MATH 3311, FALL 2025: LECTURE 1, AUGUST 25

Video link: `https://youtu.be/QBkOzRrhI6k`

What is algebra? In simple terms, it's what turns up when we notice that ostensibly different kinds of mathematical objects have the same essential structures governing their behavior. We then extract this essential structure and formalize it in the shape of axioms, which can then be used to argue more abstractly about those objects, with a view towards developing results that can apply to other objects with the same essential structure.

In this course, we will be looking examples of such essential structures, which will include *groups*, *rings* and *fields*.

Example 1. Let us step back for a moment to high school algebra and consider the equation $x^2 - 2 = 0$. This has two solutions: $\pm\sqrt{2}$. How do we tell them apart? Well, one is positive and the other negative. How do we distinguish the positive one? It is the only one of the two that has a real square-root, namely $\sqrt[4]{2}$.

Example 2. Consider instead the equation $x^2 + 1 = 0$. This also has two (imaginary) solutions $\pm i$. How do we tell these apart? We can't! Unlike in the previous example, there is no god-given choice. What we can say is simply that there are two roots, which are negatives of each other.

Example 3. To get a better feel, let us look at another equation $x^2 + x + 1$. By the quadratic formula, the zeros are

$$\frac{1 \pm i\sqrt{3}}{2}.$$

Note that the double sign $\pm$ accounts for the ambiguity in the choice of i: regardless of which choice we make, we will still get the same *pair* of complex numbers, though we don't have a way of necessarily distinguishing one from the other.

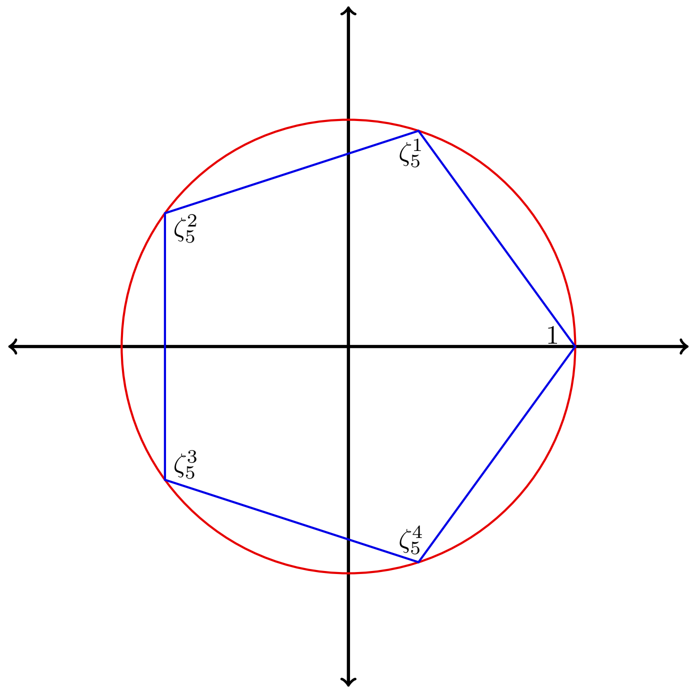
This is because there is a *symmetry*, **complex conjugation**, given by the operation $a + ib \mapsto a - ib$, which flips one zero to another, and we have no way of *breaking* the symmetry. This is true actually for any quadratic equation that does not admit real solutions: it will have two zeros each of which is the complex conjugate of the other, and we have no systematic way of breaking the symmetry between them.

Example 4. Take a more elaborate example: $x^5 - 1 = 0$. The solutions to this equation are the five fifth roots of 1: there is the obvious one, 1 itself, but also 4 more complex solutions, $\zeta_5 = e^{2\pi i/5}, \zeta_5^2, \zeta_5^3, \zeta_5^4$. The key point here is how multiplication works for complex numbers. The complex number of length 1 at an angle of θ to the real axis is $e^{i\theta}$. Moreover, we have $(e^{i\theta})^n = e^{in\theta}$; $e^{2m\pi i} = 1$, for any integers n, m.

This shows that $\zeta_5^5 = e^{2\pi i} = 1$. Similarly, $(\zeta_5^2)^5 = e^{4\pi i} = 1$, etc.

In general, if $\zeta_n = e^{2\pi i/n}$ is a solution to $x^n - 1 = 0$, as is any power of ζ_n.

Getting back to the solutions to $x^5 - 1$, they lie on the vertices of a regular pentagon circumscribed by the unit circle.



Certainly, we can distinguish 1 from the other four: It is the only *real* solution. However, it is not possible to distinguish the other four among themselves in any natural way. One can talk about the angle at which they lie, but why should the counterclockwise angle be any more natural than the clockwise one? We could have named any one of them ζ_5, and it would still be true that the non-real solutions are $\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$ (check this!).

What kinds of symmetry do these roots possess? Complex conjugation actually still gives one: It flips ζ_5 and ζ_5^4 as well as ζ_5^2 and ζ_5^3. But there is also a slightly more hidden symmetry. If we start taking cubes of the non-trivial fifth roots, we find the following:

$$\zeta_5 \mapsto \zeta_5^3 \mapsto \zeta_5^9 = \zeta_5^4 \mapsto \zeta_5^{12} = \zeta_5^2 \mapsto \zeta_5^6 = \zeta_5.$$

Here, I have repeatedly made use of the following phenomenon, which is a consequence of the fact that $\zeta_5^5 = 1$: ζ_5^m only depends on the *remainder* m leaves when you divide by 5. This is because $\zeta_5^{5q+r} = (\zeta_5^5)^q \zeta_5^r = 1^q \zeta_5^r = \zeta_5^r$.

This shows that the symmetries of the *non-trivial* zeros of $x^5 - 1$ actually interchange each of them with any other, and there is no reasonable way to break this symmetry in order to distinguish any particular subset.