# MATH 3311, FALL 2025: LECTURE 29, NOVEMBER 7

Video: `https://youtu.be/6igdJEfXE-M`

Semi-direct products

**Definition 1.** Suppose that $H, K$ are groups and that we have a homomorphism

$$\rho : H \to \mathrm{Aut}(K)$$

Then the **semi-direct product** $K \rtimes_\rho H$ is the *unique group* with underlying set $K \times H$ and with product given by

$$(k_1, h_1) \cdot (k_2, h_2) = (k_1\rho(h_1)(k_2), h_1h_2).$$

To make sense of this correctly, we have to prove:

**Proposition 1.** *$K \rtimes_\rho H$ with the above multiplication is a group with $(e, e)$ as the identity. Moreover:*

1. *$K$ is isomorphic to the* normal *subgroup $\{(k, e) : k \in K\} \leq K \rtimes_\rho H$ (we will use this to view $K$ as a normal subgroup of $K \rtimes \rho H$);*
2. *$H$ is isomorphic to the subgroup $\{(e, h) : h \in H\} \leq K \rtimes_\rho H$ (we will use this to view $H$ as a subgroup of $K \rtimes \rho H$);*
3. *$H$ is a complement for $K$ in $K \rtimes_\rho H$;*
4. *The conjugation action of $H$ on $K$ is given by the homomorphism*

$$\rho : H \to \mathrm{Aut}(K)$$

*that was part of the data for defining the semi-direct product.*

*Proof.* It's not difficult to see that we have $(e, e) \cdot (k, h) = (k, h) \cdot (e, e) = (k, h)$. The first holds because $\rho(e)(k) = k$, and the second because $\rho(h)(e) = e$.

Next, we have to check for the existence of inverses. I claim that $(k, h)^{-1} = (\rho(h^{-1})(k^{-1}), h^{-1})$. Indeed, we see

$$\begin{aligned}
(k, h)(\rho(h^{-1})(k^{-1}), h^{-1}) &= (k\rho(h)(\rho(h^{-1})(k^{-1})), hh^{-1}) \\
&= (k\rho(hh^{-1})(k^{-1}), e) \\
&= (k\rho(e)(k^{-1}), e) \\
&= (kk^{-1}, e) \\
&= (e, e).
\end{aligned}$$

The most annoying thing to check to finish the proof of the fact that $K \rtimes_\rho H$ is a group is associativity, but let's just do it.

$$\begin{aligned}
(k_1, h_1) \cdot ((k_2, h_2) \cdot (k_3, h_3)) &= (k_1, h_1) \cdot (k_2\rho(h_2)(k_3), h_2h_3) \\
&= (k_1\rho(h_1)(k_2\rho(h_2)(k_3)), h_1h_2h_3) \\
&= (k_1\rho(h_1)(k_2)\rho(h_1h_2)(k_3), h_1h_2h_3).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
((k_1, h_1) \cdot (k_2, h_2)) \cdot (k_3, h_3) &= (k_1\rho(h_1)(k_2), h_1h_2) \cdot (k_3, h_3) \\
&= (k_1\rho(h_1)(k_2)\rho(h_1h_2)(k_3), h_1h_2h_3).
\end{aligned}$$

These are clearly equal.

For (1) and (2), note that

$$(k_1, e)(k_2, e) = (k_1\rho(e)(k_2), e) = (k_1k_2, e) \; ; \; (e, h_1)(e, h_2) = (e\rho(h_1)(e), h_1h_2) = (e, h_1h_1),$$

and

$$(k_1, h_1)(k_2, e)(k_1, h_1)^{-1} = (k_1\rho(h_1)(k_2), h_1)(\rho(h_1^{-1})(k_1^{-1}), h_1^{-1}) = (k_1\rho(h_1)(k_2)k_1^{-1}, e).$$

The first pair of identities show that we have subgroups isomorphic to $K$ and $H$, and the second shows that (the subgroup isomorphic to) $K$ is normal in $K \rtimes_\rho H$.

For (3), note simply that, by construction, every element of $K \rtimes_\rho H$ can be written in the form

$$(k, e)(e, h) = (k\rho(e)(e), h) = (k, h).$$

For (4), note that by what we saw above, if $(k_1, h_1) = (e, h)$, and $k_2 = k$, then we have

$$(e, h)(k, e)(e, h)^{-1} = (\rho(h)(k), e).$$

□