MATH 3311, FALL 2025: LECTURE 30, NOVEMBER 10

Video: `https://youtu.be/MplmxboN4Hw`
Semi-direct products and complements

Definition 1. Suppose that H, K are groups and that we have a homomorphism

$$\rho : H \to \mathrm{Aut}(K)$$

Then the **semi-direct product** $K \rtimes_\rho H$ is the *unique group* with underlying set $K \times H$ and with product given by

$$(k_1, h_1) \cdot (k_2, h_2) = (k_1\rho(h_1)(k_2), h_1h_2).$$

Proposition 1. *$K \rtimes_\rho H$ with the above multiplication is a group with (e, e) as the identity. Moreover:*

(1) K is isomorphic to the normal *subgroup $\{(k, e) :\ k \in K\} \le K \rtimes_\rho H$ (we will use this to view K as a normal subgroup of $K \rtimes \rho H$);*
(2) H is isomorphic to the subgroup $\{(e, h) :\ h \in H\} \le K \rtimes_\rho H$ (we will use this to view H as a subgroup of $K \rtimes \rho H$);
(3) H is a complement for K in $K \rtimes_\rho H$;
(4) The conjugation action of H on K is given by the homomorphism

$$\rho : H \to \mathrm{Aut}(K)$$

that was part of the data for defining the semi-direct product.

We now want to relate semi-direct products and complements. First, we have the following two observations from Lecture 28.

Observation 1. Suppose that we have $K \trianglelefteq G$ and $H \le G$ is a complement for K. The conjugation action of H on K yields a homomorphism $\rho : H \to \mathrm{Aut}(K)$ satisfying $\rho(h)(k) = hkh^{-1}$ for $h \in H$ and $k \in K$.

Observation 2. The function

$$\psi : K \times H \xrightarrow{(k,h)\mapsto kh} G$$

is a bijection of groups.

Observation 3. The above function ψ is an isomorphism of groups $K \rtimes_\rho H \xrightarrow{\simeq} G$.

Proof. We have to show that, with the multiplication in $K \rtimes_\rho H$, we have

$$\psi((k_1, h_1)(k_2, h_2)) = \psi((k_1, h_1))\psi((k_2, h_2)).$$

The left hand side is

$$\psi((k_1\rho(h_1)(k_2), h_1h_2)) = k_1\rho(h_1)(k_2)h_1h_2 = k_1h_1k_2h_1^{-1}h_1h_2 = k_1h_1k_2h_2.$$

The right hand side is

$$\psi((k_1, h_1))\psi((k_2, h_2)) = (k_1h_1)(k_2h_2) = k_1h_1k_2h_2.$$

The two sides are now clearly equal. □

Definition 2. In the above situation, we will say that G **is a(n internal) semi-direct product** of K by H. It is a *non-trivial* semidirect product if ρ is non-trivial. Equivalently, if G is *not* an internal *direct* product of K and H. (Why are these equivalent?).

Example 1. Suppose that $K = \mathbb{Z}/n\mathbb{Z}$. Then we know that $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$. Therefore, semidirect products of $\mathbb{Z}/n\mathbb{Z}$ by another gruop H are produced from homomorphisms $\rho : H \to (\mathbb{Z}/n\mathbb{Z})^\times$.

(1) We could take $H = (\mathbb{Z}/n\mathbb{Z})^\times$ and ρ to be the identity. This gives us a semidirect product $(\mathbb{Z}/n\mathbb{Z}) \rtimes_\rho (\mathbb{Z}/n\mathbb{Z})^\times$.

(2) If $H - \mathbb{Z}/5\mathbb{Z}$, then giving a non-trivial homomorphism $H \to (\mathbb{Z}/n\mathbb{Z})^\times$ amounts to giving an element of order 5 in $(\mathbb{Z}/n\mathbb{Z})^\times$. For example if $n = 11$, the element $4 \in (\mathbb{Z}/11\mathbb{Z})^\times$ has order 5, and so we obtain a non-trivial semidirect product
$$\mathbb{Z}/11\mathbb{Z} \rtimes_\rho \mathbb{Z}/5\mathbb{Z}$$
where $\rho : \mathbb{Z}/5\mathbb{Z} \to (\mathbb{Z}/11\mathbb{Z})^\times$ is given by $\rho(a) = 4^a$.

This essentially ends our discussion of possibly non-abelian groups this semester. We will now move on to a better understanding of *abelian* groups, and in particular the *finitely generated* ones.

Definition 3. Suppose that G is a group and $X \subset G$ is a subset. The **subgroup generated by** X, denoted $\langle X \rangle \leq G$ is the smallest subgroup of G containing X.

Remark 1. This notion makes sense, since the intersection of any collection of subgroups containing X is once again a subgroup containing X, so $\langle X \rangle$ can be taken to be the intersection of all subgroups containing X.

Remark 2. More concretely, we have
$$\langle X \rangle = \{x_1^{\pm 1} x_2^{\pm 2} \cdots x_m^{\pm -1} :\ x_i \in X,\ m \geq 1\}.$$
That is, we take all possible products of elements of X as well as of their inverses.

Remark 3. If $X = \{x\}$ is a singleton, then $\langle X \rangle = \langle x \rangle$ is just the cyclic subgroup generated by the element x.

Definition 4. G is **finitely generated** if there is a finite subset $X \subset G$ such that $\langle X \rangle = G$. In other words, there is a finite set of symbols such that every element of G can be expressed as a product of such symbols.

We will be concerned with the problem of *classifying* finitely generated abelian groups. That is, we want a complete, non-redundant list of such groups up to isomorphism.