MATH 3311, FALL 2025: LECTURE 11, SEPTEMBER 19

Video: `https://youtu.be/vPj0_PzVBSo`

Our first goal today is to prove:

Proposition 1 (Orbit-Stabilizer I). *Suppose that G is a finite group acting on a set X. Then for $x \in X$, $\mathcal{O}(x)$ is finite, and we have*

$$|G| = |G_x| \cdot |\mathcal{O}(x)|.$$

Proof. We start by considering the function

$$\begin{aligned} \varphi_x : G &\to X \\ g &\mapsto g \cdot x. \end{aligned}$$

This takes an element g to the end-point of the path along g starting at x.

The *range* of this function is precisely the orbit $\mathcal{O}(x)$, and so we can view φ_x as a *surjective* function

$$\varphi_x : G \to \mathcal{O}(x).$$

The key now is to study the 'fibers' or pre-images of this function: What are the inputs that yield a fixed output $y \in \mathcal{O}(x)$? Another way of asking this: When do we have $\varphi_x(g_1) = \varphi_x(g_2)$?

For this, note:

$$\begin{aligned} \varphi_x(g_1) = \varphi_x(g_2) &\Leftrightarrow g_1 \cdot x = g_2 \cdot x \\ &\Leftrightarrow g_1^{-1} \cdot (g_1 \cdot x) = g_1^{-1} \cdot (g_2 \cdot x) \\ &\Leftrightarrow x = (g_1^{-1} g_2) \cdot x \\ &\Leftrightarrow g_1^{-1} g_2 \in G_x \\ &\Leftrightarrow g_1^{-1} g_2 = h \text{ for some } h \in G_x \\ &\Leftrightarrow g_2 = g_1 h \text{ for some } h \in G_x. \end{aligned}$$

Therefore, we have $\varphi_x(g_1) = \varphi_x(g_2)$ precisely when g_2 is of the form $g_1 h$ for some $h \in G_x$.

What we have here is a specific instance of the following general notion:

Definition 1. Given a subgroup $H \leq G$ and $g \in G$, the **left coset** for g with respect to H is the subset

$$gH = \{gh : \ h \in H\}.$$

In terms of this definition, what we have shown is that, for $y = g_1 \cdot x$, we have

$$\{g_2 \in G : \ \varphi(g_2) = y\} = g_1 G_x.$$

In other words, the pre-images of the map φ_x are left cosets for G_x.

All of this works without assuming anything about G. Now suppose that G is *finite*. Then we can count the number of elements of G by first fixing a possible output for φ_x, and then counting for each such output the elements in G mapping to that output. That is, we have

$$|G| = \sum_{y \in \mathcal{O}(x)} |\{g \in G : \ \varphi_x(g) = y\}|.$$

This follows from a general fact:

Fact 1. If $f : X \to Y$ is a function. Then we can write X as a disjoint union

$$X = \bigsqcup_{y \in Y} \{x \in X : \ f(x) = y\}$$

In particular, if X is finite, then we have

$$|X| = \sum_{y \in Y} |\{x \in X : \ f(x) = y\}|.$$

But now, if $y = g_1 \cdot x = \varphi_x(g_1)$, then by what we just saw above, we have

$$\{g_2 \in G : \ \varphi_x(g_2) = y\} = g_1 G_x$$

From this, we see that

$$|\{g_2 \in G : \ \varphi_x(g_2) = y\}| = |g_1 G_x| = |G_x|.$$

Here, the second equality follows because multiplication by g_1 is a bijection, and therefore preserves sizes.

Therefore, we have

$$|G| = \sum_{y \in \mathcal{O}(x)} |G_x| = |G_x| \cdot |\mathcal{O}(x)|.$$

□

The orbit-stabilizer formula can help us count sizes of groups:

Example 1. Let G be the group of rotations of a cube in three-dimensional space. It acts on the set X of the six faces of the cube. If we start with one face, then it's not hard to see that you can bring it to any other face with a series of rotations. Therefore, the orbit of any particular face is all of X. The stabilizer G_x of a particular face $x \in X$ is the group of rotations that fix the center of that face, and it is not hard to see that we have exactly 4 such rotations, all around the axis through the center of that face. This means that $|G_x| = 4$. Therefore, the orbit-stabilizer formula now says

$$|G| = |G_x| \cdot |\mathcal{O}(x)| = |G_x| \cdot |X| = 4 \cdot 6 = 24.$$

That is, we have exactly 24 rotations of the cube in three dimensional space.

Corollary 1. *If a finite group G acts on a finite set X, then we have*

$$|X| = \sum_{\text{orbits } \mathcal{O}} |\mathcal{O}|$$

where $|\mathcal{O}|$ divides $|G|$ for every orbit. Here, in the sum, we are counting each orbit exactly once.