MATH 3311, FALL 2025: LECTURE 15, OCTOBER 1

Video: `https://youtu.be/K_Q8XrbIPF8`

Recall from last time the following summary of the structure of group actions:

Proposition 1. *If $G \curvearrowright X$ is a group action, then we have*

$$X = \bigsqcup \mathcal{O}$$

where $\mathcal{O} \subset X$ are the distinct orbits. Moreover, if $x \in \mathcal{O}$ (so that $\mathcal{O} = \mathcal{O}(x)$), then there is an isomorphism of group actions

$$G/G_x \xrightarrow{\simeq} \mathcal{O}(x).$$

If we have a subgroup $H \leq G$, then we have the surjective function

$$\pi : G \xrightarrow{g \mapsto gH} G/H$$

from G onto the set of cosets of H in G.

Question 1. When does G/H have the structure of a group? More precisely, when can we view π as a *group homomorphism*?

Example 1. If $G = \mathbb{Z}$ and $H = n\mathbb{Z}$, then $\mathbb{Z}/n\mathbb{Z}$ has a group structure as an additive group such that the natural surjective function $\mathbb{Z} \xrightarrow{a \mapsto a \pmod{n}} \mathbb{Z}/n\mathbb{Z}$ is a group homomorphism.

Let us make some quick observations.

Observation 1. The only possible group structure on G/H for which π is a homomorphism is given in terms of coset representatives by

$$(g_1H)(g_2H) = g_1g_2H.$$

Observation 2. If π is a group homomorphism, then $H = \ker \pi$.

Proof. The way the group operation in G/H is supposed to work, the *identity* coset H plays the role of the identity *element*. Therefore, we have

$$\ker \pi = \{g \in G : \pi(g) = gH = H\} = H$$

where the second equality follows from the obsevation that $gH = H$ precisely when $g \in H$. □

Observation 3. For any group homomorphism $f : G \to G'$, $\ker f \leq G$ is a *normal* subgroup.

Proof. We need to verify the following properties:

(1) $e \in \ker f$: This is because $f(e) = e$.

(2) If $h_1, h_2 \in \ker f$, then $g_1g_2 \in \ker f$: This is because

$$f(h_1h_2) = f(h_1)f(h_2) = e \cdot e = e.$$

(3) If $h \in \ker f$, then $h^{-1} \in \ker f$: This is because

$$f(h^{-1}) = f(h)^{-1} = e^{-1} = e.$$

(4) (Normality) If $h \in \ker f$ and $g \in G$, then $ghg^{-1} \in \ker f$: This is because

$$f(ghg^{-1}) = f(g)f(h)f(g)^{-1} = f(g)ef(g)^{-1} = f(g)f(g)^{-1} = f(gg^{-1}) = f(e) = e.$$

□

Combining the two previous observations, we find:

Observation 4. If π is a group homomorphism, then $H = \ker \pi$ is a *normal* subgroup of G.

So normality is a *necessary* condition for π to be a homomorphism of groups; or, equivalently, for the operation $(g_1H)(g_2H) = g_1g_2H$ to be *well-defined*. Let us see an example where this fails.

Example 2. If $G = D_{2n}$ and $H \leq G$ is the subgroup generated by τ, then G/H has size n (why?). If we take the coset σH and 'multiply' it by itself we get

$$(\sigma H)(\sigma H) = \sigma^2 H.$$

But we can also represent σH as $\sigma\tau H$. If we use this instead for the first factor, then we get

$$(\sigma\tau H)(\sigma H) = \sigma\tau\sigma H.$$

Since $\sigma\tau = \tau\sigma^{-1}$, we can rewrite the right hand side as

$$\tau\sigma^{-1}\sigma H = \tau H = H.$$

Clearly, this is *not* equal to $\sigma^2 H$. This shows that the operation we wanted to define is not well-defined, and this is happening precisely because H is not normal in G.

Proposition 2 (Existence of quotient groups)**.** *The following are equivalent for a subgroup $H \leq G$:*

(1) There exists some *homomorphism $f : G \to G'$ such that $H = \ker f$;*
(2) $H \trianglelefteq G$ is a normal subgroup;
(3) The function π is a homomorphism *of groups: That is, there is a (necessarily unique) structure of a group on G/H such that π satisfies $(g_1H)(g_2H) = \pi(g_1)\pi(g_2) = \pi(g_1g_2) = g_1g_2H$.*

Proof. (1)⇒(2): This is Observation 3.

(3)⇒(1): This is Observation 2. Basically, we can take f to be the homomorphism $\pi : G \to G/H$.

To complete the circle, we must show (2)⇒(3). This amounts to the assertion that the operation

$$(g_1H)(g_2H) = g_1g_2H$$

on G/H is well-defined *independent* of the choice of coset representatives. If we replace g_1H and g_2H with g_1h_1H and g_2h_2H for $h_1, h_2 \in H$, then the product now becomes

$$\begin{aligned}(g_1h_1H)(g_2h_2H) &= g_1h_1g_2h_2H \\ &= g_1h_1g_2H \\ &= g_1g_2(g_2^{-1}h_1g_2)H \\ &= g_1g_2H,\end{aligned}$$

where in the last equality, we have used the normality of H to conclude that $g_2^{-1}h_1g_2 \in H$. □

Definition 1. When the equivalent conditions of the proposition hold, we say that G/H the **quotient group** of G by H and that $\pi : G \to G/H$ is the **quotient homomorphism**. By construction, we have

$$H = \ker \pi.$$