MATH 3311, FALL 2025: LECTURE 28, NOVEMBER 5

Video: `https://youtu.be/666COH53INo?si=8X_1PT4pNJLpmm47`

Complements

Recall from last time the notion of a complement to a normal subgroup.

Observation 1. Given $K \trianglelefteq G$ normal, and $H \leq G$, the following are equivalent:

(1) H is a complement for K;
(2) $H \xrightarrow{\pi|_H} G/K$ is an isomorphism;
(3) Every element $g \in G$ can be written uniquely in the form $g = hk$ for $h \in H$ and $k \in K$.

Example 1 (Non-example). Consider $G = \mathbb{Z}$ and $K = n\mathbb{Z}$ with $n \geq 2$. There is no subgroup $H \leq \mathbb{Z}$ with $H \xrightarrow{\simeq} \mathbb{Z}/n\mathbb{Z}$. Indeed, $\mathbb{Z}/n\mathbb{Z}$ is a finite group, and the only finite subgroup of $\mathbb{Z}$ is $\{0\}$ (why?). Therefore, this is a situation in which we have a normal subgroup with *no* complement.

We also saw that when the complement is also normal, then we actually have a *direct* product.

Proposition 1. *Suppose that $K \trianglelefteq G$ and $H \leq G$ is a complement to K. Then the following are equivalent:*

(1) $H \trianglelefteq G$ is also normal;
(2) H and K commute: $hk = kh$ for all $h \in H$ and $k \in K$;
(3) The function

$$\psi : H \times K \xrightarrow{(h,k)\mapsto hk} G$$

is an isomorphism of groups.

Definition 1. When the equivalent conditions of the proposition hold, we will say that G is an **internal direct product** of the subgroups H and K.

Remark 1. If $H \times K$ is the direct product of H and K, we can view H and K as the subgroups

$$H \simeq \{(h, e) : \ h \in H\} \ ; \ K \simeq \{(e, k) : \ k \in K\}$$

of $H \times K$. These subgroups are both normal and are complements to each other.

Remark 2. Whenever H is a complement to $K \trianglelefteq G$, the function

$$H \times K \xrightarrow{(h,k)\mapsto hk} G$$

is a *bijection*. This is essentially a reformulation of (3) of Observation 1. Note that this is not necessarily an isomorphism of *groups*, because in the direct product $H \times K$, H and K commute with each other, while this is not necessarily the case in G.

Given the previous remark, we can ask: What kind of structure does $H \times K$ have that would make this bijection an actual isomorphism? This leads to the notion of a semi-direct product.

Semi-direct products

Observation 2. If $K \trianglelefteq G$ and $H \leq G$ is a complement, then H acts on K via conjugation: $h \cdot k = hkh^{-1}$. This corresponds to a homomorphism of groups

$$\rho : H \to \mathrm{Aut}(K) \leq \mathrm{Bij}(K)$$

such that, for $h \in H$ and $k \in K$, $\rho(h)(k) = hkh^{-1} \in K$.

Proof. The main points are:

(1) The function
$$k \mapsto hkh^{-1}$$
is a bijection from K to K: That it takes K to K is because $K \trianglelefteq G$ is *normal*. It is a bijection, because it can be undone by conjugating by h^{-1}.
(2) The function above is actually a homomorphism:
$$h(k_1k_2)h^{-1} = (hk_1h^{-1})(hk_2h^{-1}).$$

□

Observation 3. The following are equivalent:
(1) ρ is trivial;
(2) H and K commute;
(3) G is an internal direct product of H and K.

Proof. The triviality of ρ is just saying that $hkh^{-1} = k$ for all $h \in H$ and $k \in K$, and this is equivalent to saying that H and K commute. The rest now follows from Proposition 1. □

Observation 4. If we have $g_1 = h_1k_1, g_2 = h_2k_2$ in G (where h_1, k_1 and h_2, k_2 are uniquely determined), then we see that
$$\begin{aligned} g_1g_2 &= (h_1k_1)(h_2k_2) \\ &= h_1h_2(h_2^{-1}k_1h_2)k_2 \\ &= h_1h_2\rho(h_2^{-1})(k_1)k_2. \end{aligned}$$

Remark 3. The h_2^{-1} showing up here is a bit annoying. So what we will do now is *switch* the order of appearance of H and K. If we can write $g = hk$, then we can also write it in the form $(hkh^{-1})h$, where $hkh^{-1} \in K$. In other words, every element of G can also be written uniquely in the form kh for some $k \in K$ and $h \in H$ (note that the k will not be the same as when we write it as product in the other order!!). From this perspective, we can rewrite the calculation in the previous observation:
$$\begin{aligned} g_1g_2 &= (k_1h_1)(k_2h_2) \\ &= k_1(h_1k_2h_1^{-1})h_1h_2 \\ &= (k_1\rho(h_1)(k_2))(h_1h_2). \end{aligned}$$

This leads to the following abstract definition.

Definition 2. Suppose that H, K are groups and that we have a homomorphism
$$\rho : H \to \mathrm{Aut}(K)$$
Then the **semi-direct product** $K \rtimes_\rho H$ is the *unique group* with underlying set $K \times H$ and with product given by
$$(k_1, h_1) \cdot (k_2, h_2) = (k_1\rho(h_1)(k_2), h_1h_2).$$