-representable. $\square$

Now observe that, for every R-module M, there is a natural map $\phi^i_M : T^iR \otimes_R M \to T^iM$. Indeed, giving such a map is, by the adjunction between tensor product and Hom, equivalent to giving a map $M \to \mathrm{Hom}_R(T^iR, T^iM)$. But, for this we can take the map $\mathrm{Hom}_R(R, M) \to Hom_R(T^iR, T^iM)$ given by the functoriality of T^i.

functors-right-exactness

Proposition 11.6.2. *The following are equivalent:*

(1) T^i *is a right exact functor.*
(2) ϕ^i_M *is an isomorphism, for all* i.
(3) ϕ^i_M *is a surjection, for all* i.

Proof. (1) ⇒ (2) is proved using the usual trick of taking a free presentation, and the using the right exactness of T^i, and the fact that ϕ^i_F is an isomorphism for all free R-modules F of finite rank. (2) ⇒ (3) is trivial, so we'll finish by proving (3) ⇒ (1). Let $v : M \to M''$ be an epimorphism; then we have the following picture:

$$\begin{array}{ccccc} T^iR \otimes_R M & \longrightarrow & T^iR \otimes_R M'' & \longrightarrow & 0 \\ \downarrow{\scriptstyle \phi^i_M} & & \downarrow{\scriptstyle \phi^i_{M'}} & & \\ T^iM & \longrightarrow & T^iM'' & \to & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

where the columns and the top row are exact. It follows that the bottom row must also then be exact, and so we see that T^i is right exact. □

proj-tfunctors-exactness

Corollary 11.6.3. *The functor* T^i *is exact if and only if it is right exact and* T^iR *is a projective* R*-module.*

Proof. It's clear from the Proposition that T^i is exact if and only if it is right exact and T^iR is flat. The statement now follows since every finitely generated flat module over a Noetherian ring is projective. □

Remark 11.6.4. The results above are very specific instances of the phenomenon that every left exact additive endo-functor of R-mod is in fact co-representable, and that every right exact additive endo-functor is essentially tensor product with a fixed R-module.

We finish with a final (as befits a finishing flourish) technical lemma.

tfunctor-technical-lemma

Lemma 11.6.5. *Let* $C^\bullet$ *be a bounded complex, and suppose* $H^i(C)$ *are finitely generated* R*-modules, for* $i \geq 0$. *Then we can find a bounded complex* $L^\bullet$ *of finite free* R*-modules and a chain map* $f : L^\bullet \to C^\bullet$, *such that* $H^\bullet(f)$ *is an isomorphism of cohomologies. If* $C^\bullet$ *is flat, then the map*

$$H^\bullet(f \otimes M) : H^\bullet(L \otimes_R M) \to H^\bullet(C \otimes_R M)$$

is an isomorphism, for all R*-modules* M.

Proof. Assume that, we have inductively constructed a complex $L_i^\bullet$ of finite free R-modules and a chain map $f_i : L_i^\bullet \to C^\bullet$ satisfying the following conditions:

(1) $H^r(f_i) : H^r(L_i) \to H^r(C)$ is an isomorphism for $r > i$.
(2) f_i^i maps $Z^i(L_i)$ onto $H^i(C)$.

For i large enough $C^i = 0$, so this is definitely possible for such i. We will now extend the complex L_i on the left and the map f_i along with it. To this end, observe that the kernel of the surjection $f_i^i|_{Z^i(L_i)}$ is $D = (f_i^i)^{-1}(B^i(C))$. Since R is Noetherian, D is a finitely generated submodule of L_i^i. Choose a set of generators $\{y_1, \dots, y_r\}$ for D, and let $\{y_1', \dots, y_r'\}$ be its image in $B^i(C)$ under f_i^i. Lift this set now to elements $\{x_1, \dots, x_r\}$ of C^{i-1}. Let $\{x_{r+1}, \dots, x_s\}$ be elements of $Z^{i-1}(C)$ whose images in $H^{i-1}(C)$ form a set of generators. Now, define the complex L_{i-1} in the following way: for $j \geq i$, set $L_{i-1}^j = L_i^j$; for $j < i-1$, set $L_{i-1}^j = 0$; finally, let L_{i-1}^{i-1} be a free R-module of rank s with basis $e_1, \dots, e_s$. It still remains to define the boundary maps. For $j \geq i$, we maintain $d_{i-1}^i = d_i^i$, so that L_i includes into L_{i-1}, and for $j = i-1$, we let d_{i-1}^{i-1} be the following map:

