MATH 3311, FALL 2025: LECTURE 16, OCTOBER 3

Video: `https://youtu.be/_HzhpdIR7xI`

Recall from last time: If we have a subgroup $H \leq G$, then we have the surjective function

$$\pi : G \xrightarrow{g \mapsto gH} G/H$$

from G onto the set of cosets of H in G.

Proposition 1 (Existence of quotient groups)**.** *The following are equivalent:*

(1) There exists some *homomorphism $f : G \to G'$ such that $H = \ker f$;*
(2) $H \trianglelefteq G$ is a normal subgroup;
(3) The function π is a homomorphism *of groups: That is, there is a (necessarily unique) structure of a group on G/H such that π satisfies $(g_1H)(g_2H) = \pi(g_1)\pi(g_2) = \pi(g_1g_2) = g_1g_2H$.*

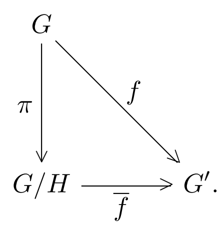
The main point here is that the operation $(g_1H)(g_2H) = g_1g_2H$ is *well-defined* exactly when H is normal in G.

Definition 1. When the equivalent conditions of the proposition hold, we say that G/H the **quotient group** of G by H and that $\pi : G \to G/H$ is the **quotient homomorphism**. By construction, we have

$$H = \ker \pi.$$

Observation 1. Suppose that $H \trianglelefteq G$ is a normal subgroup and $\pi : G \to G/H$ is the quotient homomorphism. Suppose that we have a homomorphism of groups $\overline{f} : G/H \to G'$. Then the composition $f = \overline{f} \circ \pi$ satisfies $H \leq \ker f$.

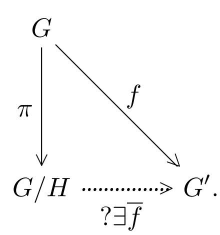
This is because π kills H, and therefore the composition $\overline{f} \circ \pi$ must also necessarily kill H. We can view this as a diagram



where the diagonal arrow kills H because the left vertical arrow does.

In fact, we can considerably strengthen the observation into the following *fundamental* result, which tells us exactly how to build homomorphisms out of the quotient group.

Proposition 2 (The Factoring Triangle)**.** *Suppose that $H \trianglelefteq G$ is a normal subgroup and that $f : G \to G'$ is a group homomorphism. Consider the following picture:*



The following are equivalent:

(1) There exists a (necessarily unique) homomorphism $\overline{f} : G/H \to G'$ such that $f = \overline{f} \circ \pi$. In this case, we will say that f ***factors through*** *π.*
(2) $f(H) = \{e\}$, or equivalently $H = \ker \pi leq \ker f$.

Consequence 1. Giving a homomorphism $\overline{f} : G/H \to G'$ is *equivalent* to giving a homomorphism $f : G \to G'$ such that $H \leq \ker f$, equivalently such that $f(H) = \{e\}$.

Slogan 1. G/H is the guardian to the world where H is dead, i.e. smushed down to the identity.

Example 1. If $H = n\mathbb{Z} \leq \mathbb{Z} = G$, then we see that giving a homomorphism $\overline{f} : \mathbb{Z}/n\mathbb{Z} \to G'$ is equivalent to giving a homomorphism $f : \mathbb{Z} \to G'$ such that $f(n\mathbb{Z}) = \{e\}$. Since n generats $n\mathbb{Z}$, this is equivalent to saying that $f(n) = e$.

Now, every homomorphism $f : \mathbb{Z} \to G'$ is given by $f(a) = g^a$ for some fixed $g \in G'$. This satisfies $f(n) = e$ precisely when $g^n = e$. In other words, we obtain a canonical bijection

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G') = \{g \in G' : \ g^n = e\}$$

This gives a quick and systematic way of recovering the results of HW 3, Problem 5. In terms of the factoring triangle, we have

$$\begin{array}{ccc} \mathbb{Z} & & \\ \Big\downarrow \pi & \searrow^{f \,:\, a \mapsto g^a \text{ such that } g^n = f(1)^n = e} & \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow[\overline{f}]{} & G'. \end{array}$$

Proof of the factoring triangle. Observation 1 shows (1)⇒(2). Let us show (2)⇒(1). The only possible definition for $\overline{f}$ is given by

$$\overline{f}(gH) = f(g) \in G'.$$

The main question here, as always when things are defined in terms of coset representatives, is if the assignment is *independent* of the representative g. If g' is another representative, then we have $g' = gh$ for some $h \in H$. But then

$$\overline{f}(g'H) = f(g') = f(gh) = f(g)f(h) = f(g)e = f(g) \in G'$$

where we have used the assumption that $f(H) = \{e\}$ to see that $f(h) = e$.

More succinctly, as sets we have $f(gH) = \{f(g)\}$ because $f(H) = \{e\}$: so the value of f on something only depends on its coset with respect to H. □

Example 2. Take $G = \mathbb{Z} \times \mathbb{Z}$, and take $H \leq G$ to be the subgroup generated by $(1, 1)$. You can easily see

$$H = \{(a, a) \in \mathbb{Z} \times \mathbb{Z}\}.$$

Since we are in an abelian situation everything is normal, and so we can make sense of the quotient G/H. To understand the quotient homomorphism, it is enough to write down some surjective homomorphism

$$f : G \to G'$$

with $\ker f = H$. You can easily check that the function

$$f : G = \mathbb{Z} \times \mathbb{Z} \xrightarrow{(a,b) \mapsto b} \mathbb{Z}$$

given by projection onto the second coordinate is such a homomorphism. Therefore, we have

$$\overline{f} : G/H \xrightarrow{\simeq} \mathbb{Z}.$$

Note that we didn't have to know anything about the explicit coset description for G/H!

Example 3. Let us look at the subgroup $H = \langle (1, 1) \rangle$ now. I claim that we have

$$(\mathbb{Z} \times \mathbb{Z})/\langle (1, 1) \rangle$$

is isomorphic to $\mathbb{Z}$.

We will do this fully in the next lecture, but for now, let us understand the following: In order to construct a *homomorphism* $(\mathbb{Z} \times \mathbb{Z})/\langle (1, 1) \rangle \to \mathbb{Z}$, we just have to construct a homomorphism

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

such that $\langle(1,1)\rangle \leq \ker f$. It is easy to see that one such homomorphism is given by $f((a,b)) = b - a$. This gives us a factoring triangle: