MATH 3311, FALL 2025: LECTURE 9, SEPTEMBER 15

Video: `https://youtu.be/CTsybBm-r3I`

Group actions

Let us recall the following from last time.

Definition 1. A **group action** or simply **action** of a group G on a set X is a group homomorphism

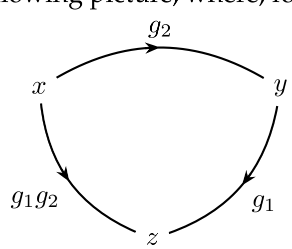
$$\rho : G \to \mathrm{Bij}(X).$$

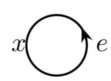
We will use the notation $G \curvearrowright X$ (read 'G acting on X') to denote that we have an action of G on X.

While this is a very compact definition, it packs a lot of information! To see this, define a function

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto \rho(g)(x) = g \cdot x. \end{aligned}$$

Then this function has several properties that fall out of the homomorphism condition, which we see in the following picture, where, for fixed $x \in X$, we view every element $g \in G$ as being a path from x to $g \cdot x$.





(1) For all $x \in X$, $g_1, g_2 \in G$, $(g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.[1] This is because we have

$$(g_1g_2) \cdot x = \rho(g_1g_2)(x) = (\rho(g_1) \circ \rho(g_2))(x) = \rho(g_1)(\rho(g_2)(x)) = g_1 \cdot (g_2 \cdot x).$$

Here, we have used the homomorphism property $\rho(g_1g_2) = \rho(g_1) \circ \rho(g_2)$.

(2) For all $x \in X$, $e \cdot x = \rho(e)(x) = \mathrm{Id}(x) = x$. This is because $\rho(e) = \mathrm{Id}$.

Conversely, suppose that we are given a function $G \times X \xrightarrow{(g,x) \mapsto g \cdot x} X$ with these properties—that is, satisfying:

(1) $(g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G$ and $x \in X$;
(2) $e \cdot x = x$ for all $x \in X$.

Consider the function

$$\rho(g) : X \xrightarrow{x \mapsto g \cdot x} X.$$

Observation 1. $\rho(g)$ is a bijection with inverse $\rho(g^{-1})$. That is, $\rho(g) \in \mathrm{Bij}(X)$.

Proof. We have

$$\begin{aligned} (\rho(g^{-1}) \circ \rho(g))(x) &= \rho(g^{-1})(g \cdot x) \\ &= g^{-1} \cdot (g \cdot x) \\ &= (g^{-1}g) \cdot x \\ &= e \cdot x \qquad\qquad = x. \end{aligned}$$

[1] From now on, we will, like in high school algebra, use concatenation of symbols (like gh) instead of bringing in the group operation every time (like $g * h$). This doesn't imply that the group operation is necessarily multiplication: For instance, it could be composition of functions, or it could be addition in a group like $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$.

Note that in the second-to and third-to-last equalities, we have used the defining properties of group actions. ☐

In fact, the first property tells us:

Observation 2. The function
$$G \xrightarrow{g \mapsto \rho(g)} \mathrm{Bij}(X)$$
is a homomorphism. Therefore, we have two *equivalent* ways of thinking about group actions $G \curvearrowright X$: Either as the data of such a homomorphism ρ or as a function $G \times X \to X$ satisfying the two properties given above.

Remark 1. One thing to take away from this is that group actions give us a systematic way of producing homomorphisms *out* of a group: any time we have $G \curvearrowright X$, we get a homomorphism $\rho : G \to \mathrm{Bij}(X)$. At this moment, we don't have many other systematic ways of producing homomorphisms.

Example 1. Every group G acts on itself! The group action is given by
$$G \times G \xrightarrow{(g,h) \mapsto gh} G.$$
The corresponding homomorphism is the homomorphism $m : G \to \mathrm{Bij}(G)$ you already saw in problem 8 on HW 2. You saw there that it was actually *injective*,

This action is called the **left multiplication action** of G on itself.

Example 2. The group S_n acts on the set $\{1, \dots, n\}$ by definition. The action is given by
$$(\sigma, i) \mapsto \sigma(i).$$
The corresponding homomorphism
$$S_n \to \mathrm{Bij}(\{1, \dots, n\}) = S_n$$
is just the *identity* homomorphism.

Example 3. More generally, any time a group G acts on the finite set $\{1, \dots, n\}$, we get a homomorphism $G \to S_n$.

Example 4. Every group G admits a *trivial action* on any set X, given by $g \cdot x = x$ for all $g \in G$ and $x \in X$. This corresponds to the trivial homomorphism $\rho : G \to \mathrm{Bij}(X)$ given by $\rho(g) = e$ for all $g \in G$. Note that this homomorphism is far from injective unless $G = \{e\}$ is itself the trivial group.

Stabilizers and orbits

Definition 2. Suppose that we have a group action $G \curvearrowright X$. For $x \in X$, the **stabilizer** of x is the subset
$$G_x = \{g \in G : \ g \cdot x = x\} \subset G.$$
If $g \cdot x = x$ so that $g \in G_x$, we will say that g **stabilizes** or **fixes** x.

Observation 3. $G_x \leq G$ is a *subgroup*.

Proof. We need to know that $e \in G_x$, which is clear since $e \cdot x = x$. We also need to know that, if $g_1, g_2 \in G_x$, then $g_1 g_2 \in G_x$. This is because
$$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = g_1 \cdot x = x,$$
where the last two equalities are using the fact that $g_1, g_2 \in G_x$. Finally, we need to know that $g \in G_x$ implies that $g^{-1} \in G_x$. This follows from
$$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = e \cdot x = x.$$
☐

Complementary to a stabilizer is:

Definition 3. The **orbit** of x is the subset
$$\mathcal{O}(x) = \{g \cdot x : \ g \in G\} \subset X.$$
This is the set of all elements of X that are 'reachable' from x via the paths provided by G.

Example 5. Consider the action of $G = S_3$ (or D_6: it's the same thing here) on the set $X = \{1, 2, 3\}$. Then the stabilizer G_1 of 1 consists of exactly two elements $G_1 = \{e, \tau\}$, where τ is the reflection across the median through the vertex 1 in the equilateral triangle. The orbit of 1 is the entire set $\mathcal{O}(1) = X$, since we can get from 1 to any other vertex via a rotation, for instance. Therefore, we have $|G_1| = 2$ and $|\mathcal{O}(1)| = 3$, and their product is $6 = |G|$.

The numerology in the previous example is no coincidence. It is a special case of the following:

Proposition 1 (Orbit-Stabilizer formula)**.** *Suppose that G is a finite group. Then $\mathcal{O}(x)$ is finite, and we have*

$$|G| = |G_x| \cdot |\mathcal{O}(x)|.$$