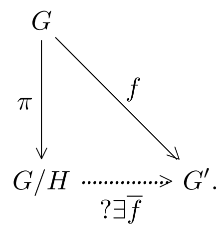
MATH 3311, FALL 2025: LECTURE 17, OCTOBER 5

Video: `https://youtu.be/6Xg66pFKVWc`

Recall from last time:

Proposition 1 (The Factoring Triangle)**.** *Suppose that $H \trianglelefteq G$ is a normal subgroup and that $f : G \to G'$ is a group homomorphism. Consider the following picture:*



The following are equivalent:

(1) There exists a (necessarily unique) homomorphism $\overline{f} : G/H \to G'$ such that $f = \overline{f} \circ \pi$.
(2) $f(H) = \{e\}$, or equivalently $H \leq \ker f$.

Before we look at applications of the triangle, let us introduce some definitions that will be helpful.

Definition 1. If $f : G \to G'$ is a homomorphism of groups, the **image** of f is given by

$$\mathrm{im}\, f = \{f(g) : g \in G\} \subset G'$$

Let us record the properties of images and kernels in the form of the following proposition.

Proposition 2. *Suppose that $f : G \to G'$ is a homomorphism of groups. Then:*

(1) $\ker f \trianglelefteq G$ is a normal subgroup of G.
(2) $\mathrm{im}\, f \leq G'$ is a subgroup of G'.
(3) f is injective if and only if $\ker f = \{e\}$.
(4) f is surjective if and only if $\mathrm{im}\, f = G'$.

Proof. (1) of course is something we've already seen several times.

As for (2), we have to verify:

(1) $e \in \mathrm{im}\, f$: This is because $e = f(e)$.
(2) If $g_1', g_2' \in \mathrm{im}\, f$, then $g_1' g_2' \in \mathrm{im}\, f$: This is because if $g_1' = f(g_1)$ and $g_2' = f(g_2)$, then

$$g_1' g_2' = f(g_1) f(g_2) = f(g_1 g_2).$$

(3) If $g' \in \mathrm{im}\, f$, then $(g')^{-1} \in \mathrm{im}\, f$: This is because if $g' = f(g)$, then

$$(g')^{-1} = f(g)^{-1} = f(g^{-1}).$$

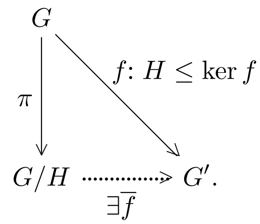
(4) is more or less immediate from the definitions, since surjectivity amounts to the statement that $\mathrm{im}\, f = f(G) = G'$.

For (3), we proceed as follows. First, if f is injective, then it is clear that $e \in G$ must be the only element satisfying $f(e) = e$. Thus, $\ker f = \{e\}$. Conversely, if $\ker f = \{e\}$, then we must see that f is injective: that is, we must show that $f(g_1) = f(g_2)$ means that $g_1 = g_2$. For this we note:

$$\begin{aligned} f(g_1) = f(g_2) &\Leftrightarrow e = f(g_1)^{-1} f(g_2) \\ &\Leftrightarrow e = f(g_1^{-1} g_2) \\ &\Leftrightarrow g_1^{-1} g_2 \in \ker f \\ &\Leftrightarrow e = g_1^{-1} g_2 \\ &\Leftrightarrow g_1 = g_2. \end{aligned}$$

In the second to last equality, we have used our hypothesis that $\ker f = \{e\}$. $\square$

Suppose that we have a factoring triangle:



Fact 1. We have $\operatorname{im} \overline{f} = \operatorname{im} f$.

Proof. This is because $\overline{f}(gH) = f(g)$ for all $g \in G$. So f and $\overline{f}$ have the exact same outputs. $\square$

Fact 2. We have

$$\ker \overline{f} = \{gH : \overline{f}(gH) = f(g) = e\} = \{gH : \ g \in \ker f\}.$$

As immediate consequences, we obtain:

Fact 3. $\overline{f}$ is surjective if and only if f is surjective.

Fact 4. $\overline{f}$ is injective if and only if $H = \ker f$.

Proof. We see that $\overline{f}(gH) = f(g) = e$ exactly when $g \in \ker f$. Now, $\overline{f}$ has trivial kernel precisely when $\overline{f}(gH) = e$ implies that $gH = H$. Combining these two observations, we see that $\overline{f}$ is injective precisely when $H = \ker f$.

Informally, everything in $\ker f$ dies in G', and everything in H dies in G/H (and no more). Therefore, $\overline{f}$ is injective precisely when everything that dies in G' is already dead in G/H. $\square$

Combining the two previous facts, we get:

Fact 5. $\overline{f}$ is an isomorphism if and only if f is surjective *and* $H = \ker f$.

Remark 1. Another way of saying this is that, if $f : G \to G'$ is a homomorphism with two properties:

(1) $\ker f = H$.
(2) f is surjective.

