MATH 3311, FALL 2025: LECTURE 27, NOVEMBER 3

Video: https://youtu.be/s0p3kYcMWX0
Complements

Definition 1. Suppose that we have a normal subgroup $K \trianglelefteq G$ of a group G. A **complement** for K in G is a subgroup $H \leq G$ such that:

(i) $G = HK = \{hk : \ h \in H, \ k \in K\}$;
(ii) $H \cap K = \{e\}$.

Remark 1. Recall that $HK \leq G$ is always a subgroup and can be realized as the pre-image of $\pi(H) \leq G/K$ via the quotient homomorphism $\pi : G \to G/K$. In fact, the factoring triangle gives us an isomorphism

$$H/(H \cap K) \xrightarrow{\simeq} \pi(H) = HK/K \leq G/K.$$

Observation 1. Given $K \trianglelefteq G$ as above, and $H \leq G$, the following are equivalent:

(1) H is a complement for K;
(2) $H \xrightarrow{\pi|_H} G/K$ is an isomorphism;
(3) Every element $g \in G$ can be written uniquely in the form $g = hk$ for $h \in H$ and $k \in K$.

Proof. (1)⇔(2): Saying that (2) holds is equivalent to saying that $HK = G$ (surjectivity) and $H \cap K = \{e\}$ (injectivity),

Let us show (1)⇔(3): Assuming (1), we see that every element $g \in G$ can be written in the form $g = hk$ for *some* $h \in H$ and $k \in K$. If we can do this in two ways, so that $g = h_1k_1 = h_2k_2$, then we get an equality $h_2^{-1}h_1 = k_2k_1^{-1}$. The left hand side is in H and the right hand side is in K, telling us that this common element is in $H \cap K = \{e\}$. But that means that $h_2 = h_1$ and $k_2 = k_1$. The proof of (3)⇒(1) is similar: If g belongs to $H \cap K$, then we can write $g = h \cdot e$ or $g = e \cdot k$ where $h \in H$ and $k \in K$. The only way for this to be possible is if $g = h = k = e$. □

Example 1. Suppose that $|G| = pq^m$ where $p < q$ are primes. Let $Q \leq G$ be the Sylow q-subgroup: this has index p and so is normal (why?). If $P \leq G$ is any Sylow p-subgroup, it will be a complement for Q. Indeed, consider the homomorphism $\pi|_P : P \to G/Q$. Both the source and target of this homomorphism have order p and so must be cyclic of order p. The homomorphism is non-trivial, since its kernel is $P \cap Q = \{e\}$, and so must in fact be an isomorphism (why?).

What happens if the complement is also normal? Things actually have to be extra simple in this case.

Proposition 1. *Suppose that $K \trianglelefteq G$ and $H \leq G$ is a complement to K. Then the following are equivalent:*

(1) $H \trianglelefteq G$ is also normal;
(2) H and K commute: $hk = kh$ for all $h \in H$ and $k \in K$;
(3) The function

$$\psi : H \times K \xrightarrow{(h,k)\mapsto hk} G$$

is an isomorphism of groups.

Proof. (1)⇒(2): This is the most interesting part. Given $h \in H$ and $k \in K$ consider the element $k^{-1}hkh^{-1} \in G$. Writing it as $k^{-1} \cdot (hkh^{-1})$ and using the normality of K shows that this element belongs to K. On the other hand, writing it as $(k^{-1}hk) \cdot h^{-1}$ and using the normalizty of H shows that it also belongs to H. But since $H \cap K = \{e\}$, we conclude that we have

$$k^{-1}hkh^{-1} = e \Leftrightarrow hkh^{-1} = k \Leftrightarrow hk = kh.$$

(2)⇒(3): That the function is a bijection is just another reformulation of (3) of Proposition 1. We need to check that it is a homomorphism:

$$\begin{aligned}\psi((h_1,k_1))\cdot f((h_2,k_2)) &= (h_1k_1)(h_2k_2)\\ &= h_1(k_1h_2)k_2\\ &= h_1(h_2k_1)k_2 && = (h_1h_2)(k_1k_2)\\ &= \psi((h_1h_2,k_1k_2)).\end{aligned}$$

Here, in the third line, we have used the fact that H and K commute.

(3)⇒(1): This is because the subgroups $H \simeq \{(h,e) : h \in H\} \leq H \times K$ and $K \simeq \{(e,k) : k \in K\}$ are both normal in $H \times K$ and these are carried via the isomorphism ψ onto the subgroups $H, K \leq G$. □

Definition 2. When the equivalent conditions of the proposition hold, we will say that G is an **internal direct product** of the subgroups H and K.