MATH 3311, FALL 2025: LECTURE 25, OCTOBER 27

Video: `https://www.youtube.com/watch?v=jbJ8KdTihOs`
<u>Conjugacy in S_n and A_n</u>

Observation 1. If $\alpha = (a_1 \ \cdot \ a_m)$ is an m-cycle and $\sigma \in S_n$ is any permutation, then $\sigma\alpha\sigma^{-1}$ is also an m-cycle, and we in fact have

$$\sigma\alpha\sigma^{-1} = (\sigma(a_1) \ \cdots \ \sigma(a_m)).$$

Remark 1. This should be thought of as an analogy with change of basis: If A is an invertible matrix and B is any square matrix of the same dimension, then ABA^{-1} can be thought of as still describing the linear transformation corresponding to B, but with respect to a different basis. Similarly, $\sigma\alpha\sigma^{-1}$ is doing the same thing as α, except that we have changed our labeling from $\{1, 2, \dots, n\}$ to $\{\sigma(1), \dots, \sigma(n)\}$.

Example 1. If $\alpha = (2\ 3\ 4)$ and $\sigma = (1\ 2)$, then

$$\sigma\alpha\sigma^{-1} = (\sigma(2)\ \sigma(3)\ \sigma(4)) = (1\ 3\ 4).$$

Remark 2. If

$$\sigma = \alpha_1\alpha_2\cdots\alpha_r$$

is the decomposition into disjoint cycles, then

$$\tau\sigma\tau^{-1} = \tau(\alpha_1\cdots\alpha_r)\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})\cdots(\tau\alpha_r\tau^{-1}).$$

What this means is that σ and $\tau\sigma\tau^{-1}$ have the *same* cycle type.

Conversely, if σ and β have the same cycle type, then we can find a τ such that $\tau\sigma\tau^{-1} = \beta$. For example, if

$$\sigma = (1\ 2\ 3)(4\ 5)\ ,\ \beta = (a\ b\ c)(d\ e),$$

then any permutation τ that satisfies $\tau(1) = a, \tau(2) = b, \tau(3) = c, \tau(4) = d, \tau(5) = e$ will work for the equality

$$\tau\sigma\tau^{-1} = (\tau(1)\ \tau(2)\ \tau(3))(\tau(4)\ \tau(5)) = \beta.$$

Definition 1 (Cycle type)**.** Given an element $\sigma \in S_n$, the **cycle type** of σ is an n-tuple $(m_1, m_2, \dots, m_n)$ of non-negative integers, where for each i, m_i is the number of i-cycles in the disjoint cycle decomposition of σ (including 1-cycles for fixed points!).

This gives us the following observation:

Observation 2. Two permutations $\sigma, \beta \in S_n$ are conjugate to each other (that is, there exists $\tau \in S_n$ such that $\tau\sigma\tau^{-1} = \beta$) *if and only if* they have the same cycle type. In other words, we have a bijection

$$\{\text{conjugacy classes in } S_n\} \leftrightarrow \{\text{cycle types}\}.$$

Recall that conjugacy classes in a group G are the orbits for the conjugation action of G on itself.

Observation 3. Orbit-stabilizer tells us that, for any group G and $h \in G$ we have a bijection,

$$G/C_G(h) \xrightarrow{\simeq} \{ghg^{-1} :\ g \in G\} = \text{Conjugacy class of } h.$$

In particular, if G is finite, then we have

$$|G| = |C_G(h)| \cdot |\text{Conjugacy class of } h|.$$

In problem 6 on HW 8, we looked at the case of a subgroup $H \leq G$ of index 2. In this case, for $h \in H$, we obtain two equalities:

$$|G| = |C_G(h)| \cdot |\text{Conjugacy class of } h \text{ in } G|;$$
$$|G|/2 = |H| = |C_H(h)| \cdot |\text{Conjugacy class of } h \text{ in } H|.$$

The main point of that problem now is that, since the factors on the right in the second equation are less than or equal to those in the first, exactly one of the following holds:

- $|C_G(h)| = 2|C_H(h)|$: In other words, there is an element $g \in G \backslash H$ such that $ghg^{-1} = h$;
- The conjugacy class of h in H is half the size of its conjugacy class in G.

This leads to:

Observation 4. The conjugacy class of an element $\sigma \in A_n$ consists of all elements of the same cycle type if and only if there is an *odd* element in the centralizer $C_{S_n}(\sigma)$.

Example 2. Consider the case of a 3-cycle $\alpha = (1\ 2\ 3) \in A_5$. Its conjugacy class in S_5, consisting of all 3-cycles, has $\binom{5}{3} \cdot 2 = 20$ elements. This shows that $|C_{S_5}(\alpha)|$ has size $|S_5|/20 = 120/20 = 6$. One can actually write down all the elements in this centralizer: They are all obtained from products of powers of $(1\ 2\ 3)$ and the odd element $(4\ 5)$. Note that the presence of the odd element ensures that the conjugacy class in A_5 is still comprised of all 3-cycles.

Example 3. Consider the case of a 5-cycle $\beta = (1\ 2\ 3\ 4\ 5)$: Its conjugacy class in S_5 consisting of all 5-cycles, has $4! = 24$ elements. Therefore, its centralizer has $120/24 = 5$ elements, which are all generated by powers of β. In particular, this centralizer has no odd elements, which tells us that the set of 5-cycles splits into two conjugacy classes in A_5.