Then $\overline{f} : G/H \to G'$ is an *isomorphism*.

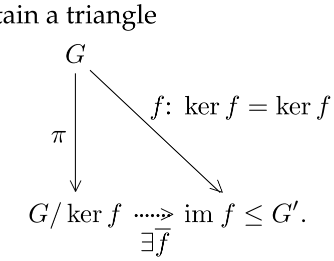
Remark 2. The above remark allows you to recognize the quotient homomorphism up to isomorphism using the two given properties. We gave an explicit construction of it using cosets, but if you had an alternate way of constructing a homomorphism with the same properties, then it will automatically be isomorphic to the construction using cosets.

We will now apply this to various contexts.

Proposition 3 (First isomorphism theorem)**.** *Suppose that $f : G \to G'$ is a homomorphism. Then there is a canonical isomorphism of groups*

$$G/\ker f \xrightarrow{\simeq} \operatorname{im} f.$$

Proof. Apply the factoring triangle to f viewed as a *surjective* homomorphism $f : G \to \operatorname{im} f \le G'$ with codomain $\operatorname{im} f$ and $H = \ker f$. We then obtain a triangle



The bottom arrow is surjective and is in fact an isomorphism by Remark 1. $\square$

Fact 6. If G is a finite group, then we obtain an equality of orders

$$|G/\ker f| = |\mathrm{im}\, f|.$$

Applying Lagrange to the left hand side, we get

$$|G| = |\ker f| \cdot |\mathrm{im}\, f|.$$

Example 1. Take $G = \mathbb{Z} \times \mathbb{Z}$, and take $H \leq G$ to be the subgroup generated by $(1, 1)$. You can easily see

$$H = \{(a, a) \in \mathbb{Z} \times \mathbb{Z} : \ a \in \mathbb{Z}\}.$$

Since we are in an abelian situation everything is normal, and so we can make sense of the quotient G/H. To understand the quotient homomorphism, it is enough to write down some surjective homomorphism

$$f : G \to G'$$

with $\ker f = H$. You can easily check that the function

$$f : G = \mathbb{Z} \times \mathbb{Z} \xrightarrow{(a,b)\mapsto a-b} \mathbb{Z}$$

is such a homomorphism. Therefore, we have

$$\overline{f} : G/H \xrightarrow{\simeq} \mathbb{Z}.$$

Note that we didn't have to know anything about the explicit coset description for G/H!

Proposition 4 (Second isomorphism theorem)**.** *Suppose that $K \trianglelefteq G$ is a normal subgroup and $H \leq G$ is another subgroup. Write $\pi|_H : H \to G/K$ for the* restriction *of the quotient homomorphism $\pi : G \to G/K$ to H. Then:*

(1) There is a canonical isomorphism

$$H/(H \cap K) \xrightarrow{\simeq} \mathrm{im}\,(\pi|_H) = \pi(H) \leq G/K.$$

(2) $HK = \{hk : \ h \in H, k \in K\} \leq G$ is a subgroup containing K and we can also describe $\mathrm{im}\,(\pi|_H)$ *as the quotient group $HK/K \leq G/K$. Therefore, there is a canonical isomorphism*

$$H/(H \cap K) \xrightarrow{\simeq} HK/K.$$

Proof. Note that, since $K = \ker \pi$, we have

$$\ker(\pi|_H) = \{h \in H : \ \pi(h) = e\} = \{h \in H : \ h \in K\} = H \cap K.$$

Therefore, (1) is just an application of Proposition 3 to $\pi|_H$.

In (2), $HK \subset G$ is the *pre-image* of $\mathrm{im}\,(\pi|_H)$ under the quotient homomorphism π. Indeed, saying that $g \in G$ is such that $\pi(g) \in \mathrm{im}\,(\pi|_H)$ is equivalent to saying that $gK = hK$ for some $h \in K$. This in turn is equivalent to saying that $h^{-1}g = k \in K$. Multiplying the equality by h now tells us that $g = hk$ for some $h \in H, k \in K$. This shows exactly the assertion about HK.

Now, it is a general fact that, under *any* homomorphism $f : G \to G'$ the pre-image of any subgroup $H' \leq G'$ is a subgroup $H \leq G$[1]

Therefore, $HK \leq G$ is a subgroup. It clearly contains K, and its image in G/K is exactly $\mathrm{im}\,(\pi|_H)$ by its construction as a pre-image! This shows (2). □

[1]Quick proof: If h_1, h_2 are in the pre-image H of H', then $f(h_1h_2) = f(h_1)f(h_2)$ is also in H', meaning that $h_1h_2 \in H'$. Clearly, $e \in H$, and if $h \in H$, then $f(h^{-1}) = f(h)^{-1} \in H'$, meaning that $h^{-1} \in H$.