$$d_{i-1}^{i-1} : e_k \mapsto \begin{cases} y_k & \text{for } 1 \leq k \leq r \\ 0 & \text{for } k \geq r+1. \end{cases}$$

We have $d_{i-1}^i y_k = f_{i-1}^{i+1} d_C^i y_k' = 0$; so this does indeed give us a boundary morphism. We now define a map $f_{i-1} : L_{i-1}^\bullet \to C^\bullet$ extending f_i in the following way: for $j \geq i$, we set $f_{i-1}^j = f_i^j$, and for $j = i-1$, we define: $f_{i-1}^{i-1}(e_k) = x_k$, for $1 \leq k \leq s$. It is now easily checked that f_{i-1}^{i-1} gives a chain map, that $H^j(f_{i-1})$ is an isomorphism for $j \geq i$, and that f_{i-1}^{i-1} maps $Z^{i-1}(L_{i-1})$ onto $H^{i-1}(C)$. Now take the union of the L_i to finish the job.

Suppose now that $C^\bullet$ is a complex of flat R-modules. Then notice that $H^i(C^\bullet \otimes_R \text{--})$ gives rise to a δ-functor from R-mod to R-mod, and that $H^i(f^\bullet \otimes_R \text{--})$ is a morphism of δ-functors.

Since both tensor product and cohomology commute with filtered colimits, and every R-module is a filtered colimit of its finitely generated submodules, it suffices to prove that the map $H^\bullet(f \otimes_R M)$ is an isomorphism for M finitely generated. Moreover, since $L^\bullet$ and $C^\bullet$ are both bounded, we can assume that $H^j(f \otimes_R \text{--})$ is an isomorphism for $j > i$, and show inductively that $H^i(f \otimes_R \text{--})$ must also be an isomorphism.

For this, choose a short exact sequence

$$0 \to N \to F \to M \to 0,$$

where F is a finite free R-module. Using the fact that $H^\bullet(f \otimes_R \text{--})$ is a morphism of δ-functors, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
H^i(L^\bullet \otimes_R F) & \longrightarrow & H^i(L^\bullet \otimes_R M) & \longrightarrow & H^{i+1}(L^\bullet \otimes_R N) & \longrightarrow & H^{i+1}(L^\bullet \otimes_R F) \\
{\scriptstyle H^i(f \otimes F)}\downarrow\cong & & {\scriptstyle H^i(f \otimes M)}\downarrow & & {\scriptstyle H^{i+1}(f \otimes N)}\downarrow\cong & & {\scriptstyle H^{i+1}(f \otimes F)}\downarrow\cong \\
H^i(C^\bullet \otimes_R F) & \longrightarrow & H^i(C^\bullet \otimes_R M) & \longrightarrow & H^{i+1}(C^\bullet \otimes_R N) & \longrightarrow & H^{i+1}(C^\bullet \otimes_R F)
\end{array}$$

The vertical arrows flanking the diagram are isomorphisms since $H^\bullet(f \otimes F)$ is an isomorphism for every free R-module F, and the second arrow from the right is an isomorphism by the inductive hypothesis. Now it follows that $H^i(f \otimes M)$ must also be an isomorphism. $\square$

6.2. Semicontinuity. Let $f : X \to Y$ be a projective morphism. Then, for $y \in Y$, X_y is a projective variety over $k(y)$, and so by (11.2.1), for any coherent sheaf $\mathscr{F}$ on X, $H^i(X_y, \mathscr{F}_y)$ is a finite dimensional $k(y)$-vector space. Observe that, by (11.4.5